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SOME TRANSCENDENTAL NUMBERS VIA EXCESS CONTINUED FRACTION

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Abstract: In ICM Hyderabad (August 2010) Professor V.M. Spinadel, University of Buones Aires, Argentina introduced excess continued fraction in a short 15 minutes paper session. In this article we use his results to produce some new transcendental numbers.

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1. Introduction

In ICM Hyderabad, August 2010, Professor V.M. Spinadel, University of Buones Aires, Argentina, introduced *Excess Continued Fraction* in short communication in section "Number Theory". We developed her results and used them to produce new transcendental numbers. We shall prove the following result.

Theorem. $a - \frac{1}{10-}\frac{1}{10^{2!}-}\frac{1}{10^{3!}-}\dots$, $a \in Z$ *is a transcendental number.*

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2. Meaning of Excess Continued Fraction

Suppose θ is an irrational number. Put $b_0 = [\theta] + 1$, then $\theta = b_0 - (b_0 - \theta) = b_0 - \frac{1}{\partial_1}$ where $\partial_1 = \frac{1}{b_0 - \theta} > 1$.Put $a_1 = [\partial_1] + 1$. Clearly $a_1 \geq 2$ and $\partial_1 = a_1 - (a_1 - \partial_1) = a_1 - \frac{1}{\partial_2}$ $\partial_2 = \frac{1}{a_1 - \partial_1} > 1$, Put $a_2 = [\partial_2] + 1$. Clearly $a_2 \geq 2$ and $\partial_2 = a_2 - (a_2 - \partial_2) = a_2 - \frac{1}{\partial_3}$ where $\partial_3 = \frac{1}{a_2 - \partial_2} > 1$ and so on.

Corresponding to an irrational no. θ this algorithm yields an object of the type $b_0 - \frac{1}{a_1 - \frac{1}{a_2 - \frac{1}{a_3 - \cdots}}}$ abbreviated as $b_0 - \frac{1}{a_1 -}\frac{1}{a_2 -}\frac{1}{a_3 -}\cdots$ or $[b_0, a_1, a_2, a_3, \cdots]$ and is said to be an excess continued fraction (ECF) of θ. Here, $b_0 \in Z$, and $a's \geq 2$. Since θ is an irrational number, $\{a_1, a_2, a_3, \cdots\}$ is an infinite sequence. Also since $x = 2 - \frac{1}{2 - \frac{1}{2 - \frac{1}{2 - \cdots}}} \Rightarrow x = 2 - \frac{1}{x} \Rightarrow x^2 - 2x + 1 = 0 \Rightarrow x = 1$, it follows that $a_n \geq 3$ frequently.

If $b_0 \geq 2$, we write $a_0 = b_0$ and if $b_0 \leq 1$; we set $b_0 = m + 2$ and $a_0 = 2$, where m is a negative integer namely $b_0 - 2$. With this setting ECF of θ appears uniquely as $m + [a_0, a_1, a_2, a_3 \cdots]$ where $a_n \geq 2 \ \forall \ n \geq 0$, m is a non positive integer and $a_n \geq 3$ frequently.

3. Meaning of $[a_0, a_1, a_2, a_3, \cdots, a_n \cdots]$, where $a_n \geq 2$, $\forall n \geq 0$ and $a_n \geq 3$ Frequently

Consider the sequence $\{p_n\}$ and $\{q_n\}$ defined as under:

$$p_{-1} = +1, p_0 = a_0, p_1 = a_1.a_0 - 1, p_n = a_n p_{n-1} - p_{n-2} \qquad (n \geq 1),$$

$$q_{-1} = 0, q_0 = 1; q_1 = a_1 = a_1.q_0 - q_{-1}, q_n = a_n q_{n-1} - q_{n-2} \qquad (n \geq 1).$$

Note that $p_1 = a_1.a_0 - 1 > a_0 + (a_0 - 1) \geq a_0 = p_0, p_2 = a_2.p_1 - p_0 \geq p_1 + (p_1 - p_0) > p_1$ etc. and so on we get that $\{p_n\}$ is a strictly increasing sequence and similarly it can be seen that $\{q_n\}$ is also a strictly increasing sequence.Moreover it follows by induction that, $\frac{p_n}{q_n} = [a_0, a_1, a_2, a_3, \cdots a_n] \ \forall n \geq 1$. As well as it can be easily checked that $p_n q_{n-1} - q_n p_{n-1} = -1$. So $0 < \frac{p_{n-1}}{q_{n-1}} - \frac{p_n}{q_n} = \frac{1}{q_n q_{n-1}} < \frac{1}{q_{n-1}^2} \leq \frac{1}{n^2}$, meaning thereby $(p_n, q_n) = 1 \, \forall n \geq 1$ and $\{\frac{p_n}{q_n}\}$ is a strictly decreasing sequence. Further since

$$\frac{p_m}{q_m} - \frac{p_{m+k}}{q_{m+k}} = \sum_{j=1}^{k} \frac{p_{m+j-1}}{q_{m+j-1}} - \frac{p_{m+j}}{q_{m+j}} = \sum_{j=1}^{k} \frac{1}{q_{m+j-1}.q_{m+j}} < \sum_{j=1}^{k} \frac{1}{q_{m+j-1}^2}$$

$$< \sum_{j=1}^{k} \frac{1}{(m+j)^2} < \sum_{j=1}^{\infty} \frac{1}{(m+j)^2} \to 0 \quad \text{as } m \to \infty.$$

we infer that $\{\frac{p_n}{q_n}\}$ is a cauchy sequence. Again since $\frac{p_{n+1}}{q_{n+1}} = \frac{a_{n+1}p_n - p_{n-1}}{a_{n+1}q_n - q_{n-1}}, \theta = \frac{a'_{n+1}p_n - p_{n-1}}{a'_{n+1}q_n - q_{n-1}}$ where $a'_{n+1} = [a_{n+1}, a_{n+2}, \cdots]$ and

$$\frac{p_n}{q_n} - \theta = \frac{1}{q_n^2(a'_{n+1} - \frac{q_{n-1}}{q_n})} < \frac{1}{q_n^2(a_{n+1} - 2)}. \tag{3.1}$$

Since $a_n \geq 3$ frequently; $\exists$ an infinite sequence $\{k_i\}$ of $+ve$ integers s.t. $a_{k_i} \geq 3$ $\forall i \geq 1$ and hence $\frac{p_{k_i-1}}{q_{k_i-1}} - \theta < \frac{1}{q_{k_i-1}^2(a_{k_i}-2)} \leq \frac{1}{q_{k_i-1}^2} \to o$ as $i \to \infty$.

Since $\{\frac{p_n}{q_n}\}$ is a cauchy sequence; $\lim_{n\to\infty} \frac{p_n}{q_n} = \theta$ or

$$\lim_{n\to\infty} [a_0, a_1, a_2, a_3 \cdots, a_n] = \theta.$$

We now provide a meaning to the expression $[a_0, a_1, a_2, a_3 \cdots a_n \cdots]$ as

$$\lim_{n\to\infty} [a_0, a_1, a_2, a_3 \cdots, a_n]$$

and write $\theta = [a_0, a_1, a_2, a_3 \cdots, a_n \cdots]$. $[a_0, a_1, a_2, a_3 \cdots, a_n \cdots]$ is an ECF expansion of θ is a matter of routine check.

4. Proof of the Theorem

It is enough to work with $a = 2$. Let $\theta = 2 - \frac{1}{10-} \frac{1}{10^{2!}-} \frac{1}{10^{3!}-} \ldots$ Here $a_0 = 2, a_n = 10^{n!}$ $\forall n \geq 1$ and $q_{n+1} = a_{n+1}q_n - q_{n-1} < a_{n+1}q_n$,we get $q_n < a_n q_{n-1}, q_{n-1} < a_{n-1}q_{n-2}$,...,$q_1 < a_1 q_0 = a_1$. Multiplying all these inequalities we get,

$$q_n < a_n a_{n-1} a_{n-2} \ldots a_1 = 10^{n!+(n-1)!+\ldots+1!} < 10^{2n!} = a_n^2.$$

So,

$$a_{n+1} - 2 = a_n^{n+1} - 2 > q_n^{(n+1)/2} - 2 = q_n^{n/2}(\sqrt{q_n} - (\frac{4}{q_n^n})^{1/2}) > q_n^{n/2}.$$

Note that $q = 10$. Hence using (1) for all $n >$ any give N: $\frac{p_n}{q_n} - \theta < \frac{1}{q_n^2(a_{n+1}-2)} < \frac{1}{a_{n+1}-2} < \frac{1}{q_n^{n/2}} < \frac{1}{q_n^{N/2}}$. Thus θ is approximable to any order howsoever large. Now Liouville's Theorem [1], Theorem 191, Chapter XI, reveals that θ is not an algebraic number. It is plain that we can replace 10 by other integers and vary the construction in many other ways.

References

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