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SOME RESULTS ON $M(f_1, f_2, f_3)_{2n+1}$-MANIFOLDS

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Abstract: The aim of this paper is to study of generalized Sasakian-space-forms (briefly $M(f_1, f_2, f_3)_{2n+1}$-manifolds). The theorems established in this paper are of general charactor and provides extension of the recently given by De and Sarkar [9].

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1. Introduction

The nature of a Riemannian manifold mostly depends on the curvature tensor R of the manifold. It is well known that the sectional curvature of a manifold determine curvature tensor completely. A Riemannian manifold with constant sectional curvature c is known as real –space forms and its curvature tensor is given by

$$R(X,Y)Z = c\{g(Y,Z)X - g(X,Z)Y\}.$$

A Sasakian manifold with constant sectional curvature is a Sasakian-space-form and it has a specific form of its curvature. Similar notion also hold for Kenmotsu and cosymplectic space forms. In order to generalized space-forms in a common frame. P.Alegre, D.E.Blair and A. Carriazo introduced the notion of generalized Sasakian-space-forms in 2004, see [1]. In this connection it should mentioned that in 1989, Z. Olszak (see [7]) studied generalized complex-

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space-forms and proved that its existence. A generalized Sasakian-space-form is defined in [1]: Given an almost contact metric manifold$M(\phi,\xi,\eta,g)$, we say that M is a generalized Sasakian-space-form if there exists three functionsf_1,f_2,f_3 on M such that the curvature tensor R is given by

$$\begin{aligned}R(X,Y)Z =&f_1\{g(Y,Z)X-g(X,Z)Y\}\\&+f_2\{g(X,\phi Z)\phi Y-g(Y,\phi Z)\phi X+2g(X,\phi Y)\phi Z\}\\&+f_3\{\eta(X)\eta(Z)Y-\eta(Y)\eta(Z)X+g(X,Z)\eta(Y)\xi-g(Y,Z)\eta(X)\xi\}.\end{aligned}$$

In such case we denote the manifold as $M(f_1,f_2,f_3)$. In [1] the authors cited the several examples of such manifolds. If $f_1=\frac{c+1}{4}$, $f_2=\frac{c-1}{4}$ and $f_3=\frac{c-1}{4}$ then a generalized Sasakian-space-form with Sasakian structure becomes Sasakian-space-forms. Generalized Sasakian-space-forms and Sasakian-space-forms have been studied by many authors (see [2], [3], [8]). Symmetry of the manifold is the most important properties among its all its geometrical properties. Symmetry of the manifold basically depends on curvature tensor and the Ricci tensor of the manifold. A Riemannian manifold is called locally symmetric if its curvature tensor R is parallel, that is $\nabla R=0$, where ∇ denotes the Levi-Civita connection. As a proper generalization of locally symmetric manifold. The notion of semi-symmetric manifold was defined by

$$(R(X,Y)\cdot R)(U,V,W)=0,\quad X,W\in\chi(M),$$

and studied by many authors, see [14]. A complete intrinsic classification of these spaces was given by Z.I. Szabo, see [14]. The notion of semi-symmetric was weakened by R.Deszcz and his coauthors (see [12], [13]) and introduced the notion of pseudo symmetric and Ricci-pseudo symmetric manifolds. We define endomorphism $R(X,Y)$ and $X\wedge Y$ by

$$R(X,Y)Z=[\nabla_X,\nabla_Y]Z-\nabla_{[X,Y]}Z, \tag{1.1}$$

$$(X\wedge Y)Z=g(Y,Z)X-g(X,Z)Y, \tag{1.2}$$

respectively, where $X,Y,Z\in\chi(M)$, $\chi(M)$ being the Lie-algebra of the vector fields on M. The present paper deals with the study of$M(f_1,f_2,f_3)_{2n+1}$-manifold satisfying certain condition.

2. Generalization Sasakian-Space-Forms

A $(2n+1)$-dimensional Riemannian manifold (M,g) is called an almost contact manifold if the following results hold [2]:

$$(a)\quad \phi^2(X)=-X+\eta(X)\xi,\qquad (b)\quad \phi\xi=0, \tag{2.1}$$

$$(a) \quad \eta(\xi) = 1, \qquad (b) \quad g(X,\xi) = \eta(X), \qquad (c) \quad \eta(\phi X) = 0, \tag{2.2}$$

$$g(\phi X, \phi Y) = g(X,Y) - \eta(X)\eta(Y), \tag{2.3}$$

$$(a) \quad g(\phi X, Y) = -g(X, \phi Y), \qquad (b) \quad g(\phi X, X) = 0, \tag{2.4}$$

$$(\nabla_X \eta)(Y) = g(\nabla_X \xi, Y). \tag{2.5}$$

An almost contact metric manifold is called contact metric manifold if $d\eta(X,Y) = \Phi(X,Y) = g(X, \phi Y)$,where Φ is called the fundamental two-form of the manifold. If ξ is a killing vector field the manifold is called a k-contact manifold. It is well known that a contact metric manifold is k-contact if and only if $\nabla_X \xi = -\phi X$, for any vector field X on(M, g). An almost contact metric manifold is Sasakian if and only if

$$(\nabla_X \phi)(Y) = g(X,Y)\xi - \eta(Y)X, \quad \text{for any vector field } X, Y.$$

In 1967, D.E. Blair introduced the notion of quasi-Sasakian manifold to unify Sasakian and cosymplectic manifolds [4]. Again in 1987, Z.Olszak introduced and characterized three-dimensional quasi-Sasakian manifolds [6]. An almost contact metric manifold of dimension three is quasi-Sasakian if and only if

$$\nabla_X \xi = -\beta \phi X, \quad \text{for all } X \in TM \text{ and a function } \beta \text{ such that } \xi\beta = 0. \tag{2.6}$$

As the consequence of (2.6), we get

$$(\nabla_X \eta)(Y) = g(\nabla_X \xi, Y) = -\beta g(\phi X, Y), \tag{2.7}$$

$$(\nabla_X \eta)(\xi) = -\beta g(\phi X, \xi) = 0. \tag{2.8}$$

Clearly such a quasi-Sasakian manifold is cosymplectic if and only if $\beta = 0$.

It is known that (see [5]) for a three-dimensional quasi-Sasakian manifold the Riemannian curvature tensor satisfies

$$R(X,Y)\xi = \beta^2 \{\eta(Y)X - \eta(X)Y\} + d\beta(Y)\phi X - d\beta(X)\phi Y, \tag{2.9}$$

for a $(2n+1)$-dimensional generalized Sasakian-space-form, we have

$$\begin{aligned} R(X,Y)Z = {} & f_1 \{g(Y,Z)X - g(X,Z)Y\} \\ & + f_2 \{g(X,\phi Z)\phi Y - g(Y,\phi Z)\phi X + 2g(X,\phi Y)\phi Z\} \\ & + f_3 \{\eta(X)\eta(Z)Y - \eta(Y)\eta(Z)X + g(X,Z)\eta(Y)\xi - g(Y,Z)\eta(X)\xi\}, \end{aligned} \tag{2.10}$$

$$R(X,Y)\xi = (f_1 - f_2)\{\eta(Y)X - \eta(X)Y\}, \tag{2.11}$$

$$R(\xi, X)Y = (f_1 - f_3)\left\{g(X, Y)\xi - \eta(Y)X\right\}, \tag{2.12}$$

$$g(R(\xi, X)Y, \xi) = (f_1 - f_3)g(\phi X, \phi Y), \tag{2.13}$$

$$R(\xi, X)\xi = (f_1 - f_3)\phi^2 X, \tag{2.14}$$

$$S(X, Y) = (2nf_1 + 3f_2 - f_3)g(X, Y) - (3f_2 + (2n-1)f_3)\eta(X)\eta(Y), \tag{2.15}$$

$$Q\xi = 2n(f_1 - f_3)\xi, \tag{2.16}$$

$$S(\phi X, \phi Y) = S(X, Y) + 2n(f_3 - f_1)\eta(X)\eta(Y), \tag{2.17}$$

$$r = 2n(2n-1)f_1 + 6nf_2 - 4nf_3. \tag{2.18}$$

Here S is the Ricci tensor and r is the scalar curvature tensor of the space-forms. A generalized Sasakian space-form of dimension greater than three is said to be conformally flat if and only if Weyl-conformal curvature tensor vanishes. It is known that [8] a $(2n+1)$-dimensional $(n > 1)$ generalized Sasakian-space-form is conformally flat if and only if $f_2 = 0$.

3. $M(f_1, f_2, f_3)_{2n+1}$-Manifolds Satisfying $R(X, Y) \cdot \tilde{P} = 0$

The Pseudo projective curvature tensor $\tilde{P}$ is defined as (see [11])

$$\begin{aligned}\tilde{P}(X, Y)Z = {} & aR(X, Y)Z + b\left[S(Y, Z)X - S(X, Z)Y\right] \\ & - \frac{r}{(2n+1)}\left[\frac{a}{2n} + b\right]\left[g(Y, Z)X - g(X, Z)Y\right].\end{aligned} \tag{3.1}$$

Here a and b are constant such that $a, b \neq 0$, R is the curvature tensor, S is the Ricci tensor and r is he scalar curvature

In view of (2.2-b), (2.10) and (3.1), we get:

$$\begin{aligned}\eta(\tilde{P}(X, Y)Z) = {} & a(f_1 - f_3)\left[g(Y, Z)\eta(X) - g(X, Z)\eta(Y)\right] + b[S(Y, Z)\eta(X) \\ & - S(X, Z)\eta(Y)] - \frac{r}{(2n+1)}\left[\frac{a}{2n} + b\right]\left[g(Y, Z)X - g(X, Z)Y\right].\end{aligned} \tag{3.2}$$

Putting $Z = \xi$ in (3.2) and using (2.2 a, b), we get

$$\eta(\tilde{P}(X, Y)\xi) = 0. \tag{3.3}$$

Again putting $X = \xi$ in (3.2), we have

$$\eta(\tilde{P}(\xi, Y)Z) = \left\{\, a(f_1 - f_3) - \frac{r}{(2n+1)}\left\{\frac{a}{2n} + b\right\}\,\right\} g(Y, Z)$$

$$+\left[\frac{r}{(2n+1)}\left\{\frac{a}{2n}+b\right\}-a(f_1+f_3)-2nb(f_1-f_3)\right]\eta(Y)\eta(Z). \quad (3.4)$$

Now

$$(R(X,Y)\cdot\tilde{P})(U,V,Z)=R(X,Y)\cdot\tilde{P}(U,V)Z-\tilde{P}(R(X,Y)U,V)Z \\ -\tilde{P}(U,R(X,Y)V)Z-\tilde{P}(U,V)R(X,Y)Z.$$

Let $(R(X,Y)\cdot\tilde{P}=0$. Then we have

$$(R(X,Y)\cdot\tilde{P})(U,V,Z)-\tilde{P}(R(X,Y)U,V)Z-\tilde{P}(U,R(X,Y)V)Z \\ -\tilde{P}(U,V)R(X,Y)Z=0.$$

Therefore

$$g[(R(\xi,Y)\cdot\tilde{P})(U,V)Z,\xi]-g[\tilde{P}(R(\xi,Y)U,V)Z,\xi] \\ -g[\tilde{P}(U,R(\xi,Y)V)Z,\xi]-g[\tilde{P}(U,V)R(\xi,Y)Z,\xi]=0.$$

From this it follows that

$$(f_1-f_3)[\tilde{P}'(U,V)Z,Y)-\eta(Y)\eta(\tilde{P}(U,V)Z)-g(Y,U)\eta(\tilde{P}(\xi,V)Z) \\ +\eta(U)\eta(\tilde{P}(V,Z)Y)-g(Y,V)\eta(\tilde{P}(U,\xi,Z))+\eta(V)\eta(\tilde{P}(U,Y)Z) \\ -g(Y,Z)\eta(\tilde{P}(U,V)\xi)+\eta(Z)\eta(\tilde{P}(U,V)Y)=0]. \quad (3.5)$$

Here $\tilde{P}'(U,V,Z,Y)=g(\tilde{P}(U,V)Z,Y)$.

Let $\{e_i\}$, $i=1,2...(2n+1)$ be the orthonormal basis of the tangent space at any point. Then putting $Y=U=e_i$ in (3.5) and sum up for $1\le i\le 2n+1$, we get

$$\eta(P(,V)Z)=\left[-a(f_1-f_3)-\frac{2r}{(2n+1)}\left\{\frac{a}{2n}+b\right\}\right]g(V,Z) \\ +\left[a(f_1-f_3)+2n(f_1-f_3)+\frac{2r}{(2n+1)}\left\{\frac{a}{2n}+b\right\}\right]\eta(V)\eta(Z) \\ -bS(V,Z). \quad (3.6)$$

In view of (3.4), (3.6),we get

$$S(V,Z)=\left[\frac{a}{b}(f_1-f_3)+\frac{r}{(2n+1)}\left\{\frac{a}{4nb}+\frac{1}{2}\right\}\right]g(V,Z)$$

$$+\left[\left\{\frac{n}{b}(b+1)+\frac{a}{b}\right\}(f_3-f_1)-\frac{r}{(2n+1)}\left\{\frac{a}{4nb}+\frac{1}{2}\right\}\right]\eta(V)\eta(Z).$$

Thus, we state the following the result.

Theorem3.1. *If in a generalized Sasakian-space-form of dimension* $(2n+1), n>0$, *the relation* $R(X,Y)\cdot\tilde{P}=0$, *holds then the manifold is an* η*-Einstein manifold provided* $f_1\neq f_3$.

4. Pseudo Projrctively Flat $M(f_1, f_2, f_3)_{2n+1}$-Manifolds

In this section we assume that $\tilde{P}=0$. Then from (3.1)

$$aR'(X,Y)Z,W)+b\left[S(Y,Z)g(X,W)-S(X,Z)g(Y,W)\right] - \frac{r}{(2n+1)}\left[\frac{a}{2n}+b\right]\left[g(Y,Z)g(X,W)-g(X,Z)g(Y,W)\right]=0. \quad (4.1)$$

Putting $X=W=\xi$ in (4.1) and using (2.12) (2.15), we get

$$S(Y,Z)=\left[-\frac{a}{b}(f_1-f_3)+\frac{r}{(2n+1)}\left\{\frac{a}{2nb}+1\right\}\right]g(Y,Z) + \left[\frac{a}{b}(f_1-f_3)+2n(f_1-f_3)+f_2+\frac{r}{(2n+1)}\left\{\frac{a}{2nb}+1\right\}\right]\eta(Y)\eta(Z). \quad (4.2)$$

Therefore the manifold is η-Einstein. Hence we can state the following result.

Theorem 4.1. *A pseudo projectively flat generalized Sasakian-space-form* $M(f_1,f_2,f_3)_{2n+1}$ *is an* η*-Einstein manifold*

5. Pseudo Projectively Recurrent $M(f_1, f_2, f_3)_{2n+1}$-Manifolds

A non-flat Riemannian manifold M is said to be pseudo projectively recurrent if its pseudo projective curvature tensor $\tilde{P}$ satisfies the condition

$$\nabla\tilde{P}=A\otimes\tilde{P}. \quad (5.1)$$

Here A is an everywhere non-zero1-form.we define a function $f^2=g(\tilde{P},\tilde{P})$ on M,where the metric g is extended to the inner product between the tensor fields. Then we know that $f(Yf)=f^2A(Y)$. This implies that

$$Yf=fA(Y) \quad (f\neq 0). \quad (5.2)$$

From (5.2), we have

$$X(Yf) - Y(Xf) = \{XA(Y) - YA(X) - A([X,Y])\} f.$$

Since the left hand side of the above equation is identically zero and $f \neq 0$ on M. Thus

$$dA(X,Y) = 0, \tag{5.3}$$

that is 1-form is A closed.

Now from $(\nabla_X \tilde{P})(U,V)Z = A(X)\tilde{P}(U,V)Z$, we have

$$(\nabla_U \nabla_V \tilde{P})(X,Y)Z = \{UA(X) + A(U)A(V)\}\tilde{P}(X,Y)Z.$$

Hence from (5.3), we have

$$(R(X,Y) \cdot \tilde{P})(U,V)Z = [2dA(X,Y)]\tilde{P}(U,V)Z = 0, \quad \forall X, Y, \tag{5.4}$$

Thus we can state the following result.

Theorem 5.1. *A pseudo projectively recurrent $M(f_1, f_2, f_3)_{2n+1}-$ manifold is an η-Einstein manifold.*

Corollary. *In a pseudo projectively recurrent $M(f_1, f_2, f_3)_{2n+1}-$ manifold the 1-form A is closed.*

6. Partially Ricci-Pseudo Symmetric $M(f_1, f_2, f_3)_{2n+1}$-Manifolds

A $M(f_1, f_2, f_3)_{2n+1}-$ manifold is said to be partially Ricci-pseudo symmetric if it satisfies

$$(R(X,Y) \cdot S)(U,V) = L_c \left[((X \wedge Y)S(U,V)\right], \tag{6.1}$$

where $L_c \in C^\infty(M)$, $(R(X,Y(S)(U,V) = -S(R(X,Y)U,V) - S(U,R(X,Y)V$, and $((X \wedge Y) \cdot S(U,V) = -S((X \wedge Y)U,V) - S(U,(X \wedge Y)V)$.

From (6.1) has the following form

$$\begin{aligned} S(R(X,Y)U,V) &+ S(U,R(X,Y)V \\ &= L_c \left[S((X \wedge Y)U,V) + S(U,(X \wedge Y)V\right]. \end{aligned} \tag{6.2}$$

We studies partially Ricci-pseudo symmetric manifolds under the restriction that $Y = V = \xi$.

Thus we have

$$S(R(X,\xi)U,\xi) + S(U,R(X,\xi)\xi) = L_c \left[S((X \wedge \xi)U,\xi) + S(U,(X \wedge \xi)\xi\right]. \tag{6.3}$$

Using (2.2 a, b), (1.3), (2.10), (2.11), (2.15) in (6.3), we get

$$[(f_1 - f_3) - L_c]\,[S(X,U) - (f_1 - f_3)g(X,U)] = 0. \tag{6.4}$$

These implies that either (a) $L_c = f_1 - f_3$ or (b) $[S(X,U) = (f_1 - f_3)g(X,U)]$ However (b) means that the manifold is an Einstein manifold.

We can state the following result.

Theorem 6.1. *A partially pseudo-Ricci symmetric $M(f_1, f_2, f_3)_{2n+1}$-manifold with never vanishing function $[L_c = f_1 - f_3]$ is an Einstein manifold.*

Corollary 1. *The necessary and sufficient condition for $M(f_1, f_2, f_3)_{2n+1}$-manifold is to be partially pseudo-Riccisymmetric is that $f_1 \neq f_3$.*

Corolary 2. *A partially pseudo-Ricci symmetric $M(f_1, f_2, f_3)_{2n+1}$-manifold is to be Ricci symmetric if $f_1 = f_3$.*

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