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# NUMERICAL APPROXIMATIONS TO THE TRANSPORT EQUATION ARISING IN NEURONAL VARIABILITY

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**Abstract:** This paper studies some finite difference approximations to find the numerical solution of first-order hyperbolic partial differential equation of mixed type, *i.e.*, transport equation with point-wise delay and advance. We are interested in the challenging issues in neuronal sciences stemming from the modeling of neuronal variability. The resulting mathematical model is a first-order hyperbolic partial differential equation involving point-wise delay and advance which models the distribution of time intervals between successive neuronal firings. We construct, analyze, and implement explicit numerical schemes for solving such type of initial and boundary-interval problems. Analysis shows that numerical approximations are conditionally stable, consistent and convergent in discrete $L^\infty$ norm. Numerical approximations works irrespective the size of point-wise delay and advance. Some numerical tests are reported to validate the computational efficiency of the numerical approximations.

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## 1. Introduction

Partial differential difference equations or more generally partial functional differential equations are of great importance since they arise in many mathematical models of control theory, mathematical biology, climate models, mathematical economics, the theory of systems which communicate through lossless channels, meteorology, and many other areas, see [1, 2]. Some differential models from population dynamics are given in [3]. Several biological phenomena can be mathematically modeled by using time dependent first-order partial differential equations of hyperbolic type which contain point-wise delay and advance or negative and positive shift in space. For instance, in Stein's model [4], the distribution of neuronal firing intervals satisfies a transport equation of mixed type with appropriate initial-boundary conditions given by

$$\begin{aligned} \frac{\partial F}{\partial t}(v,t) - (v/\tau)\frac{\partial F}{\partial v}(v,t) &= p_e[F(v-1,t) - F(v,t)] \\ &\quad + p_i[F(v+v_0,t) - F(v,t)], \\ F(v,0) &= F_0(v), \end{aligned} \tag{1.1}$$

where $0 < v < r,\ t > 0$, $F(v,t)$ is the probability that the depolarization $V_t \le v$ at time $t$, $F_0$ is initial data, $r$ is the threshold value, $p_e$ and $p_i$ are the frequencies of excitatory and inhibitory impulses, respectively. After a refractory period of some duration, an excitatory impulse produces unit depolarization, while an inhibitory impulse produces $v_0$ unit repolarization, and if the depolarization reaches a threshold of $r$ units, the neuron fires. For sub-threshold levels, the depolarization decays exponentially between impulses with the time constant $\tau$. To study the neuron variability in quantitative terms, Stein transformed this equation and obtain the characteristic function of the distribution and analyzed the mean and variance of the distribution. We refer to [4] for more detailed information about the assumptions of the model. We are interested to find the value of the unknown $F$.

We consider the following general transport equation of mixed type with an initial data $u_0$ on domain $\Omega := (0, X)$. We write the resulting equation with abstract mathematical notations. In general it reads

$$\begin{aligned} u_t + au_x &= b[u(x-\alpha,t) - u(x,t)] + c[u(x+\beta,t) - u(x,t)], \\ &\qquad x \in \Omega, \quad t > 0, \\ u(x,0) &= u_0(x), \quad x \in \bar{\Omega}, \\ u(s,t) &= \phi_1(s,t), \quad \forall\ s \in [-\alpha, 0], \quad t > 0, \\ u(s,t) &= \phi_2(s,t), \quad \forall\ s \in [X, X+\beta], \quad t > 0, \end{aligned} \tag{1.2}$$

where $a := a(x,t)$, $b := b(x,t)$, and $c := c(x,t)$ all are sufficiently smooth functions of $x$ and $t$, $\alpha$ and $\beta$ are non-zero fixed real numbers and $\Omega \subseteq \mathbb{R}$. Let $a$, $b$, and $c$ are bounded functions of $(x,t)$. We also assume that $a$ is not changing its sign in entire domain. The unknown function $u$ is defined in the underlying domain and also in the intervals $[-\alpha, 0]$ and $[X, X+\beta]$ due to the presence of point-wise delay and advance. So our extended domain is $[-\alpha, 0] \cup [0, X] \cup [X, X+\beta]$ and $t > 0$. Also the coefficients are sufficiently smooth functions in these intervals and the unknown function $u$ is as smooth as the initial data. Due to the presence of point-wise delay in equation (1.2), we need a boundary-interval condition in the left side of domain, *i.e.*, in the interval $[-\alpha, 0]$ and due to presence of advance, we need a boundary-interval condition in the interval $[X, X+\beta]$.

Equation (1.2) is a partial differential equation of mixed type, *i.e.*, equation containing point-wise delay and advance or shifts in space. Due to occurrence of both the difference arguments and non-constant coefficients, it is not difficult but also to some extent it is impossible to find the analytic solution of such type of partial differential equations by using the methods to find the exact solution of partial differential equations, see [5] pp. 97–103. Also we cannot solve such type of partial differential difference equations with classical numerical methods.

Ordinary differential equations with difference terms (delay) are quite well understood by now but there is no comparable theory for partial differential equations (i.e. for time and space dependent unknowns). A starting point to study the analysis and numerical computation of ordinary delay differential equations is Bellen and Zennaro [6]. Numerical solution of a general class of delay differential equation, including stiff problem, differential-algebraic delay equations and neutral problem is discussed by Guglielmi et al. in [7]. Implicit Runge-Kutta method is applied in modified form and possible difficulties are discussed. Asymptotic stabilities properties of implicit Runge-Kutta method for ordinary delay differential equations is considered by Hairer et al. in [8]. Partial differential equations with time delay is discussed in [2].

If delay and advance arguments are sufficiently small, the authors used the Taylor series approximations for difference arguments and proposed an explicit numerical scheme based on the finite difference method which is discussed in [9]. This method has restriction on the size of point-wise delay and advance. Equation containing only point-wise delay is considered by the authors in [10, 11] which is the case for advection equation with shift in right side in space.

The rest of the paper is organized as follows. We consider the problem (1.2) with both the shift arguments and there is no such restriction on the size of arguments and which is most realistic situation. We construct numerical schemes based on finite difference method in Section 2 and analyze the numerical schemes for consistency, stability and convergence. Even today, when finite element methods are widely dominated in the numerical study of partial differential equations and their applications, nevertheless, the applicability of the finite difference method remains valid to solve such types of transport equations. To approximate the terms containing shifts in space, we use interpolation. The numerical schemes constructed so for works nicely for sufficiently large as well as small difference arguments. In Section 3, we include some numerical experiments to validate the theoretical results derived in this paper. Finally, in Section 4, we make some concluding remarks illustrating the effect of difference arguments on the solution behavior.

## 2. Numerical Approximation

In this section, we construct numerical schemes based on the finite difference method [12, 13, 14]. We discuss some first and second order explicit numerical approximations for the given equation (1.2) based on upwind and Lax-Friedrichs finite difference approximations. The differential equation (1.2) is hyperbolic and first-order with difference terms. For space-time approximations based on finite differences, the $(x,t)$ plane is descretized by taking mesh width $h$ and time step $k$, and defining the grid points $(x_j, t_n)$ by

$$x_j = jh,\ j = 0, 1, ..., J-1, J; \quad t_n = nk,\ n = 0, 1, 2....$$

Now we look for discrete solution $u_j^n$ that approximate $u(x_j, t_n), \forall j, n$. We write $G$ for the right hand side of the equation (1.2). Also we write the closure of $\Omega_h = (x_j = jh,\ j = 1, 2, ..., J-1)$ as $\bar{\Omega}_h$ and $\bar{\Omega}_h = (x_j = jh,\ j = 0, 1, 2, ..., J)$.

### 2.1. Upwind Approximation

If $a(x,t) > 0$, we use forward difference in time and backward difference in space according the direction of characteristics. Numerical scheme for $a(x,t) > 0$ is given by

$$\frac{u_j^{n+1} - u_j^n}{k} + a_j^n \frac{u_j^n - u_{j-1}^n}{h} = G_j^n, \quad \forall\ j = 1, 2, ..., J-1 \tag{2.3}$$

together with initial and boundary-interval conditions as following

$$u_j^0 = u^0(x_j), \quad \forall\ j = 1, 2, ..., J-1 \tag{2.4a}$$

$$u(s,t_n) = \phi_1(s,t_n),\ \forall\ s \in [-\alpha, 0], \quad n = 0,1,2\ldots, \tag{2.4b}$$
$$u(s,t_n) = \phi_2(s,t_n),\ \forall\ s \in [X, X+\beta], \quad n = 0,1,2\ldots. \tag{2.4c}$$

We write $u_j^n \approx u(x_j,t_n)$, $I_\alpha^-(u_j^n) \approx u(x_j-\alpha,t_n)$, and $L_\beta^+(u_j^n) \approx u(x_j+\beta,t_n)$. Also we write $a_j^n = a(x_j,t_n)$, $b_j^n = b(x_j,t_n)$, $c_j^n = c(x_j,t_n)$. Therefore,

$$G_j^n = b_j^n[I_\alpha^-(u_j^n) - u_j^n] + c_j^n[L_\beta^+(u_j^n) - u_j^n]. \tag{2.5}$$

To find the values of terms containing point-wise shifts, we use linear interpolation. Let us suppose the term $x_j - \alpha$ lies between the two nodal points $x_{j_0-1}$ and $x_{j_0}$, the linear interpolation is defined as following

$$I(x) = \frac{x_{j_0} - x}{x_{j_0} - x_{j_0-1}} u_{j_0-1}^n + \frac{x - x_{j_0-1}}{x_{j_0} - x_{j_0-1}} u_{j_0}^n \tag{2.6}$$

Therefore, $I(x_j - \alpha) = I_\alpha^-(u_j^n)$. Similarly, assuming that $x_{j_1-1} < x_j + \beta < x_{j_1}$, we can define an interpolation $L(x)$ similar to (2.6) and $L(x_j+\beta) = L_\beta^+(u_j^n)$. We can write the numerical scheme (2.3) as following

$$u_j^{n+1} = u_j^n - \lambda a_j^n(u_j^n - u_{j-1}^n) + kG_j^n, \quad \forall\ j = 1,2,...,J-1 \tag{2.7}$$

where $\lambda = \frac{k}{h}$. Similarly for $a(x,t) < 0$, we use forward difference in time and forward difference in space. Numerical approximation in this case is given by

$$u_j^{n+1} = u_j^n - \lambda a_j^n(u_{j+1}^n - u_j^n) + kG_j^n, \quad \forall\ j = 1,2,...,J-1$$

together with initial data and boundary-interval conditions as given in (2.4) and the value of $G_j^n$ is given by the formula (2.5). Combining both cases together, the numerical approximation can be written as following

$$u_j^{n+1} = u_j^n - \lambda\left[\frac{a_j^n + |a_j^n|}{2}(u_j^n - u_{j-1}^n) + \frac{a_j^n - |a_j^n|}{2}(u_{j+1}^n - u_j^n)\right] + kG_j^n, \quad \forall\ j = 1,2,...,J-1,$$

together with initial-boundary conditions.

If one or both the difference arguments are sufficiently small, then we can use Taylor Series approximations for difference arguments. The Taylor series approximation for point-wise delay is given by

$$u(x-\alpha,t) = u(x,t) - \alpha u_x(x,t) + O(\alpha^2).$$

Similar expression can be written for the point-wise advance. Using these expansions in equation (1.2), we get

$$u_t + a(x,t)u_x = b(x,t)[-\alpha u_x + O(\alpha^2)] + c(x,t)[\beta u_x + O(\beta^2)].$$

Therefore, $u_t + [a(x,t) + \alpha b(x,t) - \beta c(x,t)]u_x + O(\alpha^2) + O(\beta^2) = 0.$

In this case, we can construct the numerical scheme as discussed in [9].

Now we discuss consistency, stability, and convergence of the numerical approximation constructed so for.

**Theorem 1.** *Let the function $u_0$ be sufficiently smooth then the explicit numerical scheme (2.3) is consistent of order 1 in the maximum norm. Also if CFL condition is satisfied, then on the arbitrary finite time interval $[0,T]$, the numerical scheme (2.3) is stable in the maximum norm, where stability constant is of the form $C = 1 + O(k)$, and has convergence of order 1.*

*Proof.* Firstly, we study the consistency of the numerical scheme (2.3) for the case $a(x,t) > 0$. Similar analysis can be applied to discuss the case when $a(x,t) < 0$.

The consistency error of the numerical scheme (2.3) is the difference between the two sides of the equation when the approximation $u_j^n$ is replaced throughout by the exact solution $u(x_j,t_n)$ of the differential equation (1.2). If $u$ is sufficiently smooth, then the consistency error $E_j^n$ of the difference scheme (2.3) is given by

$$\begin{aligned} E_j^n &= \frac{u(x_j,t_{n+1}) - u(x_j,t_n)}{k} + a_j^n \frac{u(x_j,t_n) - u(x_{j-1},t_n)}{h} - G(x_j,t_n) \\ &= [u_t + \frac{1}{2}ku_{tt} + O(k^2)]_j^n + [a(u_x - \frac{1}{2}hu_{xx}) + O(h^2)]_j^n - G(x_j,t_n) \\ &= [u_t + au_x]_j^n - G(x_j,t_n) + \frac{1}{2}[ku_{tt} - ahu_{xx}]_j^n + O(h^2) + O(k^2). \end{aligned}$$

Here we used the Taylor series approximations for the terms $u(x_j,t_{n+1})$ and $u(x_{j-1},t_n)$ w.r.t. $t$ and $x$ respectively.

As $u$ satisfy the given differential equation (1.2), $[u_t + au_x]_j^n - G(x_j,t_n) = 0$.

Hence, the consistency error is given by

$$E_j^n = \frac{1}{2}[ku_{tt} - ahu_{xx}]_j^n + O(h^2) + O(k^2).$$

Now $E_j^n \to 0$ while $(h,k) \to (0,0)$, which implies the first-order consistency of the difference scheme in spatial as well as time direction.

Now consider the finite difference scheme (2.7) and by applying the triangle inequality, we get

$$|u_j^{n+1}| \leq |(1 - a_j^n\lambda)|\ |u_j^n| + |a_j^n\lambda|\ |u_{j-1}^n| + k|G_j^n|$$

taking the norm, we get

$$\begin{aligned}\|u^{n+1}\|_{\infty,h} &= \max_j |u_j^{n+1}| \\ &\leq \max_j |(1 - a_j^n\lambda)|\ |u_j^n| + \max_j |a_j^n\lambda|\ |u_{j-1}^n| + k\max_j |G_j^n|.\end{aligned}$$

Now using CFL condition $A\lambda \leq 1$, where $|a(x,t)| \leq A$, $\forall (x,t)$; first two terms in above inequality can be combined and we get

$$\|u^{n+1}\|_{\infty,h} \leq \|u^n\|_{\infty,h} + k\|G^n\|_{\infty,h}$$

here the value $\|G^n\|_{\infty,h}$ is given by

$$\begin{aligned}\|G^n\|_{\infty,h} &= \max_j \left|b_j^n[I_\alpha^- u_j^n - u_j^n] + c_j^n[L_\beta^+ u_j^n - u_j^n]\right| \\ &\leq \max_j \left|b_j^n[I_\alpha^- u_j^n - u_j^n]\right| + \max_j \left|c_j^n[L_\beta^+ u_j^n - u_j^n]\right| \\ &\leq (2B + 2M)\|u^n\|_{\infty,h}.\end{aligned}$$

Here we assume that $b(x,t)$ and $c(x,t)$ are bounded in the domain, *i.e.*, $|b_j^n| \leq B$ and $|c_j^n| \leq M$, where $B$ and $M$ are constants and $2(B+M)k$ is of the form $O(k)$. Using these values, we get the following estimate

$$\|u^{n+1}\|_{\infty,h} \leq \|u^n\|_{\infty,h} + O(k)\|u^n\|_{\infty,h}$$

*i.e.*,

$$\|u^{n+1}\|_{\infty,h} \leq C\|u^n\|_{\infty,h}$$

which implies the stability of the numerical approximation, where $C = 1 + O(k)$.

Now set $e_j^n = u_j^n - u(x_j, t_n)$ in (2.6). We write $e_{j\alpha}^n$ and $e_{j\beta}^n$ for error containing term point-wise delay and advance respectively. Now $u_j^n$ satisfies (2.6) exactly, while the exact solution $u(x_j, t_n)$ leaves the remainder $E_j^n k$. Therefore, the error in the approximation is given by

$$e_j^{n+1} = (1 - a_j^n\lambda)e_j^n + a_j^n\lambda e_{j-1}^n + k[b_j^n(e_{j\alpha}^n - e_j^n)] + k[c_j^n(e_{j\alpha}^n - e_j^n)] - kE_j^n$$

and $\qquad e_0^n = 0.$

For $A\lambda \leq 1$,

$$\begin{aligned}\left\|e^{n+1}\right\|_{\infty,h} &= \max_j |e_j^{n+1}| \\ &\leq \|e^n\|_{\infty,h} + k\max_j |b_j^n| [\|e^n\|_{\infty,h} + \|e^n\|_{\infty,h}] + k\max_j |c_j^n| \\ &[\|e^n\|_{\infty,h} + \|e^n\|_{\infty,h}] + k\max_j |E_j^n| \\ &\leq \|e^n\|_{\infty,h} + 2kB\|e^n\|_{\infty,h} + 2kM\|e^n\|_{\infty,h} + k\max_j |E_j^n| \\ &= (1 + 2Bk + 2Mk)\|e^n\|_{\infty,h} + k\max_j |E_j^n|,\end{aligned}$$

since we are using the given initial value for $u_j^n$, so $\left\|e^0\right\|_{\infty,h} = 0$ and if we suppose that the consistency error is bounded, *i.e.*, $|E_j^n| \leq E_{max}$, then by using induction method in the above inequality

$$\left\|e^{n+1}\right\|_{\infty,h} \leq nkE_{max} \leq TE_{max},$$

where $nk = T$, which proves that the numerical scheme (2.3) is convergent provided that the solution $u$ has bounded derivatives up to second order. □

**Remark.** When both the difference arguments $\alpha$ and $\beta$ are equal or multiple of each other, we can construct a mesh such that the terms containing both the difference arguments $(x_j - \alpha)$ and $(x_j + \beta)$ belong to the discrete set of grid points. This can be done by taking a particular mesh size. In this case there is no need to use an interpolation.

### 2.2. Lax-Friedrichs Approximation

In this approximation, we approximate the time derivative by forward difference and space by centered difference and then we replace $u_j^n$ by the mean value between $u_{j+1}^n$ and $u_{j-1}^n$ for stability purpose. Numerical scheme is given by

$$\frac{u_j^{n+1} - \frac{u_{j+1}^n + u_{j-1}^n}{2}}{k} + a_j^n \frac{u_{j+1}^n - u_{j-1}^n}{2h} = G_j^n \tag{2.8}$$

*i.e.*,

$$u_j^{n+1} = \frac{1}{2}(u_{j+1}^n + u_{j-1}^n) - \frac{\lambda}{2}a_j^n(u_{j+1}^n - u_{j-1}^n) + kG_j^n, \quad \forall\, j = 1, 2, ..., J-1$$

together with initial and boundary-interval conditions as in (2.4). To tackle point-wise delay and advance in $G$, we use the interpolation as we did in previous section.

We discuss consistency, stability, and convergence of this approximation in the next Theorem.

**Theorem 2.** *Let the function $u_0$ be sufficiently smooth then the explicit numerical scheme (2.8) is consistent of order 2 in space and of order 1 in time. Also if CFL condition is satisfied, then on the arbitrary finite time interval $[0, T]$, the numerical scheme (2.8) is stable in the maximum norm where stability constant is of the form $C = 1 + O(k)$, and hence convergent.*

*Proof.* We study the consistency of the numerical scheme (2.8) as we discussed in Theorem (2.1). If $u$ is sufficiently smooth then by using Taylor approximations for the terms $u_j^{n+1}$, $u_{j+1}^n$, and $u_{j-1}^n$, consistency error $E_j^n$ of the difference scheme (2.8) is given by

$$E_j^n = \frac{k}{2}u_{tt} - \frac{h^2}{k}u_{xx} + \frac{1}{6}a_j^n h^2 u_{xxx} + O(h^4 + k^{-1}h^4 + k^2).$$

Now $E_j^n \to 0$ while $(h, k) \to (0, 0)$, which implies the numerical scheme is consistent of order 2 in space and of order 1 in time as long as $k^{-1}h^2 \longrightarrow 0$.

Proof of stability can be done via adopting similar steps as we did in previous theorem (2.1). By using CFL condition we get stability inequality of the form $\|u^{n+1}\|_{\infty,h} \leq C\|u^n\|_{\infty,h}$, where stability constant $C$ is of the form $C = 1+O(k)$.

Proof of convergence also can be done as we did in previous theorem (2.1). Order of convergence is less than 2 in space as we are using linear interpolation for right side containing point-wise delay and advance. □

## 3. Numerical Experiments

In this section, we present some numerical examples to validate the predicted results established in the paper. We perform numerical computations using MATLAB. For first example, we take data from neuronal variability model of Stein [4]. Second and third examples are taken for general mathematical interests. The maximum absolute errors for the considered examples are calculated using the half mesh principle [15] as the exact solution for the considered examples are not available. We calculate the errors by refining the grid points. The error in the numerical approximation is given by

$$E(h, k) = \max_{0\leq j\leq J,\ 0\leq n\leq N} |u_h^k(j, n) - u_{h/2}^{k/2}(2j, 2n)|$$

**Example 1.** We consider the transport equation (1.1) with the following coefficients and initial-boundary conditions:

$$\begin{aligned}&\tau = 1, \quad p_e = 1/50, \quad p_i = 1/100, \quad v_0 = 1/2, \quad r = 3,\\ &F(v,0) = 1 - e^{-2v},\\ &F(0,t) = 0, \qquad F(r,t) = 1.\end{aligned}$$

The computed solution is plotted in Figure 1 and Figure 2 with the two numerical approximations constructed in this paper at time $t = 0.5$ and $t = 1.0$, respectively. We take step sizes $h = .001$ and $k = .0001$ for both the Figures. The graphs of upwind and Lax-Friedrichs are very close to each other at time level $t = 0.5$ but as the value of $t$ is increased, the approximate solution by both the schemes has a slight difference. We draw the Error Table 1 which verify the first-order convergence of upwind scheme in both spatial as well as time direction. Error Table 2 verifies the first-order convergence of Lax-Friedrichs approximation in time and greater than one in space (not second-order because we use linear interpolation to deal with point-wise delay and advance). Both the approximations work for low frequencies of excitatory and inhibitory impulses due to the form of stability constant. Both the graphs shows the probability density function of neurons such as in integrate-and-fire models.

**Example 2.** We consider the transport equation (1.2) with the following variable coefficients and initial and boundary-interval conditions:

$$\begin{aligned}&a(x,t) = \tfrac{1+x^2}{1+2xt+2x^2+x^4}; \quad b(x,t) = 0.02; \quad c(x,t) = 0.001,\\ &u(x,0) = \exp[-10(4x-1)^2];\\ &u(x,t) = 0, \quad \forall\, x \in [-\alpha, 0];\\ &u(x,t) = 0, \quad \forall\, x \in [2, 2+\beta].\end{aligned}$$

We consider the domain $\Omega = [0,2]$ and $h = .001 = k$. We show the numerical solution at different time levels $t = 0.2$, $t = 0.4$, and $t = 0.6$ with the two numerical approximations with $\alpha = 0.8$, and $\beta = 0.3$ in Figure 3. Errors in both the approximations are shown in Table 3 and Table 4, respectively.

**Example 3.** We consider the transport equation (1.2) with the following variable coefficients and initial-boundary conditions:

$$\begin{aligned}&a(x,t) = \tfrac{1+x^2}{1+2xt+2x^2+x^4}, \quad b(x,t) = \tfrac{1}{1+2xt+x^4t^4}, \quad c(x,t) = \tfrac{1-x^2t^4}{1+4x^6},\\ &u(x,0) = \exp[-10(4x-5)^2],\\ &u(x,t) = 0, \quad \forall\, x \in [-\alpha, 0],\\ &u(x,t) = 0, \quad \forall\, x \in [2, 2+\beta].\end{aligned}$$

| $\Delta t \downarrow \Delta x \rightarrow$ | 1/100 | 1/200 | 1/400 | 1/800 |
|---|---|---|---|---|
| $\Delta x/2$ | 0.005314 | 0.002665 | 0.001337 | 0.000668 |
| $\Delta x/4$ | 0.002664 | 0.001334 | 0.000667 | 0.000334 |
| $\Delta x/8$ | 0.001332 | 0.000667 | 0.000332 | 0.000175 |
| $\Delta x/16$ | 0.000661 | 0.000335 | 0.000171 | 0.000086 |

Table 1: The maximum absolute error for example 1 by upwind scheme

| $\Delta t \downarrow \Delta x \rightarrow$ | 1/100 | 1/200 | 1/400 | 1/800 |
|---|---|---|---|---|
| $\Delta x/2$ | 0.006816 | 0.003016 | 0.001415 | 0.000689 |
| $\Delta x/4$ | 0.004324 | 0.001705 | 0.000756 | 0.000354 |
| $\Delta x/8$ | 0.003106 | 0.001083 | 0.000434 | 0.000192 |
| $\Delta x/16$ | 0.001935 | 0.000856 | 0.000273 | 0.000113 |

Table 2: The maximum absolute error for example 1 by Lax-Friedrichs scheme

| $\Delta t \downarrow \Delta x \rightarrow$ | 1/100 | 1/200 | 1/400 | 1/800 |
|---|---|---|---|---|
| $\Delta x/2$ | 0.005112 | 0.002655 | 0.001356 | 0.000659 |
| $\Delta x/4$ | 0.002567 | 0.001456 | 0.000773 | 0.000357 |
| $\Delta x/8$ | 0.001456 | 0.000558 | 0.000349 | 0.000174 |
| $\Delta x/16$ | 0.000662 | 0.000385 | 0.000168 | 0.000075 |

Table 3: The maximum absolute error for example 2 by upwind scheme

We take the domain $\Omega = [0, 2]$ and $h = .001 = k$. We show the numerical solution at different time levels $t = 0.1$, $t = 0.5$, and $t = 1$ with the two numerical approximations with $\alpha = 1$, and $\beta = 0.1$ in Figure 4. Also we show the approximate solution in Figure 5 with $\alpha = 1.3$ and $\beta = 0.3$ at $t = 0.2$, $t = 0.4$, and $t = 0.6$, respectively. As time increases the height of the graph decreases and impulse moves to right side. Error tables 5 and 6 are plotted with $\alpha = 1$ and $\beta = 0.4$. Analysis of Tables also verifies that the methods are convergent in spatial as well as time direction. Table 5 verifies first-order convergence of upwind approximation in both space and time. Table 6 verifies order of convergence greater than one in space and order one in time for Lax-Friedrichs approximation. Plotted figures also shows the effect of point-wise delay and advance on solution.

| $\Delta t \downarrow \Delta x \rightarrow$ | 1/100 | 1/200 | 1/400 | 1/800 |
|---|---|---|---|---|
| $\Delta x/2$ | 0.006614 | 0.003413 | 0.001545 | 0.000685 |
| $\Delta x/4$ | 0.004024 | 0.001766 | 0.000768 | 0.000352 |
| $\Delta x/8$ | 0.002497 | 0.000986 | 0.000424 | 0.000173 |
| $\Delta x/16$ | 0.001678 | 0.000456 | 0.000263 | 0.000097 |

Table 4: The maximum absolute error for example 2 by Lax-Friedrichs scheme

| $\Delta t \downarrow \Delta x \rightarrow$ | 1/100 | 1/200 | 1/400 | 1/800 |
|---|---|---|---|---|
| $\Delta x/2$ | 0.012855 | 0.006423 | 0.003210 | 0.001605 |
| $\Delta x/4$ | 0.006428 | 0.003211 | 0.001605 | 0.000803 |
| $\Delta x/8$ | 0.003214 | 0.001606 | 0.000803 | 0.000401 |
| $\Delta x/16$ | 0.001607 | 0.000803 | 0.000401 | 0.000201 |

Table 5: The maximum absolute error for example 3 by upwind scheme

| $\Delta t \downarrow \Delta x \rightarrow$ | 1/100 | 1/200 | 1/400 | 1/800 |
|---|---|---|---|---|
| $\Delta x/2$ | 0.014941 | 0.006650 | 0.003198 | 0.001585 |
| $\Delta x/4$ | 0.008633 | 0.003762 | 0.001665 | 0.000800 |
| $\Delta x/8$ | 0.006442 | 0.002920 | 0.000941 | 0.000416 |
| $\Delta x/16$ | 0.004553 | 0.001890 | 0.000732 | 0.000331 |

Table 6: The maximum absolute error for example 3 by Lax-Friedrichs scheme

## 4. Conclusions

The two explicit numerical schemes based on the finite difference method are constructed to find the approximate solution of the transport equation of mixed type. The proposed methods works irrespective the size of difference arguments. The stability and convergence analysis proves that proposed numerical schemes are conditionally stable, consistent and convergent in space and time. The point-wise delay and advance affects the solution which is shown via numerical examples. Order of convergence also depends on the interpolation used to tackle the difference arguments. Finally, we remark that the strategy developed here can be applied to a problem having multiple point-wise delay or advance or both.

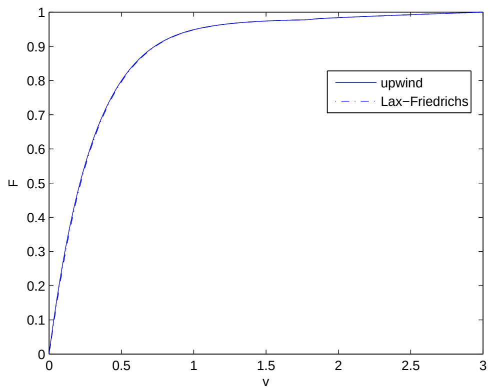


Figure 1: Example 1: The approximate solution at $t = 0.5$

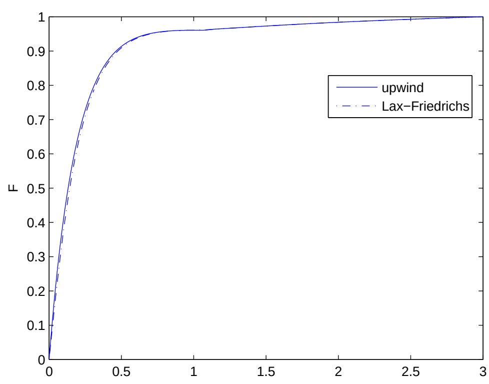


Figure 2: Example 1: The approximate solution at $t = 1$

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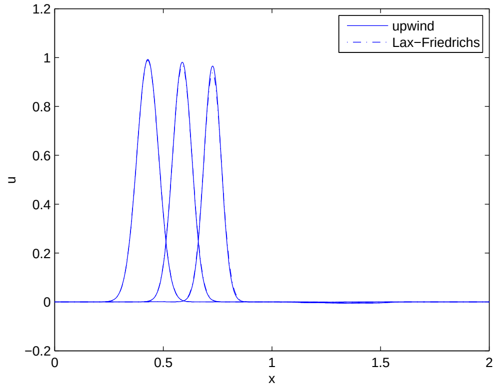


Figure 3: Example 2: The approximate solution at different time level with $\alpha = 0.8$ and $\beta = 0.3$

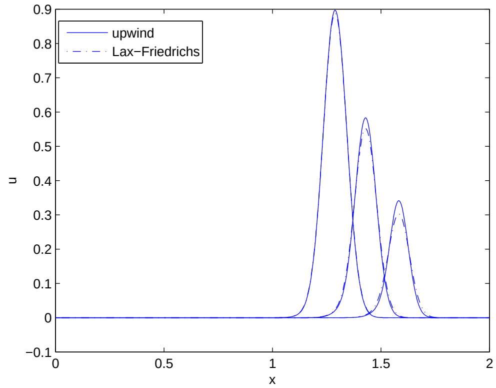


Figure 4: Example 3: The approximate solution at different time level with $\alpha = 1.0$ and $\beta = 0.1$

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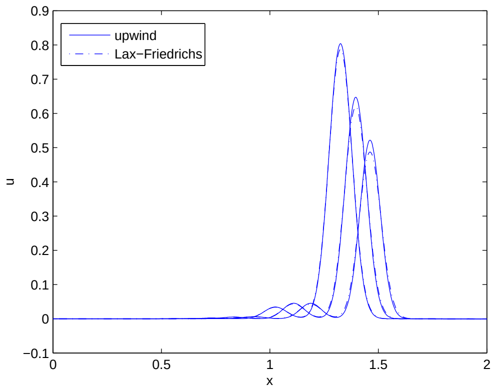


Figure 5: Example 3: The approximate solution at different time level with $\alpha = 1.3$ and $\beta = 0.3$

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