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Pfaff fields and sectional genus

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Abstract: Here we give an extension of two results of M. Corrêa and M. Jardin on Pfaff fields on projective schemes.

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1. Introduction

Esteves and Kleiman introduced the general set-up of Pfaff systems and Pfaff fields with the right amount of generality (see [4]). Let Y be an equidimensional reduced projective variety defined over an algebraically closed field $\mathbb{K}$. Set $m := \dim(Y)$. Fix an integer $s \in \{1, \dots, m\}$. A rank s Pfaff field on Y is a map $\Omega_Y^s \to L$, where $L \in \mathrm{Pic}(Y)$. For any ample line HThe H-sectional genus $g(X, H)$ of H is the unique integer such that $2g(X,H) - 2 = (\omega_X + (m-1)H) \cdot H^{m-1}$.

We first consider an extension of [2], Theorem 2.

Theorem 1. *Assume the existence of a rank m Pfaff field $\eta : \Omega_Y^m \to L$ not vanishing identically on any irreducible component of Y. Let $\mathcal{O}_Y(1)$ be any ample line bundle on Y. Let $T \subset Y$ denote the sum of all codimension 1 components of the singular set of η and of the singular scheme Σ_Y if Y with the multiplicities coming from the scheme-structure described in [3], Subsection 4.1. Then $2g(Y, \mathcal{O}_Y(1)) \le L \cdot \mathcal{O}_Y(1)^{m-1} + T \cdot \mathcal{O}_Y(1)^{m-1}$ (intersection numbers).*

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If $Y \subset \mathbb{P}^n$ and η comes from a rank m Pfaff field η_1 of $\mathbb{P}^n$ (i. e. Y is a solution of $\eta_1 = 0$ not contained in a the singular locus of η_1) then η is singular on a codimension 1 subvariety of Y (see [4], Proposition 3.4, for much more). Hence $T \neq \emptyset$ in this case. Notice that if Y is smooth, then the obvious isomorphism $\Omega_Y^m \to \omega_Y$ is a rank m Pfaff field with $T = \emptyset$.

From now on in the introduction we assume char$(\mathbb{K}) = 0$. For any $R \in \text{Pic}(X)$ let $q(R,H)$ denote the infimum of all rational number u such that there is an injective map $R \hookrightarrow M$ with $M \in \text{Pic}(X)$ and M numerically equivalent to the $\mathbb{Q}$-divisor uH. Obviously $q(R,H) \in \mathbb{R}$. We use the integer $q(R,H)$ to extend [2], Theorem 1, to the case of Pfaff fields not associated to multiples of H, i.e. we prove the following result.

Theorem 2. *Let X be a smooth and connected m-dimensional projective variety whose tangent bundle Θ_X is μ-semistable with respect to at least one polarization H. Fix an integer k such that $1 \leq k \leq m$ and assume the existence of a k-Pfaff field, i.e. the existence of a non-zero map $\Omega_X^k \to R$ with $R \in \text{Pic}(X)$. Then $mR \cdot H^{m-1} \geq -k \cdot \omega_X \cdot H^{m-1}$ and $mq(R,H)H^n \geq -k \cdot \omega_X \cdot H^{m-1}$.*

Since $2g(X,H) - 2 = (\omega_X + (m-1)H) \cdot H^{m-1}$, may use the equality $\omega_X \cdot H^{m-1} = 2g(X,H) - 2 - (m-1)H^m$ to rephrase the second inequality of Theorem 2 in terms of the sectional genus $g(X,H)$.

For several examples of smooth and connected m-dimensional projective variety with stable tangent bundle, see several papers quoted in [2]. For varieties with semistable tangent bundles we may add the Abelian varieties and varieties with an Abelian variety as an étale covering.

Theorems 1 and 2 are about the sectional genus. However, if $m \geq 2$, then they should imply bounds on the intermediate cohomology $\oplus_{t\in\mathbb{Z}} H^i(Y, \mathcal{O}_Y(t))$ and $\oplus_{t\in\mathbb{Z}} H^i(Y, L \otimes \mathcal{O}_Y(t))$, $1 \leq i \leq m-1$ (see Proposition 1 for the smooth case).

2. The Proofs

Lemma 1. *Let F be a rank k torsion free sheaf on X. Fix a rank k vector bundle E on X and an ample line bundle H on X. Then there is an integer x such that for all integers $t \geq x$ there is an inclusion $F \hookrightarrow E \otimes H^{\otimes t}$. We call $k(F,E,H)$ the integer x just introduced.*

Proof. Let y be the minimal integer x such that for all integers $t \geq x$ the vector bundle $E \otimes H^{\otimes t}$ is spanned (it exists by the definition of ample line bundle). Fix any integer $t \geq y$. For a general k-dimensional vector space

$V \subset H^0(X, E \otimes H^{\otimes t})$ the evaluation map $V \otimes \mathcal{O}_X \to E \otimes H^{\otimes t}$. Thus we may take $y = k(\mathcal{O}_X^{\otimes k}, E, H)$ (here we have $=$ and not just $\geq$ by the minimality assumption of y).

Since F is torsion free, the natural map $j_F : F \to F^{\vee\vee}$ is injective and for any vector bundle G any injective map $F \to G$ factors through j_F. Thus $k(F, E, H)$ exists if and only if $k(F^{\vee\vee}, E, H)$ exists. Moreover, if any of them exists, then they are equal. Hence it is sufficient to prove the existence of the integer $k(F^{\vee\vee}, E, H)$. Let w be the minimal integer such that $F^{\vee} \otimes H^{\otimes t}$ is spanned for all $t \geq w$. Fix any integer $t \geq w$. Let W be a general k-dimensional linear subspace of $H^0(X, F^{\vee} \otimes H^{\otimes})$. Since the evaluation map $W \otimes \mathcal{O}_X \to F^{\vee} \otimes H^{\otimes t}$ is injective and these sheaves have no torsion, taking duals we get an inclusion $F^{\vee\vee} \to W \otimes \mathcal{O}_X \otimes H^{\otimes t}$. Thus $k(F^{\vee\vee}, \mathcal{O}_X^{\oplus k}, H)$ exists and it is equal to w. It is easy to check that $k(F^{\vee\vee}, E, H) \leq k(F^{\vee\vee}, \mathcal{O}_X^{\oplus k}, H) + k(\mathcal{O}_X^{\oplus k}, E, H)$, in which the finiteness of the right hand side of this inequality proves that the integer $k(F^{\vee\vee}, E, H)$ exists. □

If H is also spanned, then $k(F, E, H)$ is the minimal integer x such that there is an injective map $F \hookrightarrow E \otimes H^{\otimes x}$.

Proof of Theorem 2. Fix a real number $\epsilon > 0$. To get the second inequality it is sufficient to prove $mq(R, H)H^n \geq -k \cdot \omega_X \cdot H^{m-1} - \epsilon$. Fix a real number $\eta > 0$. By the definition of the real number $q(R, H)$ there are $M \in \mathrm{Pic}(X)$ and an injective map $R \to M$ such that M is numerically equivalent to qH with $q \in \mathbb{Q}$ and $q \leq q(R, H) + \eta$. By assumption we have $h^0(\Omega_X^k \otimes M^*) > 0$. Since Θ is H-semistable, Ω_X^k is H-semistable (see [6], [5], Corollary 3.2.10). We have $\mu(\Omega_X^k, H) = k \cdot \mu(\Omega_X^1, H) = k \cdot \omega_X \cdot H^{m-1}/m$. Hence the non-zero maps $\Omega_X^k \to R$ and $\Omega_X^k \to M$ give $mR \cdot H^{m-1} \geq -k \cdot \omega_X \cdot H^{m-1}$ and $\binom{m-1}{k-1} qH^m \geq -\omega_X \cdot H^{m-1}$. Taking $0 < \eta \ll 1$ we conclude. □

Proof of Theorem 1. For any coherent sheaf $\mathcal{F}$ on Y let $\mathcal{T}(\mathcal{F})$ denote its torsion subsheaf. There is an injective map $j_Y : \Omega_Y^m / \mathcal{T}(\Omega_Y^m) \hookrightarrow \mathcal{I}_{\Sigma_Y, Y} \omega_Y$ (see [3], Subsection 3.1). Since L is locally free, the map $\eta : \Omega_Y^m \to L$ induces a non-zero map $\eta' : \Omega_Y^m / \mathcal{T}(\Omega_Y^m) \to L$. Set $\eta'' := \eta' \circ j : \mathcal{I}_{\Sigma_Y} \omega_Y \to L$. Notice that η'' is an isomorphism on the Zariski open dense subset of Y. Take the intersection $m - 1$ times with $\mathcal{O}_Y(1)$ and use the definition of T. □

Kodaira's vanishing gives the following result.

Proposition 1. *Assume* $\mathrm{char}(K) = 0$ *and* Y *smooth. Let* x *be the minimal integer such that* $\omega_Y \otimes \mathcal{O}_Y(x)$ *is ample,* y *the minimal integer such that* $L \otimes \mathcal{O}_Y(y)$ *is ample and* z *the minimal integer such that* $\mathcal{O}_Y(z) \otimes L^*$ *is ample. Fix an integer* i *such that* $1 \leq i \leq m - 1$. *Then* $H^i(Y, \mathcal{O}_Y(t)) = 0$ *if* *either* $t \geq x+1$ *or* $t < 0$ *and* $H^i(Y, L \otimes \mathcal{O}_Y(t)) = 0$ *if either* $t \geq x+y$ *or* $t \leq -z$.

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