**International Journal of Pure and Applied Mathematics**
**Volume 70 No. 1 2011, 113-115**

# A BOUNDEDNESS RESULT FOR PFAFF FIELDS

E. Ballico
Department of Mathematics
University of Trento
38 123 Povo (Trento) - Via Sommarive, 14, ITALY
e-mail: ballico@science.unitn.it

**Abstract:** Fix positive integers $s, d, n$ with $n \geq s+1$. Let $W \subseteq \mathbb{P}^n$ be an integral and Gorenstein projective variety of dimension $s+1$ such that $\dim(\mathrm{Sing}(W)) \leq s-1$. Fix $M, H \in \mathrm{Pic}(W)$ with $H$ ample. Here we prove the existence of an integer $x_0(H, d, M)$ with the following property. Fix any integer $x \geq x_0(H, d, M)$ and any integral $X \in |M \otimes H^{\otimes x}|$ such that $\dim(\mathrm{Sing}(X)) \leq s-2$; then there is no non-zero Pfaff field $\Omega_X^s \to \mathcal{O}_X(d)$. In particular $X$ is not a solution of a rank $s$ and degree $d$ Pfaff field on $\mathbb{P}^n$ whose singular locus does not contain $X$.

**AMS Subject Classification:** 37F75, 14J60, 58A17
**Key Words:** Pfaff field

*

Esteves and Kleiman introduced the general set-up of Pfaff systems and Pfaff fields with the right amount of generality ([5]). Let $Y$ be an equidimensional reduced projective variety defined over an algebraically closed field $\mathbb{K}$. Set $m := \dim(Y)$. Fix an integer $s \in \{1, \dots, m\}$. A rank $s$ Pfaff field on $Y$ is a map $\Omega_Y^s \to L$, where $L \in \mathrm{Pic}(Y)$. If $Y \subset \mathbb{P}^n$, and $\eta_1 : \Omega_{\mathbb{P}^n}^s \to \mathcal{O}_{\mathbb{P}^n}(d)$ is a non-zero rank $s$ Pfaff field, we say that $Y$ is a leaf of $\eta_1$ if $Y$ is not completely contained in singular locus of $\eta_1$ and $\eta_1$ factors through the natural map $\Omega_{\mathbb{P}^n}^s \to \Omega_Y^s$. A question of Poincaré aims to bounds $\deg(Y)$ in terms of $d$ for any leaf $Y$. As it stands the answer is negative and to give a positive answer we must both bounds the admissible singularities of $Y$ and the cohomology groups of $\mathcal{I}_Y$ ([2], [3], [4], [5] and references therein). In particular [2], Remark 21, gives an example

Received: February 25, 2011 

with $Y$ a smooth curve. Here we bound $\deg(Y)$ allowing only codimension 2 singularities, but prescribing that the leaf varies in a fixed $(s+1)$-dimensional variety. The case $s = n-1$ is classical (see e.g. [6], [2], [3], [4], [5]).

**Theorem 1.** *Fix an integer $s \geq 1$. Let $W$ be an integral and Gorenstein projective variety of dimension $s+1$ such that* $\dim(\mathrm{Sing}(W)) \leq s-1$. *Fix $L, M, H \in \mathrm{Pic}(W)$ with $H$ ample. Then there exists an integer $x_0(H, L, M)$ with the following property. Fix any integer $x \geq x_0(H, L, M)$ and any integral $X \in |M \otimes H^{\otimes x}|$ such that* $\dim(\mathrm{Sing}(X)) \leq s-2$. *Then there is no non-zero Pfaff field $\Omega^s_X \to L|X$.*

Taking $W \hookrightarrow \mathbb{P}^n$, $n \geq s+1$ and $L := \mathcal{O}_W(d)$ for some integer $d$ we immediately get the following result.

**Corollary 1.** *Fix positive integers $s, d, n$ with $n \geq s+1$. Let $W \subseteq \mathbb{P}n$ be an integral and Gorenstein projective variety of dimension $s+1$ such that* $\dim(\mathrm{Sing}(W)) \leq s-1$. *Fix $M, H \in \mathrm{Pic}(W)$ with $H$ ample. Then there exists an integer $x_0(H, d, M)$ with the following property. Fix any integer $x \geq x_0(H, d, M)$ and any integral $X \in |M \otimes H^{\otimes x}|$ such that* $\dim(\mathrm{Sing}(X)) \leq s-2$. *Then there is no non-zero Pfaff field $\Omega^s_X \to \mathcal{O}_X(d)$. In particular $X$ is not a solution of a rank $s$ and degree $d$ Pfaff field on $\mathbb{P}^n$ whose singular locus does not contain $X$.*

*Proof of Theorem 1.* Fix $x \in \mathbb{Z}$ such that there is $X \in |M \otimes H^{\otimes x}|$ such that $\dim(\mathrm{Sing}\,(Y)) \leq s-2$ with $\dim(\mathrm{Sing}(X)) \leq s-2$ and $\eta : \Omega^s_X \to L|X$ such that $\eta \neq 0$. Hence $\mathrm{Coker}(\eta)$ is supported by a closed subscheme of $X$ with dimension at most $s-1$. For any coherent sheaf $F$ on $X$ let $\mathcal{T}(F)$ denote the torsion subsheaf of $F$. Since $L$ has no torsion, $\eta$ induces a non-zero map $\eta' : \Omega^s_X/\mathcal{T}(\Omega^s_X) \to L$ such that $\mathrm{Im}(\eta) = \mathrm{Im}(\eta')$, i.e. $\mathrm{Coker}(\eta) = \mathrm{Coker}(\eta')$. Thus the map $\eta_* : H^s(X, \Omega^s_X/\mathcal{T}(\Omega^s_X)) \to H^s(X, L)$ is surjective. In [1], subsection 3.1, the authors defined a scheme-structure on the algebraic set $\mathrm{Sing}(X)$ using the natural map $\Omega^s_X \to \omega_X$, which gives an injective map $\Omega^s_X/\mathcal{T}(\Omega^s_X) \to \omega_X$, because $\omega_X$ has no torsion ([1], 3.1). Call $\Sigma_X$ this scheme-structure on $\mathrm{Sing}(X)$. Since $W$ is Gorenstein and $X$ is a Cartier divisor of $W$, $X$ is Gorenstein. Thus $\Omega^s_X/\mathcal{T}(\Omega^s_X) \cong \mathcal{I}_{\Sigma_X}\omega_X$. Hence $H^s(X, \Omega^s_X/\mathcal{T}(\Omega^s_X)) \cong H^s(X, \mathcal{I}_{\Sigma_X}\omega_X)$. Since $\dim(\Sigma(X) \leq s-2$, a standard exact sequence gives $H^s(X, \mathcal{I}_{\Sigma_X}\omega_X) \cong H^s, \omega_X)$. Duality gives that the latter vector space has dimension at most 1. Hence $h^s(X, L|X) \leq 1$. Thus $h^0(X, \omega_X \otimes (L|X)^*) \leq 1$ (duality). Since $X$ is a Cartier divisor of $W$, we have $\omega_X \cong \omega_W \otimes (M \otimes H^{\otimes x}|X$. Look at the exact sequence of coherent sheaves on $W$:

$$0 \to L^* \otimes \omega_W \to L^* \to L^* \otimes \omega_W \otimes M \otimes H^{\otimes x} \to \omega_X \otimes (L^*|X) \to 0 \qquad (1)$$

The integer $h^0(W, L \otimes \omega_W)$ does not depend from $x$. For $x \gg 0$ the integer $h^0(W, L^* \otimes \omega_W \otimes M \otimes H^{\otimes x})$ is arbitrarily large, because $H$ is ample. Thus for $x \gg 0$ we get $h^0(X, \omega_X \otimes (L|X)^*) \geq 2$, contradiction. □

## Acknowledgements

The author was partially supported by MIUR and GNSAGA of INdAM (Italy).

## References

[1] J.D.A.S. Cruz, E. Esteves, Bounding the regularity of subschemes invariant under Pfaff fields on projective spaces, *Comm. Math. Helv.*, To Appear, *ArXiv:* 0902.2332v1.

[2] E. Esteves, The Castelnuovo-Mumford regularity of an integral variety of a vector field on projective space, *Math. Res. Lett.*, **9** (2002), 1-15.

[3] E. Esteves, S. Kleiman, *Bounds of Leaves on One-Dimensional Foliations. Real and Complex Singularities*, Contemp. Math., **354**, Amer. Math. Soc., Providence, RI (2004).

[4] E. Esteves, S. Kleiman, Bounds on leaves of foliations in the plane, Dedicated to the 50th anniversary of IMPA, *Bull. Braz. Math. Soc.* (N.S.), **34**, No. 1 (2003), 145-169.

[5] E. Esteves, S. Kleiman, Bounding solutions of Pfaff equations, *Comm. Algebra*, **31** (2003), 3771-3793.

[6] M. Soares, The Poincaré problem for hypersurfaces invariants by one-dimensional foliations, *Invent. Math.*, **128** (1997), 495-500.

116