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AN ELEMENTARY APPLICATION OF CASTELNUOVO-SEVERI INEQUALITY

E. Ballico
Department of Mathematics
University of Trento
38 123 Povo (Trento) - Via Sommarive, 14, ITALY
e-mail: ballico@science.unitn.it

Abstract: Let $f : X \to C$ and $u : X \to Y$ morphisms between smooth curves with u a Galois covering with group G. Here we remark that if $p_a(X)$ is large, then Castelnuovo-Severi inequality shows that G acts on C f-equivariantly.

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Let X, Y, C be smooth and connected projective curves and $f : X \to C$, $u : X \to Y$, non-constant morphisms. Set $g := p_a(Y)$, $y := p_a(C)$, $z := p_a(Y)$, $x := \deg(f)$ and $e := \deg(u)$. We say that u normalizes f if there is a commutative diagram

$$\begin{array}{ccc} X & \xrightarrow{f} & C \\ u \downarrow & & \downarrow v \\ Y & \xrightarrow{m} & D. \end{array} \tag{1}$$

with $\deg(m) = x$ and $\deg(v) = e$. Outside the ramification points of v or m (1) is a cartesian diagram of schemes. Hence (1) is a fibber product if u

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is unramified. We will works on a range of integers x, y for which we from X, C, Y, f, u we get (1). Hence we will easily get $p_a(D)$ in terms of the other invariants, extending [3] to the case $x := \deg(f)$ not prime and f not a Galois covering. In [3], [4] and several related paper u (resp. v) is a Galois covering with group H (resp. G) and the assumption (as here) implies that H is a normal subgroup of $\mathrm{Aut}(X)$. Here we say that f is normalized by $\mathrm{Aut}(X)$ if a diagram (1) exists for all Galois coverings u and v is a Galois covering with Galois group isomorphic to G.

We first check the following well-known consequence of Castelnuovo-Severi inequality (see [6], Corollary at p. 20, [8], [2], Theorem 3.5).

Theorem 1. *Fix integers x, y, g, e, t such that $x \geq 2$, $y \geq 0$, $e \geq 2$, $t \geq 0$ and $g > 2xy+(x-1)^2$. Let $f : X \to C$ be a degree x separable morphism with C a smooth curve of genus e and X of genus g. Assume that f is a simple covering, i.e. assume the non-existence of a smooth curve X' and coverings $f_1 : X \to X'$, $f_2 : X' \to Y$ such that $\deg(f_i) \geq 2$, $i = 1, 2$, and $f = f_2 \circ f_1$. Assume the existence of a Galois covering $u : X \to Y$ with group G and $\sharp(G) = e$. Then a diagram (1) exists with v Galois with group isomorphic to G.*

Question 1. Is Theorem 1 true if $g = 2xy + (x-1)^2$ and $e \geq 3$?

Here we see that it is true if $y = 0$.

Proposition 1. *Let $f : X \to \mathbb{P}^1$ be a simple covering of degree $x \geq 2$ with X a smooth curve of genus $(x-1)^2$. If x is a prime assume that f is not a cyclic covering. Let H be the set of all $f \in \mathrm{Aut}(X)$ such that there is $h_1 \in \mathrm{Aut}(\mathbb{P}^1)$ with $h \circ f = f \circ h_1$. Then H has index at most two in $\mathrm{Aut}(X)$.*

Proof. Fix $h \in \mathrm{Aut}(X) \setminus H$. Since $p_a(X) = (x-1)^2$ and f is simple, X is embeeded by $(f, h \circ f)$ as a smooth curve of type (x, x) in the smooth quadric surface $\mathbb{P}^1 \times \mathbb{P}^1 \subset \mathbb{P}^3$. It is well-known that in this case X has gonality x and that it has exactly two g^1_x: the g^1_x induced by the projections of Q onto one of its factors (case K_S numerically even of [7], Theorem 3.8, or case $e = 0$ of [5], Remark 2 at page 350. Hence every $a \in \mathrm{Aut}(X)$ either fixes both rulings of X or it exchanges it. Every element of $\mathrm{Aut}(X)$ fixing the two rulings is an element of H. □

Remark 1. The simpleness of the covering $f : X \to C$ is always satisfied if y is a prime, but it is satisfied in many other cases (e.g. if there is at least one $P \in C$ such that $\sharp(f^{-1}(P)_{red}) = y - 1$).

Proof of Theorem 1. Fix $h \in G \backslash \{Id_G\}$. Notice that $h \circ f : X \to C$ is a degree y separable morphism. Since $g > 2xy + (x-1)^2$, we may apply Castelnuovo-

Severi inequality in the form of [6], Corollary at p. 26, with $d_1 = d_2 = y$ and $g_1 = g_2 = x$. We get that the morphism $\beta = (f, h \circ f) : X \to C \times C$ is not birational onto its image. Since f is simple, we get that the normalization of $\beta(X)$ is isomorphic to C. We get that h permutes the fibers of f, i.e. we get the existence of $h_1 \in \mathrm{Aut}(C)$ such that $h \circ f = f \circ h_1$.

Question 2. Is it possible to improve Castelnuovo-Severi inequalities on $p_a(X)$ when there are $s \geq 3$ morphisms $f_i : X \to C_i$, $1 \leq i \leq s$, pairwise not composed with an involution?

When $C_i \cong \mathbb{P}^1$ for all i, this is true by [1].

We work over an algebraically closed field $\mathbb{K}$ such that $\mathrm{char}(\mathbb{K}) = 0$.

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