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ON THE OSCILLATION OF CERTAIN NONLINEAR ELLIPTIC DIFFERENTIAL EQUATIONS

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Abstract: Using the integral average technique and a new function $H(r,s,l)$ defined in the sequel, some new oscillation criteria are obtained for second order elliptic differential equations with damping of the form

$$\nabla\cdot(A(x)\nabla y)+B^T(x)\nabla y+q(x)f(y)=0,\quad x\in\Omega,$$

where Ω is an exterior domain in $\mathbb{R}^N$. The main results are of a high degree of generality than many previous results.

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1. Introduction

In this paper, we consider the oscillation behavior of solutions to second order elliptic differential equations with damping of the form

$$\nabla\cdot(A(x)\nabla y)+B^T(x)\nabla y+q(x)f(y)=0, \tag{1}$$

where $x=(x_1,x_2,\cdots,x_N)\in\Omega(a_0)\subseteq\mathbb{R}^N, N\geq 2, \nabla=(\frac{\partial}{\partial x_1},\frac{\partial}{\partial x_2},\cdots,\frac{\partial}{\partial x_N}), |x|=\left[\sum_{i=1}^N x_i^2\right]^{1/2}, \Omega(a_0)=\{x\in\mathbb{R}^N:|x|\geq a_0\}$ for some $a_0>0$.

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In what follows, we always assume that:

(C_1) $A(x) = (A_{ij}(x))_{N\times N}$ is a real symmetric positive definite matrix function (ellipticity condition) with $A_{ij} \in C^{1+\mu}_{loc}(\Omega(a_0), \mathbb{R}), \mu \in (0,1), i, j = 1, \cdots, N$, $\lambda_{max}(x)$ denotes the largest (necessarily positive) eigenvalue of the matrix $A(x)$; there exists a function $\lambda \in C^1(\mathbb{R}^+, \mathbb{R}^+)$ such that

$$\lambda(r) \geq \max_{|x|=r} \lambda_{max}(x) \quad \text{for} \quad r > 0;$$

(C_2) $b_i \in C^{\mu}_{loc}(\Omega, \mathbb{R}), \mu \in (0,1), i = 1, \cdots, N$;

(C_3) $q \in C^{\mu}_{loc}(\Omega(a_0), \mathbb{R}), \mu \in (0,1)$ and $q(x) \not\equiv 0$ for $|x| \geq a_0$;

(C_4) $f \in C^1(\mathbb{R}, \mathbb{R}), yf(y) > 0$ and $f'(y) \geq k > 0$ for all $y \neq 0$ and some constant k.

A function $y \in C^{2+\mu}_{loc}(\Omega(a_0), \mathbb{R}), \mu \in (0,1)$, is said to be a solution of equation (1) in $\Omega(a_0)$, if $y(x)$ satisfies equation (1) for all $x \in \Omega(a_0)$. For the existence of solutions of equation (1), we refer the reader to the monograph [1]. We restrict our attention only to the nontrivial solution $y(x)$ of equation (1), i.e., for any $a > a_0$, $\sup\{|y(x)| : |x| > a\} > 0$. A nontrivial solution $y(x)$ of equation (1) is called oscillatory if the zero set $\{x : y(x) = 0\}$ of $y(x)$ is unbounded, otherwise it is called nonoscillatory. equation (1) is called oscillatory if all its nontrivial solutions are oscillatory.

There are a great number of papers (see, for example, [2–8] and the references quoted therein) devoted to the particular cases of equation (1), including the following second order ordinary differential equations:

$$y''(t) + q(t)y(t) = 0, \tag{2}$$

$$(r(t)y'(t))' + q(t)f(y(t)) = 0, \tag{3}$$

$$(r(t)y'(t))' + p(t)y'(t) + q(t)f(y(t)) = 0. \tag{4}$$

In 1999, Kong [13] proved the following theorem.

Theorem A. *equation (2) is oscillatory provided that for each $l \geq t_0$, there exists $\alpha > 1$ such that*

$$\limsup_{t\to\infty} \frac{1}{t^{\alpha-1}} \int_l^t (s-l)^{\alpha} q(s)ds > \frac{\alpha^2}{4(\alpha-1)}, \tag{5}$$

and

$$\limsup_{t\to\infty} \frac{1}{t^{\alpha-1}} \int_l^t (t-s)^{\alpha} q(s)ds > \frac{\alpha^2}{4(\alpha-1)}. \tag{6}$$

In 2004, by using an new kennel function, Sun [15] proved the following new Kamenev-type theorem.

Theorem B. *equation (4) with $r(t) \equiv 1$ is oscillatory provided that for each $l \geq t_0$, there exists a constant $\alpha > 1/2$ such that*

$$\limsup_{t\to\infty} \frac{1}{t^{2\alpha+1}} \int_l^t (t-s)^{2\alpha}(s-l)^2 \left[4kq(s) - p^2(s) + 4\frac{t-(1+\alpha)s+\alpha l}{(t-s)(s-l)}p(s)\right] ds > \frac{4\alpha}{(2\alpha-1)(2\alpha+1)}. \quad (7)$$

Some other similar results can be found in Sun [16], Dubé and Mingarelli [17],Yang [18].

For the semilinear elliptic equation

$$\nabla \cdot (A(x)\nabla y) + q(x)f(y) = 0, \quad (8)$$

Noussair and Swanson [10] first extended the Wintner theorem by using the following partial Riccati type transformation

$$W(x) = -\frac{\alpha(|x|)}{f(y(x))}(A\nabla y)(x), \quad (9)$$

where $\alpha \in C^2$ is an arbitrary positive function. Swanson [10] summarized the oscillation results for equation (8) up to 1979. For recent contributions, we refer the reader to Xu et al. [11, 12] and the reference therein.

This paper is motivated by [10, 12–15, 17, 18]. In section 2, by using an new kennel function $H(r,s,l)$, some oscillation criteria related to Euler integral for equation (1) are established for the case when $\frac{\partial b_i}{\partial x_i}$ does not exist for some i. Our results are extensions of the results of Kong [13], Li and Agarwal [14], Sun [15], and Yang [18].

In this work, we introduce the following two new real-valued functions which will be extensively used in the sequel.

Definition 1. Let $E_0 = \{(r,s,l) : r > s > l \geq a_0\}$ and $E = \{(r,s,l) : r \geq s \geq l \geq a_0\}$ and $H \in C^1(E,\mathbb{R})$. A real-valued function H is said to belong to Φ, denoted by $H \in \Phi$, if there exists a function $h \in C^1(E_0,\mathbb{R})$ satisfying the following conditions:

(H1) $H(r,r,l) = 0; H(r,l,l) = 0$ on E, and $H(r,s,l) > 0$ on E_0.

(H2) $H(r,s,l)$ has a continuous partial derivative on D_0 with respect to the second variable, and there is a function $h \in C(E_0;\mathbb{R})$ such that

$$\frac{\partial H(r,s,l)}{\partial s} = h(r,s,l), \quad (r,s,l) \in E_0.$$

Definition 2. Let $D = \{(r,s) : r \geq s \geq a_0\}, D_0 = \{(r,s) : r > s > a_0\}$, H_1 and $H_2 \in C^1(D,\mathbb{R})$. A pair of real-valued functions (H_1,H_2) is said to belong to Ξ, denoted by $(H_1,H_2) \in \Xi$, if there exist functions h_1 and $h_2 \in C^1(D_0,\mathbb{R})$ satisfying the following conditions:

(H1) $H_i(r,r) = 0$ for $r \geq a_0$ and $H_i(r,s) > 0$ on D_0 for $i = 1,2$;

(H2) $\frac{\partial H_1(r,s)}{\partial s} = -h_1(r,s), \forall (r,s) \in D_0$;

(H3) $\frac{\partial H_2(s,l)}{\partial s} = h_2(s,l), \forall (s,l) \in D_0$.

2. Main Results

For simplicity, we define the functions ρ, Q and g as follows: For any given function $\eta \in C^1([a_0,+\infty),\mathbb{R})$, let

$$\rho(r) = \exp\left[-\int_{a_0}^{r} \frac{k\eta(s)s^{1-N}}{\omega} ds\right],$$

$$Q(r) = \rho(r) \times \left\{\int_{S_r}\left[q(x) - \frac{1}{2k}\lambda(x)|B^T A^{-1}|^2\right] d\sigma + \frac{k}{2\omega}\lambda(r)\eta^2(r)r^{1-N} - [\lambda(r)\eta(r)]'\right\},$$

$$g(r) = \frac{2\omega}{k}\lambda(r)\rho(r)r^{N-1},$$

where $S_r = \{x \in \mathbb{R}^N : |x| = r\}, r > 0, B^T = (b_1(x),...,b_N(x)), d\sigma$ denotes the spherical integral element in $\mathbb{R}^N$, ω is the area of unit sphere in $\mathbb{R}^N$ and k is defined in (C_4).

We begin with the following lemma, the proof of this lemma is easy and thus omitted.

Lemma A. *For two n-dimensional vectors $u,v \in \mathbb{R}^N$, and a positive constant c, then*

$$cuu^T + uv^T \geq \frac{c}{2}uu^T - \frac{1}{2c}vv^T. \tag{10}$$

Theorem 2.1. *Equation (1) is oscillatory provided that for each* $l \geq a_0$, *there exist a function* $\eta \in C^1([a_0, +\infty), \mathbb{R})$ *and a function* $H \in \Phi$ *such that*

$$\limsup_{r\to\infty} \int_l^r \left[H(r,s,l)Q(s) - \frac{1}{4}\frac{g(s)h^2(r,s,l)}{H(r,s,l)} \right] ds > 0. \tag{11}$$

Proof. Suppose to the contrary that there exists a solution $y(x)$ of equation (1) such that $y(x) > 0$ for $|x| \geq a_1 \geq a$. Define

$$W(x) = \frac{1}{f(y)}(A\nabla y)(x), \qquad x \in G[a_1, \infty) \tag{12}$$

and

$$V(r) = \rho(r) \left[\int_{S_r} W(x)\cdot\gamma(x)d\sigma + \lambda(r)\eta(r) \right], \qquad x \in G[a_1, \infty), \tag{13}$$

where ∇y denotes the gradient of $y(x)$, $\gamma(x) = \frac{x}{|x|}$, $|x| \neq 0$ is the outward unit normal to S_r.

From equation (1) and (12), it follows that

$$\begin{aligned} \nabla\cdot W(x) &= -\frac{f'(y)}{f^2(y)}(\nabla y)^T A\nabla y - \frac{1}{f(y)}[q(x)f(y) + B^T\nabla y] \\ &\leq -kW^TA^{-1}W - q(x) - B^TA^{-1}W \\ &= -kW^TA^{-1}W - q(x) + B^TA^{-1}W \\ &\leq -\frac{k}{\lambda(x)}W^TW - q(x) + B^TA^{-1}W \qquad \text{(by Lemma A)} \\ &\leq -\frac{k}{2\lambda(x)}|W|^2 + \frac{1}{2k}\lambda(x)|B^TA^{-1}|^2 - q(x). \end{aligned} \tag{14}$$

Using Green's formula in (13), we get

$$\begin{aligned} V'(r) &= \frac{\rho'(r)}{\rho(r)}V(r) + \rho(r)\left\{ \int_{s_r} \operatorname{div} W(x)d\sigma + [\lambda(r)\eta(r)]' \right\} \\ &\leq \frac{\rho'(r)}{\rho(r)}V(r) - \rho(r)\frac{k}{2\lambda(r)}\int_{s_r}|W|^2 d\sigma \\ &\quad -\rho(r)\left\{ \int_{s_r}\left[q(x) - \frac{1}{2k}\lambda(x)|B^TA^{-1}|^2 \right] d\sigma - [\lambda(r)\eta(r)]' \right\}. \end{aligned} \tag{15}$$

In view of (C_1), we have $(W^T A^{-1} W)(x) \geq \lambda_{max}^{-1}(x)|W(x)|^2$. Then, by Cauchy-Schwartz inequality, we obtain

$$\int_{S_r} |W(x)|^2 d\sigma \geq \frac{r^{1-N}}{\omega}\left[\int_{S_r} W(x)\cdot\gamma(x)d\sigma\right]^2.$$

Moreover, by (13) and (15), we get

$$\begin{aligned} V'(r) &\leq \frac{\rho'(r)}{\rho(r)}V(r) - \frac{k\rho(r)r^{1-N}}{2\omega\lambda(r)}\left[\int_{S_r} W(x)\gamma(x)d\sigma\right]^2 \\ &\quad -\rho(r)\left\{\int_{S_r}\left[q(x)-\frac{1}{2k}\lambda(x)|B^TA^{-1}|^2\right]d\sigma - [\lambda(r)\eta(r)]'\right\} \\ &= \frac{\rho'(r)}{\rho(r)}V(r) - \frac{k\rho(r)r^{1-N}}{2\omega\lambda(r)}\left[\frac{V(r)}{\rho(r)} - \lambda(r)\eta(r)\right]^2 \\ &\quad -\rho(r)\left\{\int_{S_r}\left[q(x)-\frac{1}{2k}\lambda(x)|B^TA^{-1}|^2\right]d\sigma - [\lambda(r)\eta(r)]'\right\} \\ &= -Q(r) - \frac{1}{g(r)}V^2(r). \end{aligned} \tag{16}$$

Multiplying (16), with r replaced by s, by $H(r,s,l)$ and integrating from l to $r(r \geq l \geq a_1)$, after simple computation, gives

$$\begin{aligned} &\int_l^r H(r,s,l)Q(s)ds \leq -\int_l^r H(r,s,l)V'(s)ds - \int_l^r \frac{H(r,s,l)}{g(s)}V^2(s)ds \\ &= \int_l^r h(r,s,l)V(s)ds - \int_l^r \frac{H(r,s,l)}{g(s)}V^2(s)ds \\ &\leq \int_l^r \frac{1}{4}\frac{g(s)h^2(r,s,l)}{H(r,s,l)}ds \end{aligned}$$

which implies that

$$\int_l^r \left[H(r,s,l)Q(s) - \frac{1}{4}\frac{g(s)h^2(r,s,l)}{H(r,s,l)}\right]ds \leq 0, \quad r \geq l \geq a_1$$

This contradicts the assumption (11). Hence equation (1) is oscillatory. The proof is complete.

Choose $H(r,s,l) = H_1(r,s)H_2(s,l)$ where $(H_1,H_2) \in \Xi$ in Theorem 2.1, by the simple computation, we have the following theorems.

Theorem 2.2. *Equation (1) is oscillatory provided that for each* $l \geq a_0$, *there exist a function* $\eta \in C^1([a_0,+\infty),\mathbb{R})$ *and functions* $(H_1,H_2) \in \Xi$ *such* *that*

$$\limsup_{r\to\infty}\int_l^r H_1(r,s)H_2(s,l)\left\{Q(s)-\frac{1}{4}g(s)\left[-\frac{h_1(r,s)}{H_1(r,s)}+\frac{h_2(s,l)}{H_2(s,l)}\right]^2\right\}ds>0. \tag{17}$$

If we choose $H_1(r,s)=(r-s)^\alpha, H_2(s,l)=(s-l)^\beta$, for $\alpha,\beta>1$,by Theorem 2.2, we have the following oscillation result.

Theorem 2.3. *Equation (1) is oscillatory provided that for each $l\geq a_0$, there exist a function $\eta\in C^1([a_0,+\infty),\mathbb{R})$ and two constants $\alpha,\beta>1$ such that*

$$\limsup_{r\to\infty}\int_l^r (r-s)^\alpha(s-l)^\beta\left[Q(s)-g(s)\frac{[\beta r-(\alpha+\beta)s+\alpha l]^2}{4(r-s)^2(s-l)^2}\right]ds>0. \tag{18}$$

Define

$$R(r)=\int_{a_0}^r\frac{1}{g(s)}ds,\quad r\geq a_0.$$

Let $H_1(r,s)=[R(r)-R(s)]^\alpha, H_2(s,l)=[R(s)-R(l)]^\beta$ for $\alpha,\beta>1$, then by Theorem 2.2, we have the following oscillation result.

Theorem 2.4. *Equation (1) is oscillatory provided that for each $l\geq a_0$, there exist two constants $\alpha,\beta>1$, such that*

$$\limsup_{r\to\infty}\int_l^r [R(r)-R(s)]^\alpha[R(s)-R(l)]^\beta\Bigg[Q(s) - \frac{[\beta R(r)-(\alpha+\beta)R(s)+\alpha R(l)]^2}{4g(s)[R(r)-R(s)]^2[R(s)-R(l)]^2}\Bigg]ds>0. \tag{19}$$

By Theorem 2.4, we have the following theorem.

Theorem 2.5. *Assume that $\lim_{r\to\infty}R(r)=\infty$. Then equation (1) is oscillatory provided that for each $l\geq a_0$, there exist two constants $\alpha,\beta>1$ such that*

$$\limsup_{r\to\infty}\frac{1}{R^{\alpha+\beta-1}(r)}\int_l^r[R(r)-R(s)]^\alpha[R(s)-R(l)]^\beta Q(s)ds > \alpha\beta(\alpha+\beta-2)\frac{\Gamma(\beta-1)\Gamma(\alpha-1)}{4\Gamma(\alpha+\beta)}, \tag{20}$$

where the Γ is the usual gammma function.

Proof. Note that

$$\begin{aligned}
&\int_l^r [R(r)-R(s)]^{\alpha-2}[R(s)-R(l)]^{\beta-2}[\beta R(r)-(\alpha+\beta)R(s)+\alpha R(l)]^2\frac{1}{g(s)}ds \\
&= \int_l^r [R(r)-R(s)]^{\alpha-2}[R(s)-R(l)]^{\beta-2}\Big[\beta[R(r)-R(s)]-\alpha[R(s)-R(l)]\Big]^2 dR(s) \\
&= \int_0^{R(r)-R(l)} u^{\beta-2}[R(r)-R(l)-u]^{\alpha-2}\Big[\beta[R(r)-R(l)-u]-\alpha u\Big]^2 du \\
&= \int_0^v u^{\beta-2}(v-u)^{\alpha-2}[\beta(v-u)-\alpha u]^2 du = \beta^2\int_0^v u^{\beta-2}(v-u)^{\alpha}du \\
&\quad - 2\alpha\beta\int_0^v u^{\beta-1}(v-u)^{\alpha-1}du+\alpha^2\int_0^v u^{\beta}(v-u)^{\alpha-2}du. \qquad (21)
\end{aligned}$$

by setting $u=R(s)-R(a_i)$ and $v=R(b_i)-R(a_i)$. Using the following Euler's Beta function

$$\int_0^1 x^{\mu-1}(1-x)^{\nu-1}dx=\frac{\Gamma(\mu)\Gamma(\nu)}{\Gamma(\mu+\nu)}. \quad Re(\mu,\nu)>0$$

we have by setting $x=\frac{u}{v}$,

$$\begin{aligned}
\int_0^v u^{\beta-2}(v-u)^{\alpha}du &= v^{\alpha+\beta-1}\frac{\Gamma(\beta-1)\Gamma(\alpha+1)}{\Gamma(\alpha+\beta)} \\
\int_0^v u^{\beta-1}(v-u)^{\alpha-1}du &= v^{\alpha+\beta-1}\frac{\Gamma(\beta)\Gamma(\alpha)}{\Gamma(\alpha+\beta)}. \\
\int_0^v u^{\beta}(v-u)^{\alpha-2}du &= v^{\alpha+\beta-1}\frac{\Gamma(\beta+1)\Gamma(\alpha-1)}{\Gamma(\alpha+\beta)}.
\end{aligned}$$

Thus,

$$\begin{aligned}
&\int_0^v u^{\beta-2}(v-u)^{\alpha-2}[\beta(v-u)-\alpha u]^2du \\
&= \beta^2 v^{\alpha+\beta-1}\frac{\Gamma(\beta-1)\Gamma(\alpha+1)}{\Gamma(\alpha+\beta)} - 2\alpha\beta v^{\alpha+\beta-1}\frac{\Gamma(\beta)\Gamma(\alpha)}{\Gamma(\alpha+\beta)} \\
&\quad +\alpha^2 v^{\alpha+\beta-1}\frac{\Gamma(\beta+1)\Gamma(\alpha-1)}{\Gamma(\alpha+\beta)} \\
&= \frac{v^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)}\left[\beta^2\Gamma(\beta-1)\Gamma(\alpha+1)-2\alpha\beta\Gamma(\beta)\Gamma(\alpha)+\alpha^2\Gamma(\beta+1)\Gamma(\alpha-1)\right] \\
&= \frac{v^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)}\Bigg[\beta^2\alpha(\alpha-1)\Gamma(\beta-1)\Gamma(\alpha-1)
\end{aligned}$$

$$-2\alpha(\alpha-1)\beta(\beta-1)\Gamma(\beta-1)\Gamma(\alpha-1)+\alpha^2\beta(\beta-1)\Gamma(\beta-1)\Gamma(\alpha-1)\Big]$$
$$=\frac{v^{\alpha+\beta-1}\Gamma(\beta-1)\Gamma(\alpha-1)}{\Gamma(\alpha+\beta)}\alpha\beta[\beta(\alpha-1)-2(\alpha-1)(\beta-1)+\alpha(\beta-1)]$$
$$=\alpha\beta(\alpha+\beta-2)\frac{\Gamma(\alpha-1)\Gamma(\beta-1)}{\Gamma(\alpha+\beta)}v^{\alpha+\beta-1}. \tag{22}$$

Substituting back in for $v=R(r)-R(l)$, and in view of (21) and (22), we have

$$\int_l^r [R(r)-R(s)]^{\alpha-2}[R(s)-R(l)]^{\beta-2}[\beta R(r)-(\alpha+\beta)R(s)+\alpha R(l)]^2\frac{1}{g(s)}ds$$
$$=\alpha\beta(\alpha+\beta-2)\frac{\Gamma(\alpha-1)\Gamma(\beta-1)}{\Gamma(\alpha+\beta)}[R(r)-R(l)]^{\alpha+\beta-1}. \tag{23}$$

So we have that

$$\limsup_{r\to\infty}\frac{1}{R^{\alpha+\beta-1}(r)}\int_l^r [R(r)-R(s)]^{\alpha}[R(s)-R(l)]^{\beta}$$
$$\times\left[Q(s)-\frac{[\beta R(r)-(\alpha+\beta)R(s)+\alpha R(l)]^2}{4g(s)[R(r)-R(s)]^2[R(s)-R(l)]^2}\right]ds$$
$$=\limsup_{t\to\infty}\frac{1}{R^{\alpha+\beta-1}(r)}\int_l^r [R(r)-R(s)]^{\alpha}[R(s)-R(l)]^{\beta}Q(s)$$
$$-\limsup_{r\to\infty}\frac{1}{4R^{\alpha+\beta-1}(r)}\int_l^r[R(r)-R(s)]^{\alpha-2}[R(s)-R(l)]^{\beta-2}$$
$$\times[\beta R(r)-(\alpha+\beta)R(s)+\alpha R(l)]^2\frac{1}{g(s)}ds$$
$$=\limsup_{t\to\infty}\frac{1}{R^{\alpha+\beta-1}(t)}\int_l^r [R(r)-R(s)]^{\alpha}[R(s)-R(l)]^{\beta}Q(s)ds$$
$$-\alpha\beta(\alpha+\beta-2)\frac{\Gamma(\beta-1)\Gamma(\alpha-1)}{4\Gamma(\alpha+\beta)}. \tag{24}$$

From (20) and (24), we obtain

$$\limsup_{r\to\infty}\int_l^r [R(r)-R(s)]^{\alpha}[R(s)-R(l)]^{\beta}$$
$$\times\left[Q(s)-\frac{[\beta R(r)-(\alpha+\beta)R(s)+\alpha R(l)]^2}{4g(s)[R(r)-R(s)]^2[R(s)-R(l)]^2}\right]ds>0.$$

Hence, equation (1) is oscillatory by Theorem 2.4. This completes the proof of Theorem 2.5.

From Theorems 2.5, we have the following Corollaries.

Corollary 2.1. *Assume that* $\lim_{r\to\infty} R(r) = \infty$. *Then equation (1) is oscillatory provided that for each* $l \geq a_0$, *there exists a constant* $\alpha > \frac{1}{2}$ *such that*

$$\limsup_{r\to\infty} \frac{1}{R^{2\alpha+1}(r)} \int_l^r [R(r) - R(s)]^{2\alpha}[R(s) - R(l)]^2 Q(s)ds > \frac{\alpha}{(2\alpha-1)(2\alpha+1)}. \quad (25)$$

Proof. Replaced α, β by 2α and 2, respectively, in (20), we obtain

$$\begin{aligned} &\limsup_{r\to\infty} \frac{1}{R^{2\alpha+1}(r)} \int_l^r [R(r) - R(s)]^{2\alpha}[R(s) - R(l)]^2 Q(s)ds \\ &> (2\alpha)2(2\alpha+2-2)\frac{\Gamma(2-1)\Gamma(2\alpha-1)}{4\Gamma(2\alpha+2)} \\ &= \frac{\alpha(2\alpha)\Gamma(1)\Gamma(2\alpha-1)}{(2\alpha+1)(2\alpha)(2\alpha-1)\Gamma(2\alpha-1)} \\ &= \frac{\alpha}{(2\alpha-1)(2\alpha+1)}. \end{aligned} \quad (26)$$

Corollary 2.2. *Assume that* $\lim_{r\to\infty} R(r) = \infty$. *Then equation (1) is oscillatory provided that for each* $l \geq a_0$, *there exists a constant* $\beta > \frac{1}{2}$ *such that*

$$\limsup_{r\to\infty} \frac{1}{R^{2\beta+1}(r)} \int_l^r [R(r) - R(s)]^2[R(s) - R(l)]^{2\beta} Q(s)ds > \frac{\beta}{(2\beta-1)(2\beta+1)}. \quad (27)$$

Example 2.1. Consider equation (1) with

$$A = \text{diag}\left(\frac{1}{r}, \frac{1}{r}\right), \quad B^T = \left(\frac{|sinr|}{r}, \frac{|cosr|}{r}\right),$$

$$q(x) = \frac{\gamma - 12\pi^2 \ln^2 r + 4\pi^2 r^2 \ln^2 r}{8\pi^2 r^3 \ln^2 r}, \quad f(y) = y + y^5.$$

where $r = \sqrt{x_1^2 + x_2^2}, r \geq 1, N = 2, k = 1$.

Let $\eta = 2\pi$. A straightforward computation yields, for $r \geq 1$

$$g_2(r) = \frac{4\pi}{r}, \quad R_2(r) = 4\pi \ln r, \quad Q_2(r) = \frac{\gamma}{4\pi r \ln^2 r}.$$

Hance

$$\limsup_{r\to\infty} \frac{1}{R_2^{2\beta+1}(r)} \int_l^r [R_2(r) - R_2(s)]^2 [R_2(s) - R_2(l)]^{2\beta} Q_2(s) ds$$
$$= \gamma \limsup_{r\to\infty} \frac{1}{(\ln r)^{2\beta+1}} \int_l^r [\ln r - \ln s]^2 [\ln s - \ln l]^{2\beta} \frac{1}{s \ln^2 s} ds$$
$$= \frac{\gamma}{\beta(2\beta+1)(2\beta-1)}$$

Then for any $\gamma > \frac{1}{4}$, there exists a constant $\beta > \frac{1}{2}$, such that $\frac{\gamma}{\beta(2\beta+1)(2\beta-1)} > \frac{\beta}{(2\beta-1)(2\beta+1)}$. i.e., (27) holds, By Corollary 2.2, equation (1) is oscillatory when $\gamma > \frac{1}{4}$.

Remark 2.1. If the assumption (C_4) is replaced by

$$\frac{f(y)}{y} \geq k > 0, \quad y \neq 0,$$

then we can obtain the same results when $q(x) \geq 0, x \in \Omega \subset R^N$.

Remark 2.2. Theorems 2.3-2.5 and Corollaries 2.1 and 2.2 are new because we introduced a new class of kernel functions $H(r, s, l)$ which is basically a product $H_1(r, s)H_2(s, l)$ for a kernel $H_i(r, s)(i = 1, 2)$ of Philos type.

Remark 2.3. From Theorems 2.1-2.5, we can present different explicit sufficient conditions for the oscillation of the solutions to equation (1) by appropriate choice of $H(r, s, l), (H1, H2) \in \Xi$ and $\eta(s)$. For instance, if we choose $H(r, s, l) = H_1(r, s)H_2(s, l)$ where $H_1(r, s) = (r-s)^\alpha, H_2(s, l) = (s-l)^\beta, H_1(r, s) = [R(r)-R(s)]^\alpha, H_2(s, l) = [R(s)-R(l)]^\beta$, or $H_1(r, s) = [\log Q(r)/Q(s)]^\alpha, H_2(s, l) = [\log Q(s)/Q(l)]^\beta$, or $H_1(r, s) = \left[\int_s^r dz/w(z)\right]^\alpha, H_2(s, l) = \left[\int_l^s dz/w(z)\right]^\beta$, or $H_1(r, s) = \rho(r - s)$, $H_2(s, l) = \rho(s - l)$ etc., for $r \geq s \geq a_0$, where $\alpha > 1, \beta > 1$ are constants, $R(r) = \int_{a_0}^r ds/u(s), Q(r) = \int_r^\infty ds/u(s) < \infty$, for $r \geq a_0, w \in C([a_0, \infty), (0, 1))$ satisfying $\int_{a_0}^\infty \frac{1}{w(z)} dz = \infty, \rho(0) > 0, \rho(u) > 0$ and $\rho'(u) \geq 0$ for $u > 0$.

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