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THE MEDIUM DOMINATION NUMBER OF A GRAPH

Duygu Vargör[1 §], Pınar Dündar[2]
[1,2]Department of Mathematics
Faculty of Science
Ege University
Bornova, 35100, Izmir, TURKEY

Abstract: In a communication network the resistance of network is the response to any disruption in some of stations or lines. Vulnerability values measures the resistance of network in disruption of some vertices until communication breakdown. A network can be modeled by a graph whose vertices represent the stations and whose edges represent the relation between the vertices. In graph theory, some stability measures have been studied widely such as connectivity, edge-connectivity, integrity, tenacity, vertex covering and domination. These parameters take consideration into the neighborhood of edges and vertices. In a graph each vertex is capable of protecting every vertex in its neighborhood and in domination every vertex is required to be protected. In this paper, for any connected, undirected, loopless graph we define the medium domination number of a graph and study on some graph classes. The medium domination number is a notion which uses neighborhood of each pair of vertices. The main idea of this parameter is that each $u, v \in V$ must be protected. So it is needed to examine how many vertices are capable of dominating both of u and v. Also the total number of vertices that dominate every pair of vertices and average value of this is defined as "the medium domination number" of a graph. We establish some new results and relation with the other vulnerability measures and give an algorithm with the complexity of $O(n^2)$.

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§Correspondence author

1. Introduction

In a communication network, if there is a disruption on some vertices and lines, it lost its effectiveness. Generally, a network can be modeled by a graph. A more stable model is preferred in network design. Vulnerability value of a communication network is the resistance of network to disruption of some vertices until communication of network breakdown. A graph G is denoted by $G = (V, E)$ where V and E are vertices and edges sets of G respectively. The length of a shortest $u - v$ path in a connected G is called the distance from a vertex u to a vertex v. $d(u, v)$ denotes *the distance* between v and u. Two $u - v$ paths are *internally disjoint* if they have no vertices in common, other than u and v. *The degree of a vertex* v in a graph G is the number of edges of incident with v and denoted by $deg(v)$. *The minimum degree* among the vertices of a graph G is denoted by $\delta(G)$. *The maximum* degree among the vertices of a graph G is denoted by $\triangle(G)$. *The eccentricity* $e(v)$ of a vertex v in a connected graph G is the distance between v and a vertex farthest from v in G. The minimum and the maximum eccentricities among the vertices of G are called the radius $r(G)$ and the diameter $diam(G)$ of G [2].

The connectivity is defined to be the minimum number of vertices in a set whose deletion results in a disconnected graph and denoted by $k(G)$. The *edge connectivity* is defined to be the minimum number of edges in a set whose deletion results in a disconnected graph and denoted by $k'(G)$.

Menger's classical theorem tells us that in a $k-$ *connected* graph, every pair of vertices are joined by k internally disjoint paths. Two vertices u and v in a graph G are said to be $k-$ *connected* if there are k or more pairwise internally disjoint paths between them. The u and v *connectivity* of G, denoted $k(u, v)$, is defined to be the maximum value of k and also $k(u, v)$ is called local connectivity of u and v. The sum of every pair vertices $k(u, v)$ is defined by *total connectivity* of G [1].

If the order of G is n, then *the average connectivity* of G, denoted by $\overline{k}(G)$ is defined to be

$$\overline{k}(G) = \frac{\sum\limits_{\forall u,v \in V(G)} k(u, v)}{\binom{n}{2}}$$

The number of $u - v$ internally disjoint paths of length i in G is denoted by $k_i(u, v)$ [1].

A set $S \subseteq V(G)$ is a *dominating set* if every vertex not in S is adjacent to a vertex in S. The *dominating number* of G, denoted by $\gamma(G)$, is the minimum cardinality of a dominating set in G. There are the many variations of domination number such as connected, independent and total domination numbers [3, 4, 5, 6]. The connected domination number $\gamma_C(G)$ is the minimum cardinality of a connected dominating set. The independent domination number $\gamma_C(G)$ is the minimum cardinality of an independent domination set. $S \subseteq V(G)$ is a *dominating set* if $N(s) = V(G)$ for $\forall s \in S$. The total domination number $\gamma_t(G)$ is the minimum cardinality taken over minimal (total) domination sets of G.

2. The Medium Domination Number

In this section, we define the medium domination number of a graph. For any connected, undirected and loopless graph, the medium domination number is a notion which uses neighborhood of each pairs of vertices. In this paper, we think that each $u, v \in V$ must be protected and it is needed to examine how many vertices are capable of dominating both of u and v. We calculate the total number of vertices that dominate all pairs of vertices and evaluate the average of this value and call it "the medium domination number" of graph.

We examine relations between the other vulnerability measures and medium domination number of a graph. We give some theorems and results on it. We denote the number of vertices that dominate both of u and v by $dom(u, v)$. We define the medium domination number as a new vulnerability measure to check the stability of a connected, undirected, loopless graph G that has n vertices, q edge.

Definition 1. For $G = (V, E)$ and $\forall\, u, v \in V$; if u and v are adjacent they dominate each other, then at least $dom(u, v) = 1$.

Definition 2. For $G = (V, E)$ and $\forall\, u, v \in V$; the total number of vertices that dominate every pair of vertices is defined as

$$TDV(G) = \sum_{\forall u, v \in V(G)} dom(u, v).$$

Definition 3. For any connected, undirected, loopless graph G of order n, the medium domination number of G is defined as $MD(G) = \dfrac{TDV(G)}{\binom{n}{2}}$.

G_1 and G_2 that have the same number of vertices and connectivity are shown in Figure 2.1.

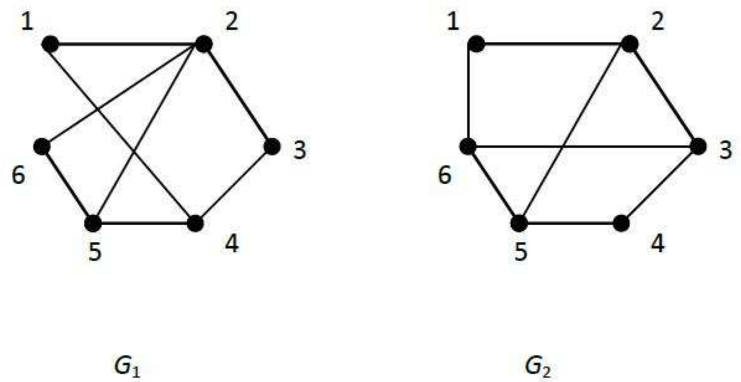

Figure 1

For graph G_1; $dom(1,2) = 1$, $dom(1,3) = 2$, $dom(1,4) = 1$, $dom(1,5) = 2$, $dom(1,6) = 1$, $dom(2,3) = 1$, $dom(2,4) = 3$, $dom(2,5) = 2$, $dom(2,6) = 2$, $dom(3,4) = 1$, $dom(3,5) = 2$, $dom(3,6) = 1$, $dom(4,5) = 1$, $dom(4,6) = 1$, $dom(5,6) = 2$.

For graph G_2; $dom(1,2) = 1$, $dom(1,3) = 2$, $dom(1,4) = 0$, $dom(1,5) = 2$, $dom(1,6) = 1$, $dom(2,3) = 1$, $dom(2,4) = 2$, $dom(2,5) = 1$, $dom(2,6) = 3$, $dom(3,4) = 1$, $dom(3,5) = 3$, $dom(3,6) = 1$, $dom(4,5) = 1$, $dom(4,6) = 2$, $dom(5,6) = 1$.

$TDV(G_1) = 23$ thus $MD(G_1) = \dfrac{23}{15}$. $TDV(G_2) = 22$ thus $MD(G_1) = \dfrac{22}{15}$. Finally, the graph G_1 prefer the graph G_2 because $MD(G_1) > MD(G_2)$.

For $G = (V, E)$ and $\forall\ u, v \in V$;

Theorem 4. *$dom(u, v) = k_1(u, v) + k_2(u, v)$, where k_1 and k_2 denotes the number of $u - v$ internally disjoint paths of length 1 and length 2 respectively.*

Proof. There are two different cases about the number of vertices that dominate both of u and v:

Case 1. If u and v are adjacent, u is capable of protecting v and v is also capable of protecting u since they are both in their neighborhood. Thus, all adjacent pairs dominate each other. From definition 2.1 it is know that the number of $u - v$ internally disjoint paths of length 1 is $k_1(u, v) = 1$, so $dom(u, v) = 1$. If u and v aren't adjacent, then there is no $u - v$ internally disjoint paths of length 1 and this is denoted by $k_1(u, v) = 0$.

Case 2. There may be more than one vertex which are both adjacent to u

and v. If we label these vertices by $w_0,\ w_1,\ w_2...$; the vertices that dominate both of u and v are $w_0,\ w_1,\ w_2...$ In this case internally disjoint paths between u and v, should be taken as $w_0,\ w_1,\ w_2...$ From case 1 and 2, the number of the vertices that both dominate u and v is equal to sum of the number of u-v internally disjoint paths of length one and two. Then, we obtain $dom(u,v) = k_1(u,v) + k_2(u,v)$ □

Result 2.1. $dom(u,v) \leq k(u,v)$.

Result 2.2. $dom(u,v) \leq \min\{deg(u), deg(v)\}$.

Result 2.3. $TDV(G) \leq K(G)$.

Proof. For $G = (V,E)$ the total number of vertices that dominate every pair of vertices (TDV) is the sum of internally disjoint paths of length one and two. TDV is the sum of internally disjoint paths. Thus, for all pairs we obtain $TDV(G) \leq K(G)$. □

Theorem 5. *For G has n vertices, q edges and for $deg(v_i) \geq 2$;*

$$TDV(G) = q + \left\{ \sum_{v_i \in V} \binom{deg\ v_i}{2} \right\}.$$

Proof. For G, the total number of vertices that dominate every pair of vertices (TDV) is the sum of internally disjoint paths of length one and two. The number of internally disjoint paths of length one is equal to the number of edges. Since every vertex v_i of G is the centre vertex of length of two, the number of internally disjoint paths of length two is equal to $\binom{deg\ v_i}{2}$. Thus for all pairs, we obtain $TDV(G) = q + \left\{ \sum_{v_i \in V} \binom{deg\ v_i}{2} \right\}$. □

Result 2.4. For any tree T with n vertices ;

$$TDV(T) = n - 1 + \left\{ \sum_{v_i \in V} \binom{deg\ v_i}{2} \right\}.$$

Theorem 6. $\dfrac{4n-6}{n(n-1)} \leq MD(G) \leq n-1.$

Proof. Since the total number of vertices that dominate every pair of vertices (TDV) consists of all adjacent pairs, the minimum value of $TDV(G)$ must be the number of edges that constructs a path graph and the maximum value of $TDV(G)$ is $n-1$ that is a complete graph. If all pairs are examined, we obtain $\frac{4n-6}{n(n-1)} \leq MD(G) \leq n-1$. □

Theorem 7. *For T is tree that has n vertices;* $\frac{4n-6}{n(n-1)} \leq MD(G) \leq 1$.

Proof. The minimum value of $MD(G)$ is proved before. For a tree, since there is no cycle, there is only one disjoint path between each vertex. Thus, the number of internally disjoint paths between all pairs is $\binom{n}{2}$. From the definition of $MD(G)$, we obtain $\frac{4n-6}{n(n-1)} \leq MD(G) \leq 1$. □

Result 2.5. The results on the relations between $MD(G)$ and the degrees of vertices are given.

1. $MD(G) \geq \frac{\sum_{i=1}^{n} deg\ (v_i)}{n(n-1)}$

2. $MD(G) \geq \frac{\overline{d}(G)}{n-1}$

3. $MD(G) \geq \frac{\delta(G)}{n-1}$

4. $(n-1)MD(G) \geq \delta(G) \geq k'(G) \geq k(G)$.

Theorem 8. $MD(G) \leq \triangle(G)$.

Proof. For all vertices $dom(u,v) \leq \min\{deg(u), deg(v)\} \leq \triangle(G)$ can be taken. When all pairs are considered, $MD(G) \leq \triangle(G)$. □

3. Results on the Medium Domination Number of Basic Graph Classes

In this section, we find the medium domination number of special graph classes by using the theorems that proved in second section. We give the theorems about special graph classes.

Theorem 9. *Let P_n be the path graph with n vertices; then we have*

$$MD(P_n) = \frac{2n-3}{\binom{n}{2}}.$$

Theorem 10. *Let K_n be the complete graph with n vertices; then we have*

$$MD(G) = n - 1.$$

Theorem 11. *Let C_n be the cycle graph with n vertices; then we have*

$$MD(G) = \frac{2n}{\binom{n}{2}}.$$

Theorem 12. *Let $K_{1,n}$ be the star graph with n vertices; then we have*

$$MD(G) = 1.$$

Theorem 13. *Let $K_{m,n}$ be the complete bipartite graph with n vertices; then we have*

$$MD(G) = \frac{\binom{m}{2}n + \binom{n}{2}m + mn}{\binom{m+n}{2}}.$$

Theorem 14. *Let $W_{1,n}$ be the wheel graph with n vertices; then we have*

$$MD(G) = \frac{\frac{n^2+9n}{2}}{\binom{n+1}{2}}.$$

Theorem 15. *Let G be the r-regular graph with n vertices; then we have*

$$MD(G) = \frac{r^2}{n-1}.$$

4. An Algorithm for Computing Medium Domination Number of a Graph

In this section, we present an algorithm computing the medium domination number of a graph. The complexity of the given algorithm is $O(n^2)$.

N: The number of vertices of graph
A[I, J]: The adjacency matrix of graph
DEG[I]: The degree of vertices
SUM1: The sum of the degree of vertices
EDGES: The number of edges of graph
TDV: The total number of vertices that dominate every pair of vertices
MD: The Medium Domination number

```
BEGIN
WRITE ”ENTER THE NUMBER OF VERTICES OF THE GRAPH”;
READLN(N);
FOR I:=1 TO N DO
   BEGIN
         FOR J:=1 TO N DO
         A[I, J]:=0; DEG[I]:=0;
END;
FOR I:=1 TO N DO
         BEGIN
         READLN(A[I, J]);
         DEG[I]:= DEG[I] + A[I, J];
         END;
         SUM1:= SUM1+ DEG[I];
         IF DEG[I]>1 THEN SUM2:= (SUM2+ (DEG[I]*( DEG[I]-1)))/2;
END;
EDGES:=SUM1/2;
TBS:= EDGES+SUM2;
MD:= TDV/((N*(N-1)/2);
WRITE ”MEDIUM DOMINATION NUMBER=”, MD;
END.
```

5. Conclusion

In graph theory, there are many stability parameters such as the connectivity number, the edge- connectivity number, the independence number, the vertex domination number and the domination number. The neighborhood of edges and vertices is important in these parameters. The domination number of a graph is a new vulnerability measure that considers the neighborhood of vertices. From the definition of domination, every vertex of a graph must be protected by its neighborhood .In this search, the main ideas is each $u, v \in V$ must be protected and to calculate the number of vertices that are capable of dominating both of u and v. The total number of vertices that dominate every pair of vertices is examined and the average of this value is calculated and called "the medium domination number" of a graph. Some theorems and results on the medium domination number of a graph and basic graph classes are given. In forth section, an algorithm computing the medium domination number of a graph is given. The complexity of the given algorithm is $O(n^2)$.

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