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M^*-CONTINUITY AND PRODUCT MINIMAL STRUCTURE ON MINIMAL STRUCTURES

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Abstract: We introduce the concept of M^*-continuity and investigate characterizations for M^*-continuity by using the new interior operator and closure operator defined in minimal structures. Also we introduce the concept of product minimal structures and study some related properties of such structures.

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1. Introduction

In [7], Popa and Noiri introduced the notion of minimal structure which is a generalization of a topology on a given nonempty set. And they introduced the notion of M-continuous functions as functions defined between minimal structures. They showed that the M-continuous functions on minimal structures have properties similar to those of continuous functions between topological

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spaces. They introduced the notions of m-T_2, m-compactness, m-closed graph and m-strongly closed graph. In this paper, we introduce the notion of M^*-continuity, new interior operator and closure operator in minimal structures. We obtain some characterizations of M^*-continuous functions by using the new interior operator and closure operator. We introduce the notion of product minimal structure and study some properties of such structures.

2. Preliminaries

Throughout the present paper X and Y are topological spaces. Let S be a subset of X. The closure (resp., interior) of S will be denoted by $cl(S)$ (resp., $int(S)$). A subset S of X is called a *preopen* set [3] (resp., α-set [6], *semi-open* [1]) if $S \subset int(cl(S))$ (resp., $S \subseteq int(cl(int(S))), S \subseteq cl(int(S))$). The complement of a preopen set (resp., α-set, *semi-open*) is called a *preclosed* set (resp., α-closed set, *semi-closed*). The family of all preopen sets (resp., α-sets, semi-open sets) in X will be denoted by $PO(X)$ (resp., $\alpha(X), SO(X)$). We know the family $\alpha(X)$ is a topology finer than the given topology on X.

Definition 2.1. (see [7]) A subfamily m_X of the power set $P(X)$ of a nonempty set X is called a *minimal structure* on X if $\emptyset \in m_X$ and $X \in m_X$. By (X, m_X), we denote a nonempty set X with a minimal structure m_X on X. Simply we call (X, m_X) a minimal structure on X. Set $M(x) = \{U \in m_X : x \in U\}$.

Remark 2.2. Let (X, τ) be a topological space. Then τ, $PO(X)$, $\alpha(X)$ and $SO(X)$ are all minimal structures on X.

Definition 2.3. (see [2]) Let (X, m_X) be a minimal structure. For a subset A of X, the closure of A and the interior of A are defined as the following:

(1) $m - Int(A) = \cup\{U : U \subseteq A, U \in m_X\}$.

(2) $m - Cl(A) = \cap\{F : A \subseteq F, X - F \in m_X\}$.

Theorem 2.4. (see [7]) *Let (X, m_X) be a minimal structure and $A \subseteq X$. $x \in m - Cl(A)$ if and only if $A \cap V \neq \emptyset$ for all $V \in M(x)$.*

Theorem 2.5. (see [2]) *Let (X, m_X) be a minimal structure and $A \subset X$. Then:*

(1) $X = m - Int(X)$ and $\emptyset = m - Cl(\emptyset)$.

(2) $m - Int(A) \subseteq A$ and $A \subset m - Cl(A)$.

(3) If $A \in m_X$, then $m - Int(A) = A$ and if $X - F \in m_X$, then $m - Cl(F) = F$.

(4) If $A \subseteq B$, then $m-Int(A) \subseteq m-Int(B)$ and $m-Cl(A) \subseteq m-Cl(B)$.

(5) $m-Int(A \cap B) = m-Int(A) \cap m-Int(B)$ and $m-Cl(A \cup B) = m-Cl(A) \cup m-Cl(B)$.

(6) $m-Int(m-Int(A)) = m-Int(A)$ and $m-Cl(m-Cl(A)) = m-Cl(A)$.

(7) $m-Cl(X-A) = X-(m-Int(A))$ and $m-Int(X-A) = X-(m-Cl(A))$.

Definition 2.6. (see [7]) Let (X, m_X) and (Y, m_Y) be two minimal structures. Then $f : X \to Y$ is said to be *M-continuous* if for $x \in X$ and $V \in M(f(x))$, there is $U \in M(x)$ such that $f(U) \subseteq V$.

Theorem 2.7. (see [7]) *Let $f : X \to Y$ be a function on two w-spaces (X, m_X) and (Y, m_Y). Then the following statements are equivalent:*

(1) f is M-continuous.

(2) $f(m-Cl(A)) \subseteq m-Cl(f(A))$ for $A \subseteq X$.

(3) $m-Cl(f^{-1}(B)) \subseteq f^{-1}(m-Cl(B))$ for $B \subseteq Y$.

(4) $f^{-1}(m-Int(B)) \subseteq m-Int(f^{-1}(B))$ for $B \subseteq Y$

3. M^*-Continuous Functions

Lemma 3.1. *Let (X, m_X) be a minimal structure and $A \subset X$. Then $x \in m-Int(A)$ if and only if there exists an element $U \in M(x)$ such that $U \subset A$.*

Proof. From Definition 2.3, it is obvious. □

Definition 3.2. Let $f : X \to Y$ be a function between minimal structures (X, m_X) and (Y, m_Y). Then f is said to be M^**-continuous* if for every $A \in m_Y$, $f^{-1}(A) \in m_X$.

Theorem 3.3. *Let $f : X \to Y$ be a function between minimal structures (X, m_X) and (Y, m_Y).*

(i) Then the following are equivalent:

(1) f is M^*-continuous.

(2) For every $Y - F \in m_Y$, $X - f^{-1}(F) \in m_X$.

(ii) If f is M^*-continuous, then M-continuous.

Proof. It is obvious. □

M-continuous function may not be M^*-continuous as shown in the next example.

Example 3.4. Let $X = Y = \{a, b, c\}$. Consider two minimal structures defined as follows: $m_X = \{\emptyset, \{a\}, \{b\}, X\}$, $m_Y = \{\emptyset, \{a, b\}, Y\}$.

Let $f : (X, m_X) \to (Y, m_Y)$ be a function defined by $f(x) = x$, for $x \in X$. Then f is M-continuous, but not M^*-continuous.

Definition 3.5. Let (X, m_X) be a minimal structure on X and $A \subset X$. The m-interior and m-closure of A on m_X (denoted by $I_m(A)$, $C_m(A)$, respectively) are defined as the following:

(1) $I_m(A) = \{x \in A : A \in M(x)\}$.

(2) $Cl_m(A) = \{x \in X : X - A \notin M(x)\}$.

Theorem 3.6. *Let (X, m_X) be a minimal structure and $A, B \subseteq X$. Then the following hold.*

(1) $I_m(A) \subset m - Int(A) \subseteq A$ and $A \subseteq m - Cl(A) \subseteq Cl_m(A)$.

(2) $I_m(\emptyset) = \emptyset$ and $Cl_m(X) = X$.

(3) $I_m(A) = X - Cl_m(X - A)$ and $Cl_m(A) = X - I_m(X - A)$.

Proof. (1) and (2) are obvious.

For (3), let $x \in X - Cl_m(X - A)$. Then

$$x \in X - Cl_m(X - A) \Leftrightarrow X - (X - A) \in M(x) \Leftrightarrow x \in I_m(A).$$

Hence we have $I_m(A) = X - Cl_m(X - A)$. Similarly, we have $Cl_m(A) = X - I_m(X - A)$. □

Theorem 3.7. *Let (X, m_X) be a minimal structure and $A \subseteq X$. Then the following hold.*

(1) $A \in m_X$ if and only if $I_m(A) = A$.

(2) $X - F \in m_X$ if and only if $Cl_m(F) = F$.

Proof. (1) Let $A \in m_X$ and $x \in A$. Then $A \in M(x)$, so $A \subseteq I_m(A)$. This implies $I_m(A) = A$.

Suppose that $I_m(A) = A$. For $x \in A$, from $x \in I_m(A)$, it follows $A \in M(x)$. So $A \in m_X$.

(2) It is similar to (1). □

Theorem 3.8. *Let $f : X \to Y$ be a function between minimal structures (X, m_X) and (Y, m_Y). Then the following are equivalent:*

(1) f is M^*-continuous.

(2) $f^{-1}(I_m(B)) \subseteq I_m(f^{-1}(B))$ for $B \subseteq Y$.

(3) $Cl_m(f^{-1}(B)) \subseteq f^{-1}(Cl_m(B))$ for $B \subseteq Y$.

(4) $f(Cl_m(A)) \subseteq Cl_m(f(A))$ for $A \subseteq X$.

Proof. (1) $\Rightarrow$ (2) Suppose f is M^*-continuous. Let $x \in f^{-1}(I_m(A))$. From definition of m-interior operator, it follows $A \in M(f(x))$. It is $f^{-1}(A) \in M(x)$ by the M^*-continuity, so that $x \in I_m(f^{-1}(A))$.

(2) $\Rightarrow$ (1) Let $B \in m_Y$; then $B = I_m(B)$ and so $f^{-1}(B) = I_m(f^{-1}(B))$ by (2). From Theorem 3.7, it follows f is M^*-continuous.

(2) $\Leftrightarrow$ (3) It is obvious by Theorem 3.6 and Theorem 3.7.

(3) $\Rightarrow$ (4) From (3), it follows $Cl_m(f^{-1}(f(A))) \subseteq f^{-1}(Cl_m(f(A)))$ for $A \subseteq X$. Hence $Cl_m(A) \subseteq f^{-1}(Cl_m(f(A)))$.

(4) $\Rightarrow$ (3) By (4), $f(Cl_m(f^{-1}(B))) \subseteq Cl_m(f(f^{-1}(B))) \subseteq Cl_m(B)$ for $B \subseteq Y$. Hence we have (3). □

Let X be a nonempty set. We recall that a collection $\mathbf{H}$ of subsets of X is called an *m-family* [4] on X if $\cap \mathbf{H} \neq \emptyset$.

Definition 3.9. Let (X, m_X) be a minimal structure and $\mathbf{H}$ an m-family on X. Then we say that an m-family $\mathbf{H}$ *M-converges* to $x \in X$ if $\mathbf{H}$ is finer than $M(x)$ i.e., $M(x) \subset \mathbf{H}$.

Let $f : X \to Y$ be a function; then it is obvious $f(\mathbf{H}) = \{f(F) : F \in \mathbf{H}\}$ is an m-family on Y.

Theorem 3.10. *Let (X, m_X) be a minimal structure. Then:*

(1) Let $A \in m_X$. Then $x \in A$ if and only if $A \in \mathbf{H}$ whenever an m-family $\mathbf{H}$ M-converges to x.

(2) Let $X - A \in m_X$. Then $x \in A$ if and only if there exists an m-family $\mathbf{H}$ such that $\mathbf{H}$ M-converges to x and $X - A \notin \mathbf{H}$.

Proof. (1) Let $A \in m_X$ and $x \in A$. Since $A = I_m(A)$, we have $A \in M(x)$. Hence if an m-family $\mathbf{H}$ M-converge to x, then it follows that $A \in \mathbf{H}$ since $M(x) \subseteq \mathbf{H}$.

Suppose that for every m-family $\mathbf{H}$ M-converging to x, $A \in \mathbf{H}$. Then since $M(x)$ M-converges to x, by hypothesis, we get $A \in M(x)$ and from Theorem 3.7, it follows $x \in I_m(A) = A$.

(2) Let $x \in A = Cl_m(A)$; then $X - A \notin M(x)$. Let $\mathbf{H} = M(x)$; then $\mathbf{H}$ is an m-family satisfying the condition.

For the converse, let $\mathbf{H}$ be an m-family M-converging to x and $X - A \notin \mathbf{H}$. Since $M(x) \subseteq \mathbf{H}$, we get $X - A \notin M(x)$ and so $x \in Cl_m(A) = A$. □

Theorem 3.11. *Let $f : X \to Y$ be a bijective function between minimal structures (X, m_X) and (Y, m_Y). Then f is M^*-continuous if and only if for an m-family $\mathbf{H}$ M-converging to $x \in X$, $f(\mathbf{H})$ M-converges to $f(x)$.*

Proof. Suppose f is M^*-continuous and $\mathbf{H}$ is an m-family M-converging to $x \in X$. By hypothesis and surjectivity, we get $M(f(x)) \subseteq f(M(x)) \subseteq f(\mathbf{H})$, so that $f(\mathbf{H})$ M-converges to $f(x)$.

For the converse, let $U \in M(f(x))$ for $U \subseteq Y$. Since $M(x)$ M-converges to x, by hypothesis, we get $M(f(x)) \subseteq f(M(x))$ for $x \in X$. From f is injectivity, it follows $f^{-1}(U) \in M(x)$. □

Let (X, m_X) be a minimal structure. X is said to be *m-compact* [7] if for every cover of X by sets of m_X has a finite subcover. A subset K of X is said to be *m-compact* if every cover of K by subsets of m_X has a finite subcover.

Theorem 3.12. *Let (X, m_X) be a minimal structure.*

(1) If X is m-compact and $X - K \in m_X$, then K is m-compact in X.

(2) If $f : (X, m_X) \to (X, m_Y)$ is M^*-continuous and A is a m-compact subset of X, then $f(A)$ is m-compact in Y.

Proof. (1) Let (X, m_X) be m-compact and $X - K \in m_X$. Let $\{U_i \in m_X : i \in J\}$ be a cover of K; then $(X - K) \cup \{U_i \in m_X : i \in J\} = X$. Since X is m-compact, there exists a finite subset J_0 of J such that $(X - K) \cup \{U_i \in m_X : i \in J_0\} = X$. Then $K \subseteq \cup\{U_i \in m_X : i \in J_0\}$ and so K is m-compact in X.

(2) Let $\{U_i \in m_Y : i \in J\}$ be a cover of $f(A)$. Then $A \subseteq \cup\{f^{-1}(U_i) : i \in J\}$, where $f^{-1}(U_i) \in m_X$. Since A is m-compact, there exists a finite subset J_0 of J such that $A \subseteq \cup\{f^{-1}(U_i) \in m_X : i \in J_0\}$. Then $f(A) \subseteq \cup\{U_i \in m_Y : i \in J_0\}$ and so $f(A)$ is m-compact in Y. □

4. Product Minimal Structure

Lemma 4.1. *The intersection of any minimal structures is a minimal structure.*

Proof. It is straightforward. □

Theorem 4.2. *Let $f : X \to (Y, m_Y)$ be a function from a nonempty set X to a minimal (Y, m_Y). Then there is the weakest minimal structure on X for which f is M^*-continuous.*

Proof. Set $M = \{m : m$ is a weak structure in X and $f : (X, m) \to (Y, m_Y)$ is M^*-continuous $\}$. Since $M \neq \emptyset$, from Lemma 4.1, it follows $m_o = \cap\{m : m \in M\}$ is a weak structure which is the required minimal structure on X. □

Theorem 4.3. *Let (X, m_X) and (Y, m_Y) be minimal structures. Then $m_{X\times Y} = \{(U, V) : U \in m_X, V \in m_Y\}$ is a minimal structure on $X \times Y$.*

Proof. It is obvious that $\emptyset$, $X \times Y \in m_{X\times Y}$. Hence $m_{X\times Y}$ is a minimal structure on $X \times Y$. □

Lemma 4.4. *Let (X, m_X) and (Y, m_Y) be minimal structures. Then a subset A of the minimal structure $m_{X\times Y}$ on $X \times Y$ is $A = m-Cl(A)$ in $X \times Y$ if for each $(x, y) \in (X \times Y) - A$, there exist $U \in m_X$ and $V \in m_Y$ containing x and y, respectively, such that $(U \times V) \cap A = \emptyset$.*

Proof. By Theorem 2.4, it is obvious. □

Theorem 4.5. *Let (X_1, m_1) and (X_2, m_2) be minimal structures and let $\pi_i : X_1 \times X_2 \to X_i$ be a projection map, $i = 1, 2$. Then $\mathbf{m} = \{(U, V) : U \in m_1, V \in m_2\}$ is the weakest minimal structure on $X_1 \times X_2$ for which π_i is M^*-continuous for each $i = 1, 2$.*

Proof. Let $\mathbf{m_i}^*$ be the weakest minimal structure on $X_1 \times X_2$ for which π_i is M^*-continuous for each $i = 1, 2$. By Lemma 4.1 and Theorem 4.2, $\mathbf{m_1}^* \cap \mathbf{m_2}^*$ is the weakest minimal structure such that π_i is M^*-continuous. It is obvious $\mathbf{m} = \mathbf{m_1}^* \cap \mathbf{m_2}^*$. Hence the proof is completed. □

Definition 4.6. Let (X_i, m_i) be a minimal structure for $i \in J$. Set $\mathbf{S} = \{\pi_i^{-1}(U_i) : U_i \in m_i$ for $i \in J\}$ where $\pi_i : \prod X_i \to X_i$ is an i-th projection map. We call $\mathbf{m} = \{\cap\mathbf{B} : \mathbf{B} \subseteq \mathbf{S}$ and $\mathbf{B}$ is finite $\}$ the *product minimal structure* on $X = \prod X_i$.

Theorem 4.7. *Let (X_i, m_i) be a minimal structure for $i \in J$ and $(\prod X_i, \mathbf{m})$ the product minimal structure. Then $\pi_i : (\prod X_i, \mathbf{m}) \to (X_i, m_i)$ is M^*-continuous.*

Proof. Obvious. □

Theorem 4.8. *Let (X_i, m_i) be a minimal structure for $i \in J$ and $(\prod X_i, \mathbf{m})$ the product minimal structure. Then $f : (Y, m_Y) \to (\prod X_i, \mathbf{m})$ is M^*-continuous if and only if $\pi_i \circ f$ is M^*-continuous.*

Proof. Obvious. □

Theorem 4.9. *Let $\{(X_i, m_i) : i \in J\}$ be a family of minimal structures and $\mathbf{m}$ the product minimal structure. If $\prod X_i$ is m-compact, then each (X_i, m_i) is m-compact.*

Proof. From each π_i is a M^*-continuous surjection, it follows each (X_i, m_i) is m-compact. □

Let (X, m_X) be a minimal structure. Then X is said to be m-T_2 [7] if for every two distinct points x and y in X, there exist two disjoint sets U and V in m_X such that $x \in U$ and $y \in V$.

Theorem 4.10. *Let $\{(X_i, m_i) : i \in J\}$ be a family of minimal structures and $\prod X_i$ the product minimal structure. If (X_i, m_i) is a m-T_2 space for each $i \in J$, then $\prod X_i$ is m-T_2.*

Proof. Let $x = (x_i)$ and $y = (y_i)$ be two distinct points in $\prod X_i$. Then for each $i \in J$, there exist two disjoint sets U_i and V_i in m_i such that $x_i \in U_i$ and $y_i \in V_i$. Since π_i is a M^*-continuous surjection, $\pi_i^{-1}(U_i)$ and $\pi_i^{-1}(V_i)$ are in the product minimal structure $\mathbf{m}$ containing $x = (x_i)$ and $y = (y_i)$, respectively. Hence the product minimal structure $(\prod X_i, \mathbf{m})$ is m-T_2. □

Theorem 4.11. *Let (X, m_X) and (Y, m_Y) be minimal structures. If $f : X \to Y$ is a M^*-continuous function and Y is m-T_2, then $A = \{(x_1, x_2) : f(x_1) = f(x_2)\} = m - Cl(A)$ in the product minimal structure $m_{X \times Y}$.*

Proof. It is sufficient to show that $A^c \subseteq m - Cl(A)^c$. Let $(x_1, x_2) \notin A$; then $f(x_1) \neq f(x_2)$. Since Y is m-T_2, there exist $U, V \in m_Y$ such that $U \cap V = \emptyset$ and $f(x_1) \in U$, $f(x_2) \in V$. Since f is M^*-continuous, $f^{-1}(U)$ and $f^{-1}(V)$ are in m_X. Since $f^{-1}(U) \cap f^{-1}(V) = \emptyset$, by Definition 4.6, $f^{-1}(U) \times f^{-1}(V)$ is in the product minimal structure $m_{X \times Y}$ such that $(x_1, x_2) \in f^{-1}(U) \times f^{-1}(V)$ and $f^{-1}(U) \cap f^{-1}(V) = \emptyset$. This implies $f^{-1}(U) \times f^{-1}(V) \cap A = \emptyset$, and so $(x_1, x_2) \notin m - Cl(A)$. □

Let (X, m_X) and (Y, m_Y) be minimal structures. We call a function $f : X \to Y$ has a *strongly m-closed graph (resp., m-closed graph)* [7] if for each $(x, y) \notin G(f)$, there exist two sets $U \in m_X$ and $V \in m_Y$ containing x and y, respectively, such that $(U \times m - Cl(V)) \cap G(f) = \emptyset$ (resp., $(U \times V) \cap G(f) = \emptyset$).

Theorem 4.12. *Let (X, m_X) and (Y, m_Y) be between minimal structures. A function $f : X \to Y$ has a m-closed graph if and only if the graph $G(f) = \{(x, f(x)) : x \in X\} = m - Cl(G(f))$ in the product minimal structure $(X \times Y, m_{X\times Y}$.*

Proof. For each $(x, y) \notin G(f)$. Then since f has an m-closed graph, there exist two sets $U \in m_X$ and $V \in m_Y$ such that $(U \times V) \cap G(f) = \emptyset$. Hence $(x, y) \notin m - Cl(G(f))$.

For the converse, let $(x, y) \in (X \times Y) - G(f)$. Since $G(f) = m - Cl(G(f))$, there exist two sets $U \times V \in m_{X\times Y}$ containing (x, y) where $U \in M(x)$ and $V \in M(y)$. Hence f has an m-closed graph. □

Theorem 4.13. (see [7]) *Let (X, m_X) and (Y, m_Y) be between minimal structures. If $f : X \to Y$ is M-continuous and Y is an m-T_2 space, then f has an m-closed graph.*

Corollary 4.14. *Let (X, m_X) and (Y, m_Y) be between minimal structures. If $f : X \to Y$ is W^*-continuous and Y is an m-T_2 space, then f has an m-closed graph.*

Proof. It follows from Theorem 3.3 and Theorem 4.13. □

Theorem 4.15. *Let (X, m_X) and (Y, m_Y) be between minimal structures. If $f : X \to Y$ is a M-continuous injection with an m-closed graph, then X is m-T_2.*

Proof. Let x_1 and x_2 be two distinct elements in X, then $f(x_1) \neq f(x_2)$. This implies that $(x_1, f(x_2)) \in (X \times Y) - G(f)$. Since $G(f) = m - Cl(G(f))$, there exist two sets $U \in m_X$ and $V \in m_Y$ containing x_1 and $f(x_2)$, respectively, such that $(U \times V) \cap G(f) = \emptyset$. Since f is M-continuous, there exist a set $W \in m_X$ containing x_2 such that $f(W) \subset V$. It follows $f(W) \cap f(U) = \emptyset$. Hence $W \cap U = \emptyset$ and so X is a m-T_2 space. □

Corollary 4.16. *Let (X, m_X) and (Y, m_Y) be between minimal structures. If $f : X \to Y$ is M^*-continuous injection with an m-closed graph, then X is m-T_2.*

Proof. It follows from Theorem 3.3 and Theorem 4.15. □

Lemma 4.17. (see [7]) *Let (X, m_X) and (Y, m_Y) be between minimal structures. Then a function $f : X \to Y$ has a strongly m-closed graph if for each $(x, y) \notin G(f)$, there exist two sets $U \in m_X$ and $V \in m_Y$ containing x and y, respectively, such that $f(U) \cap m - Cl(V) = \emptyset$.*

Definition 4.18. Let (X, m_X) and (Y, m_Y) be between minimal structures. A function $f : X \to Y$ is said to be M^*-open if for every $G \in m_X$, $f(G)$ is in m_Y.

Theorem 4.19. *Let $f : X \to Y$ be a function between between minimal structures (X, m_X) and (Y, m_Y).*

(1) f is W^*-open;

(2) $m - Int(f^{-1}(B)) \subseteq f^{-1}(m - Int(B))$ for each $B \subseteq Y$;

(3) $m - Int(f(A)) = f(A)$ for A in m_X.

Then (1) $\Rightarrow$ (2) $\Leftrightarrow$ (3).

Proof. (1) $\Rightarrow$ (2) Let $B \subseteq Y$ and $x \in m - Int(f^{-1}(B))$. Then by Lemma 3.1, there is a set $U \in m_X$ containing x such that $x \in U \subseteq f^{-1}(B)$. Since f is W^*-open, $f(x) \in m - Int(B)$ and so $x \in f^{-1}(m - Int(B))$.

(2) $\Rightarrow$ (3) Let A be a set in m_X. Then by hypothesis,

$$A = m - Int(A) \subseteq m - Int(f^{-1}(f(A))) \subseteq f^{-1}(m - Int(f(A))).$$

Hence $m - Int(f(A)) = f(A)$.

(3) $\Rightarrow$ (2) Let $x \in m - Int(f^{-1}(B))$ for $B \subseteq Y$. Then there is a set $U \in m_X$ containing x such that $x \in U \subseteq f^{-1}(B)$. By hypothesis,

$$f(x) \in f(U) = m - Int(f(U)) \subseteq m - Int(f(f^{-1}(B))) \subseteq m - Int(B)$$

and so $x \in f^{-1}(m - Int(B))$. □

Theorem 4.20. *Let $f : X \to Y$ be a function between between minimal structures (X, m_X) and (Y, m_Y). Then*

f is W^*-open if and only if $I_m(f^{-1}(B)) \subseteq f^{-1}(I_m(B))$ for each $B \subset Y$.

Proof. Let $x \in I_m(f^{-1}(B))$ for $B \subset Y$. Then by Lemma 3.1, there is a set $U \in M(x)$ containing x such that $x \in U \subseteq f^{-1}(B)$. Since f is W^*-open, $f(x) \in I_m(B)$. Hence we have $I_m(f^{-1}(B)) \subseteq f^{-1}(I_m(B))$

For the converse, let A be a set in m_X. Then by hypothesis,

$$A = I_m(A) \subseteq I_m(f^{-1}(f(A))) \subseteq f^{-1}(I_m(f(A))).$$

So $I_m(f(A)) = f(A)$, by Theorem 3.7, $f(A) \in m_Y$. □

Theorem 4.21. *Let (X, m_X) and (Y, m_Y) be between minimal structures. Then a function $f : X \to Y$ is M^*-open and it has an m-closed graph, then f has a strongly m-closed graph.*

Proof. Let $x \in X$ and $y \in Y$ such that $y \neq f(x)$. Since f has an m-closed graph, there exist two sets $U \in m_X$ and $V \in m_Y$ containing x and y, respectively, such that $f(U) \cap V = \emptyset$. Since f is M^*-open, $f(U) \in m_Y$ and so $f(U) \cap m - Cl(V) = \emptyset$. Therefore by Lemma 4.17, f has a strongly m-closed graph. □

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