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NONOSCILLATORY SOLUTIONS OF A HIGHER-ORDER NONLINEAR NEUTRAL DELAY DIFFERENTIAL EQUATION

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Abstract: A higher-order nonlinear neutral delay differential equation

$$\Big\{r_n(t)\cdots\Big[r_2(t)\big[r_1(t)[x(t)+P(t)x(t-\tau)]'\big]'\Big]'\cdots\Big\}' + F\big(t,x(t-\sigma_1),x(t-\sigma_2),\cdots,x(t-\sigma_m)\big)=0,\quad t\geq t_0,$$

where $\tau>0,\sigma_1,\sigma_2,\cdots,\sigma_m\geq 0$, $F\in C([t_0,+\infty)\times\mathbb{R}^m,\mathbb{R})$, $P,r_i\in C([t_0,\infty),\mathbb{R})$, $1\leq i\leq n$, is studied in this paper, and some sufficient conditions for the existence of nonoscillatory solutions of this equation are established by using Krasnoselskii fixed point theorem and expressed through five theorems according to the range of the value of the function $P(t)$.

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1. Introduction and Preliminaries

Recently, many researchers have payed their attention to second-order neutral delay differential equations, refer to [1,3∼5,8,9,11∼16]. In 1998, Kulenovic and Hadziomerspahic [8] studied the following second-order linear neutral delay differential equation with positive and negative coefficients:

$$\frac{d^2}{dt^2}\big(x(t)+px(t-\tau)\big)+Q_1(t)x(t-\sigma_1)-Q_2(t)x(t-\sigma_2)=0, \quad t\geq t_0, \quad (1.1)$$

where $\tau>0,\sigma_1,\sigma_2\geq 0, p\in\mathbb{R}=(-\infty,+\infty)$, $Q_1,Q_2\in C([t_0,+\infty),\mathbb{R}^+)$ with $\mathbb{R}^+=[0,+\infty)$, and gave some sufficient conditions which guarantee the existence of nonoscillatory solutions of equation (1.1). In 2004, Cheng and Annie [3] established the existence of solutions of equation (1.1) under weaker conditions and improved the works of Kulenovic and Hadziomerspahic [8]. In 2005, Yu and Wang [15] extended the results of Kulenovic and Hadziomerspahic [8] and investigated the existence of nonoscillatory solutions for the following second-order nonlinear neutral delay differential equation with positive and negative coefficients:

$$\Big(r(t)\big(x(t)+P(t)x(t-\tau)\big)'\Big)' \\ +Q_1(t)f\big(x(t-\sigma_1)\big)-Q_2(t)g\big(x(t-\sigma_2)\big)=0, \quad t\geq t_0, \quad (1.2)$$

where $\tau>0,\sigma_1,\sigma_2\geq 0, P,r\in C([t_0,+\infty),\mathbb{R})$, $Q_1,Q_2\in C([t_0,+\infty),\mathbb{R}^+)$, $f,g\in C(\mathbb{R},\mathbb{R})$. In 2007, Zhou [16] studied the following second-order nonlinear neutral delay differential equation:

$$\Big(r(t)\big(x(t)+P(t)x(t-\tau)\big)'\Big)'+\sum_{i=1}^{m}Q_i(t)f_i\big(x(t-\sigma_i)\big)=0, \quad t\geq t_0, \quad (1.3)$$

where $\tau>0,\sigma_i\geq 0, P,r,Q_i\in C([t_0,+\infty),\mathbb{R})$, $f_i\in C(\mathbb{R},\mathbb{R}), i=1,2,\cdots,m$, which improved the results of Kulenovic, Hadziomerspahic [8] and Yu, Wang [15]. For further knowledge of existence and uniqueness of solutions of neutral delay differential equations, pay the respect to [2,6,7].

Motivated by the papers mentioned above, in this paper we investigate the following higher-order nonlinear neutral delay differential equation

$$\Big\{r_n(t)\cdots\Big[r_2(t)\big[r_1(t)[x(t)+P(t)x(t-\tau)]'\big]'\Big]'\cdots\Big\}' \\ +F\big(t,x(t-\sigma_1),x(t-\sigma_2),\cdots,x(t-\sigma_m)\big)=0, \quad t\geq t_0, \quad (1.4)$$

where $\tau > 0, \sigma_i \geq 0, F \in C([t_0, +\infty) \times \mathbb{R}^m, \mathbb{R}), P, r_j \in C([t_0, \infty), \mathbb{R})$ for $i \in \{1, 2, \cdots, m\}, j \in \{1, 2, \cdots, n\}$, and establish the existence of nonoscillatory solutions for equation (1.4) by using Krasnoselskii fixed point theorem. Clearly equation (1.1) $\sim$ (1.3) are special cases of equation (1.4).

Lemma 1.1. (Krasnoselskii Fixed Point Theorem) *Let S be a bounded closed convex subset of a Banach space X and $T_1, T_2 : S \to X$ satisfy $T_1x+T_2y \in S$ for each $x, y \in S$. If T_1 is a contraction mapping and T_2 is a completely continuous mapping, then the equation $T_1x + T_2x = x$ has at least one solution in S.*

A solution of equation (1.4) is said to be oscillatory if it neither eventually positive nor eventually negative, and otherwise it is called nonoscillatory. Let $u \in C([t_0 - \rho, +\infty), \mathbb{R})$, where $\rho = \max\{\tau, \sigma_j\}$, where $1 \leq j \leq m$, be a given function and y_0 a given constant. From the method of steps, it follows that equation (1.4) has a unique solution $x \in C([t_0 - \rho, +\infty), \mathbb{R})$ if $x(t)+P(t)x(t-\tau)$, $r_1(t)[x(t)+P(t)x(t-\tau)]'$, $r_2(t)\big[r_1(t)[x(t)+P(t)x(t-\tau)]'\big], \cdots,$ $r_n(t)\Big[r_{n-1}(t)\cdots\Big[r_2(t)\big[r_1(t)[x(t)+P(t)x(t-\tau)]'\big]'\Big]'\cdots\Big]$ are continuously differentiable for $t \geq t_0$, $x(t)$ satisfies the equation (1.4) and

$$x(s) = u(s) \quad \text{for} \quad s \in [t_0 - \rho, t_0],$$
$$\big(x(t) + P(t)x(t-\tau)\big)'_{t=t_0} = y_0.$$

Throughout this paper, we assume that $\rho = \max\{\tau, \sigma_j\}$, where $1 \leq j \leq n$, and X denotes the set of all continuous and bounded functions on $[t_0, +\infty)$ with the sup norm and

$$S = \big\{x \in X : M \leq x(t) \leq N, t \geq t_0\big\}$$

for $N > M > 0$. Obviously, S is a nonempty closed convex subset of the Banach space X. For $P \in C([t_0, +\infty), \mathbb{R})$, put

$$\overline{P} = \limsup_{t\to+\infty} P(t) \quad \text{and} \quad \underline{P} = \liminf_{t\to+\infty} P(t).$$

2. Existence of Nonoscillatory Solutions

In this section, a few sufficient conditions of the existence of nonoscillatory solutions for equation (1.4) are given.

Theorem 2.1. *Assume that there exist constants P_0, M and N with $N > M > 0$ and functions $h, q \in C([t_0, +\infty), \mathbb{R}^+)$ and $P, r_i \in C([t_0, +\infty), \mathbb{R})$, where $1 \le i \le m$, such that for $t \in [t_0, +\infty)$*

$$|P(t)| \le P_0 < \frac{N-M}{2N}, \text{ eventually,} \tag{2.1}$$

$$\begin{aligned}\big|F(t, u_1, u_2, \cdots, u_m) - F(t, v_1, v_2, \cdots, v_m)\big| \\ \le h(t) \max\big\{|u_i - v_i| : u_i, v_i \in [M, N], 1 \le i \le m\big\},\end{aligned} \tag{2.2}$$

$$\big|F(t, u_1, u_2, \cdots, u_m)\big| \le q(t), \; u_i \in [M, N], 1 \le i \le m, \tag{2.3}$$

$$\int_{t_0}^{+\infty} \max\big\{\frac{1}{|r_i(t)|}, h(t), q(t) : 1 \le i \le m\big\} dt < +\infty. \tag{2.4}$$

Then equation (1.4) has a nonoscillatory solution.

Proof. Choose $L \in (M + P_0N, N - P_0N)$. By (2.1) and (2.4), there exists a sufficiently large $l > t_0 + \rho$ such that

$$|P(t)| \le P_0 < \frac{N-M}{2N}, \; \forall t \ge l \tag{2.5}$$

and

$$\begin{aligned}\int_l^{+\infty} \int_{s_1}^{+\infty} \cdots \int_{s_n}^{+\infty} \frac{q(s)}{\left|\prod_{i=1}^n r_i(s_i)\right|} ds ds_n \cdots ds_1 \\ \le \min\{L - P_0N - M, N - P_0N - L\}.\end{aligned} \tag{2.6}$$

Define two mappings $T_1, T_2 : S \to X$ by

$$(T_1x)(t) = \begin{cases} L - P(t)x(t-\tau), & t \ge l, \\ (T_1x)(l), & t_0 \le t < l, \end{cases} \tag{2.7}$$

$$(T_2x)(t) = \begin{cases} (-1)^n \int_t^{+\infty} \int_{s_1}^{+\infty} \cdots & \\ \quad \int_{s_n}^{+\infty} \frac{F\left(s, x(s-\sigma_1), \cdots, x(s-\sigma_m)\right)}{\prod_{i=1}^n r_i(s_i)} ds ds_n \cdots ds_1, & t \ge l \\ (T_2x)(l), & t_0 \le t < l, \end{cases} \tag{2.8}$$

for all $x \in S$.

(i) It is claimed that $T_1x + T_2y \in S$ for all $x, y \in S$.

In fact, for each $x, y \in S$ and $t \geq l$, it follows from (2.3), (2.5) and (2.6) that

$$
\begin{aligned}
&(T_1x)(t) + (T_2y)(t) \\
&\quad \geq L - P_0x(t-\tau) - \int_t^{+\infty}\int_{s_1}^{+\infty}\cdots \\
&\qquad\qquad \int_{s_n}^{+\infty} \frac{\big|F\big(s, y(s-\sigma_1), \cdots, y(s-\sigma_m)\big)\big|}{\big|\prod_{i=1}^{n} r_i(s_i)\big|} ds ds_n \cdots ds_1 \\
&\quad \geq L - P_0N - \int_l^{+\infty}\int_{s_1}^{+\infty}\cdots\int_{s_n}^{+\infty} \frac{q(s)}{\big|\prod_{i=1}^{n} r_i(s_i)\big|} ds ds_n \cdots ds_1 \\
&\quad \geq M
\end{aligned}
$$

and

$$
\begin{aligned}
&(T_1x)(t) + (T_2y)(t) \\
&\quad \leq L + P_0N + \int_l^{+\infty}\int_{s_1}^{+\infty}\cdots\int_{s_n}^{+\infty} \frac{q(s)}{\big|\prod_{i=1}^{n} r_i(s_i)\big|} ds ds_n \cdots ds_1 \\
&\quad \leq N.
\end{aligned}
$$

Thus, $T_1x + T_2y \in S$ for any $x, y \in S$.

(ii) It is declared that T_1 is a contraction mapping on S.

In reality, for any $x, y \in S$ and $t \geq l$, it is easy to derive that

$$\big|(T_1x)(t) - (T_1y)(t)\big| \leq |P(t)|\big|x(t-\tau) - y(t-\tau)\big| \leq P_0||x-y||,$$

which implies that

$$||T_1x - T_1y|| \leq P_0||x-y||.$$

$P_0 < \frac{N-M}{2N} < 1$ ensures that T_1 is a contraction mapping on S.

(iii) It can be asserted that T_2 is completely continuous.

Firstly, we show T_2 is continuous. Let $\{x_k\}_{k\geq 0} = \{x_k(t)\}_{k\geq 0} \subset S$ be a sequence such that $x_k \to x = x(t)$ as $k \to +\infty$. Since S is closed, $x \in S$. For

$t \geq l$, (2.2) guarantees that

$$\begin{aligned}
&\big|(T_2x_k)(t) - (T_2x)(t)\big| \\
&\leq \int_t^{+\infty}\int_{s_1}^{+\infty}\cdots\int_{s_n}^{+\infty} \\
&\quad \frac{h(s)\max\big\{|x_k(s-\sigma_i) - x(s-\sigma_i)| : 1 \leq i \leq m\big\}}{\big|\prod_{i=1}^n r_i(s_i)\big|}dsds_n\cdots ds_1 \\
&\leq \|x_k - x\| \int_t^{+\infty}\int_{s_1}^{+\infty}\cdots\int_{s_n}^{+\infty}\frac{h(s)}{\big|\prod_{i=1}^n r_i(s_i)\big|}dsds_n\cdots ds_1.
\end{aligned}$$

This above inequality together with (2.4) implies that T_2 is continuous.

Next, we prove T_2S is relatively compact. It suffices to show that the family of functions $\{T_2x : x \in S\}$ is uniformly bounded and equicontinuous on $[t_0, +\infty)$. The uniform boundedness is obvious. For the equicontinuity, according to Levitan's result [10], it is only need to prove that, for any given $\varepsilon > 0$, $[t_0, +\infty)$ can be decomposed into finite subintervals in such a way that on each subinterval all functions of the family have change of amplitude less than ε. By (2.4), for any $\varepsilon > 0$, take $l' \geq l$ large enough so that

$$\int_{l'}^{+\infty}\int_{s_1}^{+\infty}\cdots\int_{s_n}^{+\infty}\frac{q(s)}{\big|\prod_{i=1}^n r_i(s_i)\big|}dsds_n\cdots ds_1 < \frac{\varepsilon}{2}. \tag{2.9}$$

Then, for any $x \in S$ and $t_2 > t_1 \geq l'$, (2.9) ensures that

$$\begin{aligned}
&\big|(T_2x)(t_2) - (T_2x)(t_1)\big| \\
&\leq \int_{t_2}^{+\infty}\int_{s_1}^{+\infty}\cdots\int_{s_n}^{+\infty}\frac{\big|F\big(s, x(s-\sigma_1), \cdots, x(s-\sigma_m)\big)\big|}{\big|\prod_{i=1}^n r_i(s_i)\big|}dsds_n\cdots ds_1 \\
&\quad + \int_{t_1}^{+\infty}\int_{s_1}^{+\infty}\cdots\int_{s_n}^{+\infty}\frac{\big|F\big(s, x(s-\sigma_1), \cdots, x(s-\sigma_m)\big)\big|}{\big|\prod_{i=1}^n r_i(s_i)\big|}dsds_n\cdots ds_1 \\
&\leq \int_{l'}^{+\infty}\int_{s_1}^{+\infty}\cdots\int_{s_n}^{+\infty}\frac{q(s)}{\big|\prod_{i=1}^n r_i(s_i)\big|}dsds_n\cdots ds_1 \\
&\quad + \int_{l'}^{+\infty}\int_{s_1}^{+\infty}\cdots\int_{s_n}^{+\infty}\frac{q(s)}{\big|\prod_{i=1}^n r_i(s_i)\big|}dsds_n\cdots ds_1 \\
&< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.
\end{aligned}$$

For any $x \in S$ and $l \leq t_1 < t_2 \leq l'$, there exists $\delta > 0$ such that if $0 < t_2 - t_1 < \delta$, then

$$\begin{aligned}
&\big|(T_2x)(t_2)-(T_2x)(t_1)\big| \\
&\quad\le \int_{t_1}^{t_2}\int_{s_1}^{+\infty}\cdots\int_{s_n}^{+\infty}\frac{\big|F\big(s,x(s-\sigma_1),\cdots,x(s-\sigma_m)\big)\big|}{\big|\prod_{i=1}^{n} r_i(s_i)\big|}ds ds_n\cdots ds_1 \\
&\quad\le \int_{t_1}^{t_2}\int_{s_1}^{+\infty}\cdots\int_{s_n}^{+\infty}\frac{q(s)}{\big|\prod_{i=1}^{n} r_i(s_i)\big|}ds ds_n\cdots ds_1 \\
&\quad<\varepsilon.
\end{aligned}$$

For any $x\in S$ and $t_0\le t_1<t_2\le l$, it is easy to get that

$$\big|(T_2x)(t_2)-(T_2x)(t_1)\big|=0<\varepsilon.$$

Consequently $\{T_2x : x\in S\}$ is uniformly bounded and equicontinuous on $[t_0,+\infty)$. Therefore T_2S is relatively compact. It follows from Lemma 1.1 that there is $x_0\in S$ such that $T_1x_0+T_2x_0=x_0$. Obviously, $x_0(t)$ is a nonoscillatory solution of equation (1.4). This completes the proof.

Theorem 2.2. *Assume that there exist constants M and N with $N>\frac{2-\underline{P}}{1-\overline{P}}M>0$ and functions $h,q\in C([t_0,+\infty),\mathbb{R}^+)$ and $P,r_i\in C([t_0,+\infty),\mathbb{R})$, where $1\le i\le m$, satisfying (2.2) $\sim$ (2.4) and*

$$P(t)\ge 0,\ \textit{eventually, and } 0\le\underline{P}\le\overline{P}<1. \tag{2.10}$$

Then equation (1.4) has a nonoscillatory solution in S.

Proof. Choose $L\in(M+\frac{1+\overline{P}}{2}N, N+\frac{\underline{P}}{2}M)$. By (2.10) and (2.4), a sufficiently large $l>t_0+\rho$ can be chosen such that

$$\frac{\underline{P}}{2}\le P(t)\le\frac{1+\overline{P}}{2},\ \forall t\ge l \tag{2.11}$$

and

$$\int_{l}^{+\infty}\int_{s_1}^{+\infty}\cdots\int_{s_n}^{+\infty}\frac{q(s)}{\big|\prod_{i=1}^{n} r_i(s_i)\big|}ds ds_n\cdots ds_1 \le \min\Big\{L-M-\frac{1+\overline{P}}{2}N, N-L+\frac{\underline{P}}{2}M\Big\}. \tag{2.12}$$

Define two mappings $T_1,T_2 : S\to X$ as (2.7) and (2.8). The rest proof is analogous to that in Theorem 2.1. This completes the proof.

Similar to the proof of Theorem 2.2, we have

Theorem 2.3. *Assume that there exist constants M and N with $N > \frac{2+\overline{P}}{1+\underline{P}} M > 0$ and functions $h, q \in C([t_0, +\infty), \mathbb{R}^+)$ and $P, r_i \in C([t_0, +\infty), \mathbb{R})$, where $1 \le i \le m$, satisfying (2.2) $\sim$ (2.4) and*

$$P(t) \le 0, \text{ eventually, and } -1 < \underline{P} \le \overline{P} \le 0. \tag{2.13}$$

Then equation (1.4) has a nonoscillatory solution in S.

Theorem 2.4. *Assume that there exist constants M and N with $N > \frac{\underline{P}(\overline{P}^2-\underline{P})}{\overline{P}(\underline{P}^2-\overline{P})} M > 0$ and functions $h, q \in C([t_0, +\infty), \mathbb{R}^+)$ and $P, r_i \in C([t_0, +\infty), \mathbb{R})$, where $1 \le i \le m$, satisfying (2.2) $\sim$ (2.4) and*

$$P(t) > 1, \text{ eventually, } 1 < \underline{P} \text{ and } \overline{P} < \underline{P}^2 < +\infty. \tag{2.14}$$

Then equation (1.4) has a nonoscillatory solution in S.

Proof. Take $\epsilon \in (0, \underline{P} - 1)$ sufficiently small satisfying

$$1 < \underline{P} - \epsilon < \overline{P} + \epsilon < (\underline{P} - \epsilon)^2 \tag{2.15}$$

and

$$\big((\overline{P} + \epsilon)(\underline{P} - \epsilon)^2 - (\overline{P} + \epsilon)^2\big)N > \big((\overline{P} + \epsilon)^2(\underline{P} - \epsilon) - (\underline{P} - \epsilon)^2\big)M. \tag{2.16}$$

Choose $L \in \big((\overline{P} + \epsilon)M + \frac{\overline{P}+\epsilon}{\underline{P}-\epsilon}N, (\underline{P} - \epsilon)N + \frac{\underline{P}-\epsilon}{\overline{P}+\epsilon}M\big)$. By (2.15) and (2.4), there exists a sufficiently large $l > t_0 + \rho$ such that

$$\underline{P} - \epsilon < P(t) < \overline{P} + \epsilon, \quad \forall t \ge l \tag{2.17}$$

and

$$\int_l^{+\infty} \int_{s_1}^{+\infty} \cdots \int_{s_n}^{+\infty} \frac{q(s)}{|\prod_{i=1}^n r_i(s_i)|} ds ds_n \cdots ds_1 \\ \le \min\Big\{\frac{\underline{P} - \epsilon}{\overline{P} + \epsilon} L - (\underline{P} - \epsilon)M - N, \frac{\underline{P} - \epsilon}{\overline{P} + \epsilon} M + (\underline{P} - \epsilon)N - L\Big\}. \tag{2.18}$$

Define two mappings $T_1, T_2 : S \to X$ by

$$(T_1 x)(t) = \begin{cases} \frac{L}{P(t+\tau)} - \frac{x(t+\tau)}{P(t+\tau)}, & t \ge l \\ (T_1 x)(l), & t_0 \le t < l \end{cases} \tag{2.19}$$

$$(T_2x)(t) = \begin{cases} \frac{(-1)^n}{P(t+\tau)} \int_{t+\tau}^{+\infty} \int_{s_1}^{+\infty} \cdots \\ \quad \int_{s_n}^{+\infty} \frac{F\left(s, x(s-\sigma_1), \cdots, x(s-\sigma_m)\right)}{\prod_{i=1}^{n} r_i(s_i)} ds ds_n \cdots ds_1, & t \geq l \\ (T_2x)(l), & t_0 \leq t < l \end{cases} \tag{2.20}$$

for all $x \in S$. The rest proof is similar to that in Theorem 2.1. This completes the proof.

As the proof of Theorem 2.4, we have

Theorem 2.5. *Assume that there exist constants M and N with $N > \frac{1+\underline{P}}{1+\overline{\overline{P}}} M > 0$ and functions $h, q \in C([t_0, +\infty), \mathbb{R}^+)$ and $P, r_i \in C([t_0, +\infty), \mathbb{R})$, where $1 \leq i \leq m$, satisfying* (2.2) $\sim$ (2.4) *and*

$$P(t) < -1, \text{ eventually, } -\infty < \underline{P} \text{ and } \overline{P} < -1. \tag{2.21}$$

Then equation (1.4) has a nonoscillatory solution in S.

Remark 2.1. Theorems 2.1$\sim$2.5 extend and improve Theorem A of Cheng and Annie [3], Theorem of Kulenovic and Hadziomerspahic [8], Theorems 1-3 of Yu and Wang [14], and Theorem 1 of Zhou [16].

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