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ON APPROXIMATION OF CONJUGATE OF A FUNCTION BELONGING TO WEIGHTED $W\left(L_r, \xi\left(t\right)\right)$ CLASS BY PRODUCT MEANS

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Abstract: A good amount of work has been done on degree of approximation of functions belonging to $Lip\alpha, Lip\left(\alpha, r\right), Lip\left(\xi\left(t\right), r\right)$ and $W\left(L_r, \xi\left(t\right)\right)$ classes using Cesàro, Nörlund and generalised Nörlund single summability methods by a number of researchers (see [1], [2], [3], [4], [6], [7], [8], [9], [10]). But till now, nothing seems to have been done so far to obtain the degree of approximation of conjugate of a function using $(N, p_n)(C, 1)$ product summability method of its conjugate Fourier series. Therefore, the purpose of present paper is to establish a quite new theorem on degree of approximation of a function $\tilde{f}$, conjugate to a $2\pi-$periodic function f belonging to weighted, i.e. $W\left(L_r, \xi\left(t\right)\right)$ class, $r \geq 1$, by $(N, p_n)(C, 1)$ product summability means of its conjugate Fourier series.

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1. Introduction

Let f be a $2\pi-$ periodic function and Lebesgue integrable. The Fourier series associated with f at a point x is defined by

$$f\left(x\right) \sim \frac{1}{2}a_0 + \sum_{n=1}^{\infty}\left(a_n \cos nx + b_n \sin nx\right) \equiv \sum_{n=1}^{\infty} A_n\left(x\right) \tag{1.1}$$

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with n-th partial sum $s_n(f;x)$. The conjugate series of Fourier series (1.1) of f is given by

$$\sum_{n=1}^{\infty}(a_n \sin nx - b_n \cos nx) \equiv \sum_{n=1}^{\infty} B_n(x) \tag{1.2}$$

with n-th partial sum $\tilde{s}_n(f;x)$.

Throughout this paper, we shall call (1.2) as conjugate Fourier series of function f.

L_r- norm is defined by

$$\|f\|_r = \left(\int_0^{2\pi} |f(x)|^r dx\right)^{\frac{1}{r}}, \ r \geq 1 \tag{1.3}$$

$L_\infty-$ norm of a function $f : R \to R$ is defined by

$$\|f\|_\infty = \sup\{\,|f(x)\,| : x \in R\} \tag{1.4}$$

The degree of approximation of a function $f : R \to R$ by a trigonometric polynomial t_n of order n under sup norm $\|\cdot\|_\infty$ is defined by

$$\|t_n - f\|_\infty = \sup\{\,|t_n - f(x)| : x \in R\} \qquad \text{(see Zygmund [11])}$$

and $E_n(f)$ of a function $f \in L_r$ is given by

$$E_n(f) = \min_{t_n} \|t_n - f\|_r \tag{1.5}$$

This method of approximation is called trigonometric fourier approximation (TFA).

A function $f \in \mathrm{Lip}\,\alpha$ if

$$|f(x+t) - f(x)| = O(\,|\,t\,|^\alpha) \quad \text{for } 0 < \alpha < 1, \tag{1.6}$$

$f(x) \in \mathrm{Lip}\,(\alpha, r)$ for $0 \leq x \leq 2\pi$, if

$$\left(\int_0^{2\pi} |f(x+t) - f(x)|^r dx\right)^{\frac{1}{r}} = O|\,t\,|^\alpha, \quad 0 < \alpha \leq 1, \ r \geq 1 \tag{1.7}$$

(Definition 5.38 of Mc Fadden [5]).

Given a positive increasing function $\xi(t)$ and an integer $r \geq 1$, $f \in \mathrm{Lip}\,(\xi(t), r)$, if

$$\left(\int_0^{2\pi} |f(x+t) - f(x)|^r dx\right)^{\frac{1}{r}} = O(\xi(t)) \tag{1.8}$$

and that $f \in W(L_r, \xi(t))$ if

$$\left(\int_0^{2\pi} \left|\{f(x+t) - f(x)\} \sin^\beta x\right|^r dx\right)^{\frac{1}{r}} = O(\xi(t)), \ \beta \geq 0 \tag{1.9}$$

In case $\beta = 0$, we find that $W(L_r, \xi(t))$ class reduces to the Lip $(\xi(t), r)$ class and if $\xi(t) = t^\alpha$ then Lip $(\xi(t), r)$ class reduces to the Lip (α, r) class and if $r \to \infty$ then Lip (α, r) class reduces to the $\mathrm{Lip}\,\alpha$ class. We observe that

$$\mathrm{Lip}\,\alpha \subseteq \mathrm{Lip}\ (\alpha, r) \subseteq \mathrm{Lip}\ (\xi(t), r) \subseteq W(L_r, \xi(t)) \quad \text{for} \quad 0 < \alpha \leq 1,\ r \geq 1.$$

Let $\sum\limits_{n=0}^{\infty} u_n$ be a given infinite series with the sequence of its n-th partial sums $\{s_n\}$. The $(C,1)$ transform is defined as the n-th partial sum of $(C,1)$ summability and is given by

$$t_n = \frac{s_0 + s_1 + s_2 + ... + s_n}{n+1} = \frac{1}{n+1} \sum_{k=0}^{n} s_k \to s \quad \text{as } n \to \infty, \tag{1.10}$$

then the infinite series $\sum\limits_{n=0}^{\infty} u_n$ is summable to the definite number s by $(C,1)$ method. Let $\{p_n\}$ be a non-negative, non increasing sequence such that

$$P_n = p_0 + p_1 + ... + p_n \to \infty \quad \text{as} \quad n \to \infty, \ \ P_{-1} = p_{-1} = 0.$$

The product of (N, p_n) summability and $(C,1)$ summability defines $(N, p_n)(C,1)$ summability and we denote it by $N_n^p C_n^1$. Thus if

$$N_n^p C_n^1 = \frac{1}{P_n} \sum_{k=0}^{n} p_k C_k^1 \to s \quad \text{as} \quad n \to \infty \tag{1.11}$$

where N_n^p denotes the (N, p_n) transform of s_n and C_n^1 denotes the $(C,1)$ transform of s_n, then the series $\sum\limits_{n=0}^{\infty} u_n$ is said to be summable by $(N, p_n)(C,1)$ means or summable $(N,p_n)(C,1)$ to a definite number s. The (N,p_n) is a regular method of summability:

$$s_n \to s \Rightarrow C_n^1(s_n) = t_n = \frac{1}{n+1}\sum_{k=0}^{n} s_k \to s, \text{ as } n\to\infty \quad C_n^1 \text{ method is regular}$$
$$\Rightarrow N_n^p\left(C_n^1(s_n)\right) = N_n^p C_n^1 \to s, \text{ as } n\to\infty \quad N_n^p \text{ method is regular}$$
$$\Rightarrow N_n^p C_n^1 \text{ method is regular.}$$

We use the following notations:

$$\psi(t) = f(x+t) - f(x-t),$$

$$\tilde{M}_n(t) = \frac{1}{2\pi P_n}\sum_{k=0}^{n}\left\{p_k\left(\frac{1}{1+k}\right)\sum_{\nu=0}^{k}\frac{\cos\left(\nu+\frac{1}{2}\right)t}{\sin\frac{t}{2}}\right\}.$$

2. Main Theorem

We prove the following theorem

Theorem 2.1. *Let (N,p_n) be a regular Nörlund method defined by a positive, monotonic, non-increasing sequence$\{p_n\}$. Let f be a 2π-periodic function, Lebesgue integrable on $[0,2\pi]$ and is belonging to $W(L_r,\xi(t))$ class, $r\geq 1$, then the degree of approximation of f by $N_n^p C_n^1$ means of its conjugate Fourier series (1.2) is given by*

$$\left\|N_n^p C_n^1 - \tilde{f}\right\|_r = O\left[(n+1)^{\beta+\frac{1}{r}}\xi\left(\frac{1}{n+1}\right)\right], \tag{2.1}$$

provided $\xi(t)$ satisfies the following conditions:

$$\left\{\frac{\xi(t)}{t}\right\} \text{ be a decreasing sequence,} \tag{2.2}$$

$$\left\{\int_0^{\frac{1}{n+1}}\left(\frac{t\,|\psi(t)|}{\xi(t)}\right)^r \sin^{\beta r} t\,dt\right\}^{\frac{1}{r}} = O\left\{\frac{1}{(n+1)}\right\}, \tag{2.3}$$

and

$$\left\{ \int_{\frac{1}{n+1}}^{\pi} \left(\frac{t^{-\delta} \left|\psi\left(t\right)\right|}{\xi\left(t\right)} \right)^{r} dt \right\}^{\frac{1}{r}} = O\left\{ \left(n+1\right)^{\delta} \right\}, \tag{2.4}$$

where δ *is an arbitrary number such that* $s\left(1-\delta\right)-1>0,\ \ \frac{1}{r}+\frac{1}{s}=1,\ \ 1\leq r\leq\infty$, *conditions (2.3)and(2.4)hold uniformly in x and*

$$\tilde{f}\left(x\right) = -\frac{1}{2\pi}\int_{0}^{2\pi} \psi\left(t\right)\cot\left(\frac{t}{2}\right) dt.$$

3. Lemmas

For the proof of our theorem, we require following lemmas.

Lemmas 3.1. $\left|\tilde{M}_n\left(t\right)\right| = O\left(\frac{1}{t}\right)$ for $0\leq t\leq\frac{1}{n+1}$.

Proof. For $0\leq t\leq\frac{1}{n+1}$, $\sin\left(\frac{t}{2}\right)\geq\frac{t}{\pi}$ and $\left|\cos nt\right|\leq 1$. Then

$$\begin{aligned}
\left|\tilde{M}_n\left(t\right)\right| &= \frac{1}{2\pi\, P_n}\left|\sum_{k=0}^{n}\left[p_k\left(\frac{1}{1+k}\right)\sum_{\nu=0}^{k}\frac{\cos\left(\nu+\frac{1}{2}\right)t}{\sin\frac{t}{2}}\right]\right| \\
&\leq \frac{1}{2\pi\, P_n}\sum_{k=0}^{n}\left[p_k\left(\frac{1}{1+k}\right)\sum_{\nu=0}^{k}\frac{\left|\cos\left(\nu+\frac{1}{2}\right)t\right|}{\left|\sin\frac{t}{2}\right|}\right] \\
&= \frac{1}{2t\, P_n}\sum_{k=0}^{n}\left[p_k\left(\frac{1}{k+1}\right)\sum_{\nu=0}^{k}\left(1\right)\right] \\
&= \frac{1}{2t\, P_n}\sum_{k=0}^{n}p_k \\
&= O\left[\frac{1}{t}\right].
\end{aligned}$$

Lemmas 3.2. $\left|\tilde{M}_n\left(t\right)\right| = O\left(\frac{1}{t}\right)$ *for* $\frac{1}{n+1}\leq t\leq\pi$.

Proof. For $\frac{1}{n+1}\leq t\leq\pi$, by applying Jordan's lemma, $\sin\frac{t}{2}\geq\frac{t}{\pi}$

$$\left|\tilde{M}_n\left(t\right)\right| = \frac{1}{2\pi P_n}\left|\sum_{k=0}^{n}\left[p_k\left(\frac{1}{1+k}\right)\sum_{\nu=0}^{k}\frac{\cos\left(\nu+\frac{1}{2}\right)t}{\sin\frac{t}{2}}\right]\right|$$

$$\leq \frac{1}{2tP_n}\left|\sum_{k=0}^{n}\left[p_k\left(\frac{1}{1+k}\right)\mathrm{Re}\left\{\sum_{\nu=0}^{k}e^{i\left(\nu+\frac{1}{2}\right)t}\right\}\right]\right|$$

$$\leq \frac{1}{2tP_n}\left|\sum_{k=0}^{n}\left[p_k\left(\frac{1}{1+k}\right)\mathrm{Re}\left\{\sum_{\nu=0}^{k}e^{ivt}\right\}\right]\right|\left|e^{\frac{it}{2}}\right|$$

$$\leq \frac{1}{2tP_n}\left|\sum_{k=0}^{n}\left[p_k\left(\frac{1}{1+k}\right)\mathrm{Re}\left\{\sum_{\nu=0}^{k}e^{ivt}\right\}\right]\right|$$

$$= \frac{1}{2tP_n}\left|\sum_{k=0}^{\tau-1}\left[p_k\left(\frac{1}{1+k}\right)\mathrm{Re}\left\{\sum_{\nu=0}^{k}e^{ivt}\right\}\right]\right|$$

$$+ \frac{1}{2tP_n}\left|\sum_{k=\tau}^{n}\left[p_k\left(\frac{1}{1+k}\right)\mathrm{Re}\left\{\sum_{\nu=0}^{k}e^{ivt}\right\}\right]\right|. \tag{3.1}$$

Now considering first term of (3.1)

$$\frac{1}{2t\,P_n}\left|\sum_{k=0}^{\tau-1}\left[p_k\left(\frac{1}{1+k}\right)\mathrm{Re}\left\{\sum_{\nu=0}^{k}e^{ivt}\right\}\right]\right|$$

$$\leq \frac{1}{2t\,P_n}\left|\sum_{k=0}^{\tau-1}\left[p_k\left(\frac{1}{1+k}\right)\left\{\sum_{\nu=0}^{k}1\right\}\right]\right|\left|e^{ivt}\right|$$

$$\leq \frac{1}{2t\,P_n}\left|\sum_{k=0}^{\tau-1}(p_k)\right| \tag{3.2}$$

Now considering second term of (3.1) and using Abel's lemma,

$$\frac{1}{2t\,P_n}\left|\sum_{k=\tau}^{n}\left[p_k\left(\frac{1}{1+k}\right)\mathrm{Re}\left\{\sum_{\nu=0}^{k}e^{ivt}\right\}\right]\right|$$

$$\leq \frac{1}{2t\,P_n}\sum_{k=\tau}^{n}p_k\left(\frac{1}{1+k}\right)\max_{0\leq m\leq k}\left|\sum_{\nu=0}^{m}e^{ivt}\right|$$

$$\leq \frac{1}{2t\,P_n}\sum_{k=\tau}^{n}p_k\left(\frac{1}{1+k}\right)(1+k)$$

$$= \frac{1}{2t\,P_n}\sum_{k=\tau}^{n}p_k \tag{3.3}$$

Combining (3.1),(3.2) and (3.3),

$$\left|\tilde{M}_n(t)\right| = \frac{1}{2t\,P_n}\sum_{k=\tau}^{\tau-1} p_k + \frac{1}{2t\,P_n}\sum_{k=0}^{\tau} p_k = O\left(\frac{1}{t}\right)$$

4. Proof of Theorem 2.1

It is well known that

$$\tilde{s}_n(f;x) - \tilde{f}(x) = \frac{1}{2\pi}\int_0^{\pi} \psi(t)\,\frac{\cos\left(n+\frac{1}{2}\right)t}{\sin\frac{t}{2}}\,dt$$

Using (1.9), the $(C,1)$ transform C_n^1 of $\tilde{s}_n(f;x)$ is given by

$$C_n^1 - \tilde{f}(x) = \frac{1}{2\pi(n+1)}\int_0^{\pi}\psi(t)\sum_{k=0}^{n}\frac{\cos\left(k+\frac{1}{2}\right)t}{\sin\frac{t}{2}}dt$$

Now denoting $(N,p_n)(C,1)$ transform of $\tilde{s}_n(f;x)$ by $N_n^pC_n^1$, we write

$$\begin{aligned}
N_n^pC_n^1 - \tilde{f}(x) &= \frac{1}{2\pi\,P_n}\sum_{k=0}^{n}\left[p_k\left(\frac{1}{k+1}\right)\int_0^{\pi}\frac{\psi(t)}{\sin\frac{t}{2}}\left\{\sum_{\nu=0}^{k}\cos\left(k+\frac{1}{2}\right)t\right\}dt\right] \\
&= \int_0^{\pi}\psi(t)\;\tilde{M}_n(t)\;dt \\
&= \left[\int_0^{\frac{1}{n+1}} + \int_{\frac{1}{n+1}}^{\pi}\right]\psi(t)\;\tilde{M}_n(t)\;dt \\
&= I_{1.1} + I_{1.2} \qquad \text{(say)}. \qquad (4.1)
\end{aligned}$$

We have

$$|\psi(x+t) - \psi(x)| \le |f(u+x+t) - f(u+x)| + |f(u-x-t) - f(u-x)|.$$

Hence, by Minkowiski's inequality,

$$\left[\int_0^{2\pi}\left|\{\psi(x+t)-\psi(x)\}\sin^{\beta}x\right|^r dx\right]^{\frac{1}{r}}$$

$$\leq\left[\int_0^{2\pi}\left|\{f(u+x+t)-f(u+x)\}\sin^{\beta}x\right|^r dx\right]^{\frac{1}{r}}$$

$$+\left[\int_0^{2\pi}\left|\{f(u-x-t)-f(u-x)\}\sin^{\beta}x\right|^r dx\right]^{\frac{1}{r}}=O\{\xi(t)\}.$$

Then $f\in W(L_r,\xi(t))\Rightarrow\psi\in W(L_r,\xi(t))$.

$$\text{We consider } |I_{1.1}|\leq\int_0^{\frac{1}{n+1}}|\psi(t)|\left|\tilde{M}_n(t)\right|dt$$

Using Hölder's inequality and the fact that $\psi(t)\in W(L_r,\xi(t))$,

$$|I_{1.1}|\leq\left[\int_0^{\frac{1}{n+1}}\left\{\frac{t\,|\psi(t)|\sin^{\beta}t}{\xi(t)}\right\}^r dt\right]^{\frac{1}{r}}\left[\int_0^{\frac{1}{n+1}}\left\{\frac{\xi(t)\left|\tilde{M}_n(t)\right|}{t\sin^{\beta}t}\right\}^s dt\right]^{\frac{1}{s}}$$

$$=O\left(\frac{1}{n+1}\right)\left[\int_0^{\frac{1}{n+1}}\left\{\frac{\xi(t)\left|\tilde{M}_n(t)\right|}{t\sin^{\beta}t}\right\}^s dt\right]^{\frac{1}{s}}\quad by(2.3)$$

Since $\sin t\geq(2t/\pi)$ and using Lemma 3.1

$$I_{1.1}=O\left(\frac{1}{n+1}\right)\left[\int_0^{\frac{1}{n+1}}\left\{\frac{\xi(t)}{t^{2+\beta}}\right\}^s dt\right]^{\frac{1}{s}}$$

Since $\xi(t)$ is a positive increasing function, and using second mean value theorem for integrals,

$$I_{1.1}=O\left\{\frac{1}{n+1}\xi\left(\frac{1}{n+1}\right)\right\}\left[\int_{\varepsilon}^{\frac{1}{n+1}}\frac{dt}{t^{(2+\beta)s}}\right]^{\frac{1}{s}}\quad\text{for some } 0<\varepsilon<\frac{1}{n+1}$$

$$= O\left\{\frac{1}{n+1}\xi\left(\frac{1}{n+1}\right)\right\}\left[\left\{\frac{t^{-(2+\beta)s+1}}{-(2+\beta)s+1}\right\}_{\varepsilon}^{\frac{1}{n+1}}\right]^{\frac{1}{s}}$$

$$= O\left\{\frac{1}{n+1}\xi\left(\frac{1}{n+1}\right)\right\}\left[\{n+1\}^{2+\beta-\frac{1}{s}}\right]$$

$$= O\left\{(n+1)^{\beta+1-\frac{1}{s}}\xi\left(\frac{1}{n+1}\right)\right\}$$

$$I_{1.1} = O\left[\{n+1\}^{\beta+\frac{1}{r}}\xi\left(\frac{1}{n+1}\right)\right], \quad \text{since} \quad \frac{1}{r}+\frac{1}{s}=1 \tag{4.2}$$

Using Hölder's inequality $|\sin t| \leq 1, \sin t \geq (2t/\pi)$ conditions (2.2), (2.4), Lemma 3.2 and second mean value theorem for integrals,

$$|I_{1.2}| \leq \left[\int_{\frac{1}{n+1}}^{\pi}\left\{\frac{t^{-\delta}|\psi(t)|\sin^{\beta}t}{\xi(t)}\right\}^{r}dt\right]^{\frac{1}{r}}\left[\int_{\frac{1}{n+1}}^{\pi}\left\{\frac{\xi(t)\left|\tilde{M}_n(t)\right|}{t^{-\delta}\sin^{\beta}t}\right\}^{s}dt\right]^{\frac{1}{s}}$$

$$\leq \left[\int_{\frac{1}{n+1}}^{\pi}\left\{\frac{t^{-\delta}|\psi(t)|}{\xi(t)}\right\}^{r}dt\right]^{\frac{1}{r}}\left[\int_{\frac{1}{n+1}}^{\pi}\left\{\frac{\xi(t)\left|\tilde{M}_n(t)\right|}{t^{-\delta}\sin^{\beta}t}\right\}^{s}dt\right]^{\frac{1}{s}}$$

$$= O\left\{(n+1)^{\delta}\right\}\left[\int_{\frac{1}{n+1}}^{\pi}\left\{\frac{\xi(t)\left|\tilde{M}_n(t)\right|}{t^{-\delta}\sin^{\beta}t}\right\}^{s}dt\right]^{\frac{1}{s}}$$

$$= O\left\{(n+1)^{\delta}\right\}\left[\int_{\frac{1}{n+1}}^{\pi}\left\{\frac{\xi(t)}{t^{\beta+1-\delta}}\right\}^{s}dt\right]^{\frac{1}{s}}$$

$$= O\left\{(n+1)^{\delta}\right\}\left[\int_{\frac{1}{\pi}}^{n+1}\left\{\frac{\xi\left(\frac{1}{y}\right)}{y^{\delta-\beta-1}}\right\}^{s}\frac{dy}{y^2}\right]^{\frac{1}{s}} \quad (\text{Putting } t=\frac{1}{y})$$

$$= O\left\{(n+1)^{\delta}\xi\left(\frac{1}{n+1}\right)\right\}\left[\int_{\eta}^{n+1}\frac{1}{y^{s(\delta-1-\beta)+2}}dy\right]^{\frac{1}{s}}, \text{ for some } \frac{1}{\pi} \leq \eta \leq (n+1)$$

$$= O\left\{(n+1)^{\delta}\xi\left(\frac{1}{n+1}\right)\right\}\left[\int_{1}^{n+1}\frac{1}{y^{s(\delta-1-\beta)+2}}dy\right]^{\frac{1}{s}}, \text{ for some } \frac{1}{\pi}\leq 1\leq (n+1)$$

$$= O\left\{(n+1)^{\delta}\xi\left(\frac{1}{n+1}\right)\right\}\left[(n+1)^{(\beta+1-\delta)-\frac{1}{s}}\right]$$

$$= O\left\{(n+1)^{\beta+1-\frac{1}{s}}\xi\left(\frac{1}{n+1}\right)\right\}$$

$$I_{1.2} = O\left\{(n+1)^{\beta+\frac{1}{r}}\xi\left(\frac{1}{n+1}\right)\right\}, \quad \text{since} \quad \frac{1}{r}+\frac{1}{s}=1. \tag{4.3}$$

Now combining (4.1), (4.2) and (4.3), we get

$$\left|N_n^pC_n^1 - \tilde{f}(x)\right| = O\left\{(n+1)^{\beta+\frac{1}{r}}\xi\left(\frac{1}{n+1}\right)\right\}.$$

Now using L_r- norm, we get

$$\left\|N_n^pC_n^1 - \tilde{f}\right\|_r = \left\{\int_0^{2\pi}\left|N_n^{\tilde{p}}C_n^1 - f(x)\right|^r dx\right\}^{\frac{1}{r}}$$

$$= \left\{\int_0^{2\pi}\left\{(n+1)^{\beta+\frac{1}{r}}\xi\left(\frac{1}{n+1}\right)\right\}^r dx\right\}^{\frac{1}{r}}$$

$$= O\left\{(n+1)^{\beta+\frac{1}{r}}\xi\left(\frac{1}{n+1}\right)\right\}\left\{\int_0^{2\pi}dx\right\}^{\frac{1}{r}}$$

$$= O\left\{(n+1)^{\beta+\frac{1}{r}}\xi\left(\frac{1}{n+1}\right)\right\}.$$

This completes the proof of the theorem.

5. Corollaries

Following corollaries can be derived from our main theorem.

Corollary 6.1. *If* $\xi(t) = t^{\alpha}$, $0 < \alpha \leq 1$, *then the weighted* $W(L_r, \xi(t))$ *class,* $r \geq 1$, *reduces to the class* $\mathrm{Lip}(\alpha, r)$ *and the degree of approximation of*

$\tilde{f}$, *conjugate to a* 2π *- periodic function* $f \in \mathrm{Lip}\,(\alpha, r)$, $\frac{1}{r} < \alpha \leq 1$, *is given by*

$$\left| N_n^p C_n^1 - \tilde{f} \right| = O\left(\frac{1}{(n+1)^{\alpha - \frac{1}{r}}} \right).$$

Proof. The result follows by setting $\beta = 0$ in (2.1).

Corollary 6.2. *If* $r \to \infty$ *in corollary (6.1), then the class* Lip (α, r) *reduces to the class* $\mathrm{Lip}\,\alpha$ *and the degree of approximation of* $\tilde{f}$, *conjugate to a* 2π *- periodic function* $f \in \mathrm{Lip}\,\alpha, 0 \leq \alpha < 1$ *is given by*

$$\left\| N_c^p C_n^1 - f \right\|_\infty = O\left\{ \frac{1}{(n+1)^\alpha} \right\}.$$

References

[1] G. Alexits, *Convergence Problems of Orthogonal Series*, International Series of Monograms in Pure and Applied Mathematics, **20**, Pergamon Press, New York-Oxford-Paris (1961), Translated from German by I. Földer.

[2] Prem Chandra, Trigonometric approximation of functions in L^p norm, *J. Math. Anal. Appl.*, **275**, No. 1 (2002), 13-26.

[3] H.H. Khan, On degree of approximation of functions belonging to the class Lip (α, p), *Indian J. Pure Appl. Math.*, **5**, No. 2 (1974), 132-136.

[4] Lászaló Leindler, Trigonometric approximation in L^p norm, *J. Math. Anal. Appl.*, **302** (2005).

[5] Leonard McFadden, Absolute Nörlund summability, *Duke Math. J.*, **9** (1942), 168-207.

[6] K. Qureshi, On the degree of approximation of a periodic function f by almost Nörlund means, *Tamkang J. Math.*, **12**, No. 1 (1981), 35-38.

[7] K. Qureshi, On the degree of approximation of a function belonging to the class $\mathrm{Lip}\,\alpha$, *Indian J. pure Appl. Math.*, **13**, No. 8 (1982), 898-903.

[8] K. Qureshi, H.K. Neha, A class of functions and their degree of approximation, *Ganita*, **41**, No. 1 (1990), 37-42.

[9] B.E. Rhaodes, On degree of approximation of functions belonging to lipschitz class by Hausdorff means of its Fourier series, *Tamkang Journal of Mathematics*, **34**, No. 3 (2003), 245-247.

[10] B.N. Sahney, D.S. Goel, On the degree of continuous functions, *Ranchi University Math. Jour.*, **4** (1973), 50-53.

[11] A. Zygmund, *Trigonometric series*, 2-nd Rev. Ed., Cambridge Univ. Press, Cambridge (1959).