International Journal of Pure and Applied Mathematics
Volume 70 No. 3 2011, 349-365

HYPERBOLIC EQUATION WITH A RESISTANCE TERM IN A NONCYLINDRICAL DOMAIN

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Abstract: In this paper we investigate the existence and uniqueness of solution for the equation $u'' - \Delta u = -\nabla p$ in a noncylindrical domain $\widehat{Q}$, with div $u = 0$ in $\widehat{Q}$, under some initial and boundary conditions. This model, in a cylindrical domain, was originally proposed by J.L. Lions and it is related to the dynamics elasticity for incompressible materials.

AMS Subject Classification: 35L20, 35L65
Key Words: hyperbolic equation, noncylindrical domain, Galerkin method

1. Introduction

The objective of the present article is to investigate existence and uniqueness of solution for a hyperbolic model with a resistance term, set in a domain whose boundary is moving in time.

Let T be a positive real number and $\{\Omega_t\}_{t\in[0,T]}$ be a family of bounded open sets of $\mathbb{R}^n$ with regular boundary Γ_t. We denote by $\widehat{Q}$ the noncylindrical domain of $\mathbb{R}^{n+1}$ defined by:

$$\widehat{Q} = \bigcup_{0<t<T} \{\Omega_t \times \{t\}\}$$

with regular lateral boundary

Received: January 17, 2011

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$$\widehat{\Sigma} = \bigcup_{0<t<T} \{\Gamma_t \times \{t\}\}.$$

Therefore we consider the problem of obtaining a function $u : \widehat{Q} \longrightarrow \mathbb{R}$ which verifies

$$\begin{cases} u'' - \Delta u = -\nabla p, & \text{in } \widehat{Q} \\ \operatorname{div} u = 0, & \text{in } \widehat{Q} \\ u = 0 & \text{on } \widehat{\Sigma} \\ u(0) = u_0, \quad u'(0) = u_1, & \text{in } \Omega_0. \end{cases} \tag{1}$$

In problem (1), we represent by u the vector function

$$u(x,t) = (u_1(x,t), u_2(x,t), \ldots, u_n(x,t)),$$

for $(x,t) \in \widehat{Q}$ and $u_i(x,t) \in \mathbb{R}$, for $i = 1, 2, \ldots, n$.

The derivatives are in the sense of the theory of distributions,

$$\operatorname{div} u = \sum_{i=1}^{n} \frac{\partial u_i}{\partial x_i} \quad \text{and} \quad \nabla p = \left(\frac{\partial p}{\partial x_1}, \ldots, \frac{\partial p}{\partial x_n}\right).$$

The problem (1), defined in a cylindrical domain, was initially investigated by Lions [8] and in accordance with [1] it physically represents small deformations and displacements suffered by a solid body, constituted of an elastic and incompressible material. For this model the following results have been recently obtained: in Santos [14], the author obtains the exact controllability, in [3], the decay of energy is obtained considering a local nonlinear damping and in Antunes [2] the simultaneous controllability related to the system is investigated.

Evolution equations in noncylindrical domains have been widely studied in recent years. For the study of the existence of solutions to the wave equation in noncylindrical domains we can cite the following references: J.L. Lions [9], L.A. Medeiros [12], C. Bardos, J. Cooper [4] and A. Inoue [6], for the Navier Stokes equation we have J.L. Lions [9], O.A. Ladyzhenskaya [7], R. Temam [15], Y. Ebihara, L.A. Medeiros [13] and M. Milla Miranda, J.L. Ferrel [11].

The methodology to obtain results of existence and uniqueness for (1), cf. Lions [10], is to turn it into an equivalent problem in a cylinder Q by a diffeomorphism, then solve it.

The plan of this work is as follows. In Section 2, we set the notation, assumptions and present the equivalent problem defined in the cylinder and the Section 3 is dedicated to prove the results of existence and uniqueness of solution.

2. The Cylindrical Problem

Initially, we will describe the noncilyndrical domain $\widehat{Q} = \bigcup_{0<t<T} \{\Omega_t \times \{t\}\}$ of our problem. Given $\Omega \subset \mathbb{R}^n$ an open and bounded set, let us consider the subsets Ω_t of $\mathbb{R}^n$ defined by:

$$\Omega_t = \{x \in \mathbb{R}^n,\ x = K(t)y,\ y \in \Omega\}, \quad 0 \leq t \leq T,$$

where $K(t) = k(t)M$, $M = (m_{ij})_{n\times n}$ is a non-singular matrix, whose components are real numbers and $k : [0, T] \longrightarrow \mathbb{R}$ is a real function that satisfies $k \in C^3([0, T])$ and $k(t) \geq k_0 > 0$.

To transform (1) into a problem in the cylinder $Q = \Omega \times (0, T)$, let us consider the diffeomorphism $\tau : \widehat{Q} \longrightarrow Q$ defined by

$$\tau(x, t) = (y, t), \text{ with } y = K^{-1}x.$$

The matrices $K(t)$ and $K^{-1}(t)$ will be denoted, respectively, by $(\alpha_{ij}(t))_{1\leq i,j\leq n}$ and $(\beta_{ij}(t))_{1\leq i,j\leq n}$.

The inverse $\tau^{-1} : Q \to \widehat{Q}$ is defined by $\tau^{-1}(y, t) = (K(t)y, t)$.

The following remark aims to justify the choice of the matrix $K(t)$ associated to diffeomorphism τ.

Remark 1. If we took a general non singular matrix $K(t)$ instead of $K(t) = k(t)M$, the test functions $\psi \in \mathcal{V}$ must satisfy the divergence free condition

$$\mathrm{div}(K^{-1}(t)\psi) = 0,\ \forall t \in (0, T).$$

This divergence free condition may be rewritten as

$$\rho(t) \cdot G(y) = 0,\ \forall t \in (0, T) \tag{2}$$

with $\rho(t) = (c_1^{tr}(t), \ldots, c_n^{tr}(t))$ and $G(y) = (\nabla^{tr}\psi_1(y), \ldots, \nabla^{tr}\psi_n(y))$ where $c_j(t)$ is the j-th column of $K^{-1}(t)$ and the exponent tr denotes the matrix transpose.

Let $\mathcal{R} \subset \mathbb{R}^{n^2}$ be the subspace spanned by the vectors $\rho(t)$ for $t \in (0, T)$ and denote by $\mathbf{d}$ its dimension. Using (2), we see that the vector $G(y)$ belongs to $\mathcal{R}^\perp$. Let $\{w_1, \ldots, w_{n^2-\mathbf{d}}\}$ be a basis of $\mathcal{R}^\perp$. We denote

$$w_k = (w_{k,1}, \ldots, w_{k,n^2}) = (w_k^{(1)}, \ldots, w_k^{(n)})$$

where $w_k^{(i)} = (w_{k,(i-1)n+1}, \ldots, w_{k,in})$. Thus each component $\psi_i(y)$ of ψ satisfy

$$\nabla\psi_i(y) = \sum_{k=1}^{N} (\nabla\psi_i(y) \cdot \beta_k^{(i)})\beta_k^{(i)}$$

where $\{\beta_1^{(i)},\ldots,\beta_N^{(i)}\}$ is a orthonormal basis of the subspace spanned by $\{w_1^{(i)}, \ldots, w_{n^2-\mathbf{d}}^{(i)}\}$. If $\{\beta_1^{(i)},\ldots,\beta_N^{(i)},\gamma_1^{(i)},\ldots,\gamma_{n-N}^{(i)}\}$ is a orthonormal basis of $\mathbb{R}^n$, then $\psi_i(y)$ satisfy

$$\nabla\psi_i(y)\cdot\gamma_j^{(i)}=0,\ \ j=1,\ldots,n-N$$

and arguing as in the transport equation (see Evans [5, Ch. 2, p. 18]), we may deduce that ψ_i is a $C_0^\infty(\Omega)$ function given by

$$\psi_i(y)=g_i\left(\sum_{k=1}^{N}(y\cdot\beta_k^{(i)})\beta_k^{(i)}\right),$$

for some regular real fuction g_i. Therefore, the dimension d of the subspace generated by the the vectors $\rho(t)$, $t\in(0,T)$ may imply strong restrictions on the space $\mathcal{V}$. Notice that in our case, we kave $K(t)=k(t)M$. Thus $d=1$ and it corresponds to the least restrictive case.

Then by the change of variables $u(x,t)=v(y,t)$ and $p(x,t)=q(y,t)$, where $y=K^{-1}x$, $y\in\Omega$ and $x\in\Omega_t$, we transform the noncylindrical problem (1), into the following problem in the cylinder Q :

$$\begin{cases} v''+A(t)v+C_0(t)\nabla v\cdot y+C_1(t)\nabla v'\cdot y=-K(t)\nabla q, & \text{in } Q,\\ \operatorname{div}\left(K^{-1}(t)v\right)=0, & \text{in } Q,\\ v=0 & \text{on } \Sigma,\\ v(0)=v_0,\quad v'(0)=v_1 & \text{in } \Omega, \end{cases} \tag{3}$$

where

$$\begin{aligned} &v_0=u_0(K(0)y), v_1(y)=u_1(K(0)y)+\frac{k'(0)}{k(0)}\nabla v_0\cdot y,\\ &A(t)v=-\sum_{k,l=1}^{n}\frac{\partial}{\partial y_l}\left(a_{kl}(y,t)\frac{\partial v}{\partial y_l}\right),\\ &a_{kl}=\sum_{j=1}^{n}\beta_{kj}\beta_{lj}-\left[\frac{k'(t)}{k(t)}\right]^2 y_k y_l,\\ &C_0(t)=-\frac{k''(t)k(t)+(n-1)(k'(t))^2}{k^2(t)},\quad C_1(t)=-\frac{2k'(t)}{k(t)} \end{aligned} \tag{4}$$

and $\Sigma=\Gamma\times(0,T)$ is the lateral boundary of the cylinder Q.

In order to study the problems (1) and (3) we introduce the spaces

$$\begin{aligned} &\mathcal{V}_t=\{\psi\in(\mathcal{D}(\Omega_t))^n;\ \operatorname{div}\psi=0\},\\ &V(\Omega_t)=\{u\in(H_0^1(\Omega_t))^n,\ \operatorname{div}u=0\},\\ &H(\Omega_t)=\left\{u\in\left(L^2(\Omega_t)\right)^n,\ \operatorname{div}u=0\right\}, \end{aligned}$$

defined in the noncylindrical domain Ω_t and the spaces

$$\begin{aligned} \mathcal{V} &= \{\psi \in (\mathcal{D}(\Omega))^n;\ \operatorname{div}(M^{-1}\psi^{tr}) = 0\}, \\ V = V(\Omega) &= \{\psi \in (H_0^1(\Omega))^n,\ \operatorname{div}(M^{-1}\psi^{tr}) = 0\}, \\ H = H(\Omega) &= \left\{\psi \in \left(L^2(\Omega)\right)^n,\ \operatorname{div}(M^{-1}\psi^{tr}) = 0\right\}, \end{aligned}$$

defined in Ω.

3. Existence and Uniqueness of Solution

In this section will be presented the results of existence and uniqueness of solution to the problem (1), which are given by the theorem below.

Theorem 2. *Let $u_0 \in V(\Omega_0) \cap (H^2(\Omega_0))^n$ and $v_1 \in V(\Omega_0)$. Then there exists a function $p : \widehat{Q} \to \mathbb{R}$, $p \in L^\infty(0,T,H^1(\Omega_t))$ and an unique function u satisfying*

$$u \in L^\infty(0,T,V(\Omega_t) \cap (H^2(\Omega_t))^n), \quad u' \in L^\infty(0,T,V(\Omega_t)),$$
$$u'' \in L^\infty(0,T,H(\Omega_t))$$

such that u is the solution of (1).

As the problems (1) and (3) are equivalent, we obtain the above result by proving the existence and uniqueness results for the problem defined in the cylinder. Thus we need to define the concept of solution for (3).

Definition 3. A function $v : Q \longrightarrow \mathbb{R}$ is a weak solution of the problem (3) if v is such that

$$v \in L^\infty(0,T,V), \quad v' \in L^\infty(0,T,H),$$

and satisfies the equation

$$\begin{aligned} -\int_0^T (v',\xi')dt + \int_0^T a(t,v,\xi)dt + \int_0^T (C_0(t)\nabla v \cdot y, \xi)\,dt \\ + \int_0^T \left\langle C_1(t)\nabla v' \cdot y, \xi\right\rangle dt = 0 \end{aligned} \tag{5}$$

where

$$\xi \in L^2(0,T,V \cap (L^2(\Omega))^n), \quad \xi' \in L^2(0,T,H), \quad \xi(0) = \xi(T) = 0$$

and the initial conditions $v(0) = v_0$, $v'(0) = v_1$.

Now we are able to present the theorem which gives the results of existence and uniqueness for the cylindrical case.

Theorem 4. *Let $v_0 \in V \cap (H^2(\Omega))^n$ and $v_1 \in V$. Then there exists a function $q : Q \to \mathbb{R}$, $q \in L^\infty(0,T,H^1(\Omega))$ and a unique function v satisfying*

$$v \in L^\infty(0,T,V \cap (H^2(\Omega))^n), \quad v' \in L^\infty(0,T,V),$$
$$v'' \in L^\infty(0,T,H),$$

such that v is the solution of (3) *in the sense of above definition.*

To prove the Theorem 4 it is essential to achieve the coercivity of the form defined by

$$\langle A(t)v,\xi\rangle = a(t,v,\xi) = \sum_{k,l=1}^{n} \int_\Omega a_{kl}(y,t)\frac{\partial v}{\partial y_k}\frac{\partial \xi}{\partial y_l}dy.$$

Before we introduce the following notation

$$M_0 = \sup_{\|y\|=1} \left\|M^{tr}y\right\|_{\mathbb{R}^n}, \quad \gamma = \max_{[0,T]}\left|k'(t)\right|, \quad D = meas(\Omega),$$

and consider the hypothesis

$$M_0\gamma D < 1. \tag{6}$$

The coercivity of the $a(t,v,\xi)$, is given by the following Lemma.

Lemma 5. *The bilinear form $a(t,v,w)$ is coercive, that is,*

$$a(t,v,v) \geq a_0\|v\|^2, \ \forall\ v \in V$$

where a_0 is a positive constant.

Proof. In fact, we have for $\xi = (\xi_1,\xi_2,\ldots,\xi_n) \in \mathbb{R}^n$

$$\begin{aligned}\sum_{l,r=1}^{n} a_{lr}\xi_l\xi_r &= \sum_{l,r=1}^{n}\left[\sum_{j=1}^{n}\beta_{lj}\beta_{rj}\xi_l\xi_r - \left(\frac{k'(t)}{k(t)}\right)^2 y_l y_r \xi_l\xi_r\right]\\ &= \sum_{j=1}^{n}\left(\sum_{l=1}^{n}\beta_{lj}\xi_l\right)^2 - \left(\frac{k'(t)}{k(t)}\right)^2\left(\sum_{l=1}^{n} y_l\xi_l\right)^2\\ &= \left\|\left(k^{-1}(t)\right)^T\xi\right\|_{\mathbb{R}^n}^2 - \left(\frac{k'(t)}{k(t)}\right)^2\langle y,\xi\rangle_{\mathbb{R}^n}^2.\end{aligned} \tag{7}$$

Now, it is sufficient to note that:

- If $\eta = \left(K^{-1}(t)\right)^T \xi$ then $\xi = K(t)^T \eta$ and

$$\|\xi\|^2_{\mathbb{R}^n} = \left\|K(t)^T \left(K^{-1}(t)\right)^T \xi\right\|^2_{\mathbb{R}^n} = |k(t)|^2 \left\|M^T \left(K^{-1}(t)\right)^T \xi\right\|^2_{\mathbb{R}^n} \leq |k(t)|^2 M_0^2 \left\|\left(K^{-1}(t)\right)^T \xi\right\|^2_{\mathbb{R}^n},$$

which implies

$$\left\|\left(K^{-1}(t)\right)^T \xi\right\|^2_{\mathbb{R}^n} \geq \frac{1}{|k(t)|^2 M_0^2} \|\xi\|^2_{\mathbb{R}^n} > \frac{1}{k_0^2 M_0^2} \|\xi\|^2_{\mathbb{R}^n}.$$

- $$\left(\frac{k'(t)}{k(t)}\right)^2 \langle y, \xi\rangle^2_{\mathbb{R}^n} \leq \left(\frac{k'(t)}{k(t)}\right)^2 \|y\|^2_{\mathbb{R}^n} \|\xi\|^2_{\mathbb{R}^n} \leq \left(\frac{k'(t)}{k(t)}\right)^2 D^2 \|\xi\|^2_{\mathbb{R}^n} \leq \frac{\gamma^2}{k_0^2} D^2 \|\xi\|^2_{\mathbb{R}^n}.$$

Using the estimates obtained above in (7), we obtain

$$\sum_{l,r=1}^{n} a_{lr}\xi_l\xi_r \geq \left[\frac{1}{k_0^2 M_0^2} - \frac{\gamma^2}{k_0^2} D^2\right] \|\xi\|^2_{\mathbb{R}^n} = \frac{1-\gamma^2 M_0^2 D^2}{k_0^2 M_0^2} \|\xi\|^2_{\mathbb{R}^n} = a_0 \|\xi\|^2_{\mathbb{R}^n} = a_0 \sum_{l=1}^{n} \xi_l^2.$$

Note that, from the hypothesis (6), we have $a_0 = \dfrac{1-\gamma^2 M_0^2 D^2}{k_0^2 M_0^2} > 0$. □

Proof of Theorem 4. We employ the Faedo-Galerkin method. From the compact immersion of $\left(H_0^1(\Omega)\right)^n$ in $\left(L^2(\Omega)\right)^n$, we can solve the spectral problem

$$\left|\begin{array}{l} ((w,v))_H = \lambda(w,v), \quad \forall v \in V \\ div\ w = 0 \text{ in } \Omega. \end{array}\right. \tag{8}$$

Let (w_μ) and (λ_μ) be the solution to (8). Represent by V_m the subspace generated by $\{w_1,\ w_2,\ w_3, \ldots, w_m\}$, and let us consider the approximate problem:

$$\begin{cases} \text{Find } v_m \in V_m \text{ solution of} \\ (v''_m(t), w_j) + a(t, v_m(t), w_j) + \left(C_0(t) \sum\limits_{k=1}^{n} \dfrac{\partial v_m(t)}{\partial y_k} y_k, w_j\right) + \\ + \left(C_1(t) \sum\limits_{k=1}^{n} \dfrac{\partial v'_m(t)}{\partial y_k} y_k, w_j\right) = 0, \qquad j = 1, \ldots, m \\ v_m(0) = v_{0m} \to v_0 \text{ strongly in } V \cap (H^2(\Omega))^n \\ v'_m(0) = v_{1m} \to v_1 \text{ strongly in } \left(H_0^1(\Omega)\right)^n \end{cases} \tag{9}$$

Note that if $v_m \in V_m$, then $v_m(y,t) = \sum_{j=1}^{m} g_{jm}(t)\, w_j(y)$. It follows that (9) is a system of ordinary differential equations in the unknowns $g_{jm}(t)$, $j = 1, 2, \ldots$ and therefore has a local solution on $[0, t_m)$. The extension to interval $[0, T]$ is a consequence of the following estimates.

First Estimate. Multiplying both sides of $(9)_1$ by g'_{jm} and adding from $j = 1$ to $j = m$, we obtain

$$\frac{1}{2}\frac{d}{dt}|v'_m(t)|^2 + a(t, v_m(t), v'_m(t)) + \left(C_1(t) \sum_{k=1}^{n} \frac{\partial v'_m(t)}{\partial y_k} y_k, v'_m(t) \right) + \left(C_0(t) \sum_{k=1}^{n} \frac{\partial v_m(t)}{\partial y_k} y_k, v'_m(t) \right) = 0. \quad (10)$$

We will analyze the terms of the equality in (10).

Observe that

$$\frac{d}{dt} a(t, z, z) = a'(t, z, z) + 2a\left(t, z, z'\right),$$

and therefore, taking $z = v_m$, we get

$$a(t, v_m, v'_m) = \frac{1}{2}\frac{d}{dt} a(t, v_m, v_m) - \frac{1}{2} a'(t, v_m, v_m). \quad (11)$$

Now we note that,

$$a'(t, v_m, v_m) = \sum_{k,l=1}^{n} \int_\Omega a'_{kl}(y,t) \frac{\partial v_m}{\partial y_k} \frac{\partial v_m}{\partial y_l} dy,$$

and from $(4)_2$, it follows

$$a'_{kl}(y,t) = -\frac{2k'(t)}{k^3(t)} \sum_{j=1}^{n} m_{kj}^{-1} m_{lj}^{-1} - \left[\frac{2k'(t)}{k(t)} \left(\frac{k''(t)\, k(t) - (k'(t))^2}{k^2(t)} \right) \right] y_l y_k,$$

where m_{ij}^{-1} denote the elements of the inverse matrix M^{-1}.

In this way, from the hypothesis about $k(t)$, we guarantee that $a'_{kl}(y,t)$ is bounded over $\overline{Q}$, that is, $\max_{(y,t)\in\overline{Q}} |a'_{kl}(y,t)| \equiv k_1$.

Thus,

$$\frac{1}{2}a'(t, v_m(t), v_m(t)) \leq \frac{n^2 k_1}{2a_0} \int_\Omega \sum_{k=1}^n a_0 \left| \frac{\partial v_m(t)}{\partial y_k} \right|^2 dy. \tag{12}$$

From (12) and the coercivity of $a(t, v, w)$, we obtain

$$\frac{1}{2}a'(t, v_m(t), v_m(t)) \leq k_2 a(t, v_m(t), v_m(t)). \tag{13}$$

Furthermore, from the Gauss' Lemma, as $v'_m = 0$ on Γ, we have

$$\begin{aligned}\left(C_1(t) \sum_{k=1}^n \frac{\partial v'_m(t)}{\partial y_k} y_k, v'_m(t) \right) &= C_1(t) \int_\Omega \sum_{k=1}^n \frac{1}{2} \frac{\partial \left(v'^2_m(t) \right)}{\partial y_k} y_k dy \\ &= -\frac{n C_1(t)}{2} \left| v'_m(t) \right|^2.\end{aligned} \tag{14}$$

Returning in (10) with the terms obtained in (11) and (14) we obtain

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\left(|v'_m(t)|^2 + a(t, v_m(t), v_m(t)) \right) &= \frac{1}{2}a'(t, v_m(t), v_m(t)) \\ + \frac{n C_1(t)}{2} \left| v'_m(t) \right|^2 &- \left(C_0(t) \sum_{k=1}^n \frac{\partial v_m(t)}{\partial y_k} y_k, v'_m(t) \right).\end{aligned} \tag{15}$$

Now we observe that

$$\begin{aligned}&-\left(C_0(t) \sum_{k=1}^n \frac{\partial v_m(t)}{\partial y_k} y_k, v'_m(t) \right) \\ &\qquad \leq |C_0(t)| D \int_\Omega \sum_{k=1}^n \left| \frac{\partial v_m(t)}{\partial y_k} \right| \left| v'_m(t) \right| dy.\end{aligned} \tag{16}$$

It follows, from Holder's inequality and some calculations that

$$\begin{aligned}&-\left(C_0(t) \sum_{k=1}^n \frac{\partial v_m(t)}{\partial y_k} y_k, v'_m(t) \right) \\ &\qquad \leq \frac{n |C_0(t)| D}{2a_0} \left(\int_\Omega \sum_{k=1}^n a_0 \left| \frac{\partial v_m(t)}{\partial y_k} \right|^2 dy + a_0 \int_\Omega \left| v'_m(t) \right|^2 dy \right).\end{aligned}$$

Therefore, the coercivity of $a(t, v, w)$, implies that

$$-\left(C_0(t) \sum_{k=1}^n \frac{\partial v_m(t)}{\partial y_k} y_k, v'_m(t) \right)$$

$$\leq \frac{n\left|C_0(t)\right| D}{2a_0} a\left(t, v_m(t), v_m(t)\right) + \frac{n\left|C_0(t)\right| D}{2}\left|v_m'(t)\right|^2,$$

that is,

$$-\left(C_0(t)\sum_{k=1}^{n}\frac{\partial v_m(t)}{\partial y_k}y_k, v_m'(t)\right) \\ \leq k_3\left(a\left(t, v_m(t), v_m(t)\right) + \left|v_m'(t)\right|^2\right). \tag{17}$$

Returning in (15) with the terms obtained in (13) and (17) we get

$$\frac{1}{2}\frac{d}{dt}\left(\left|v_m'(t)\right|^2 + a\left(t, v_m(t), v_m(t)\right)\right) \\ \leq\leq k_4\left(\left|v_m'(t)\right|^2 + a\left(t, v_m(t), v_m(t)\right)\right). \tag{18}$$

From (18) we have

$$\left|v_m'\right|^2 + a\left(t, v_m, v_m\right) \leq \left(\left|v_m'(0)\right|^2 + a\left(0, v_m(0), v_m(0)\right)\right) + \\ +2k_4\int_0^t\left(\left|v_m'(s)\right|^2 + a\left(s, v_m, v_m\right)\right)ds. \tag{19}$$

From the convergence of initial data, we obtain:

- $\left|v_m'(0)\right|^2 = \left|v_{1m}\right|^2 \leq k_5$.
- $a\left(0, v_m(0), v_m(0)\right)$

$$\leq k_6\left[\int_\Omega\left(\sum_{k=1}^{n}\left|\frac{\partial v_{0m}}{\partial y_k}\right|\right)^2 dy\right]^{1/2}\left[\int_\Omega\left(\sum_{l=1}^{n}\left|\frac{\partial v_{0m}}{\partial y_l}\right|\right)^2 dy\right]^{1/2}$$

$$\leq k_6 n^2\left[\int_\Omega\sum_{k=1}^{n}\left|\frac{\partial v_{0m}}{\partial y_k}\right|^2 dy\right]^{1/2}\left[\int_\Omega\sum_{l=1}^{n}\left|\frac{\partial v_{0m}}{\partial y_l}\right|^2 dy\right]^{1/2}$$

$$= k_6 n^2\left|\nabla v_{0m}\right|^2 \leq k_7,$$

where $k_6 = \max_{(y,t)\in\overline{Q}}\left|a_{kl}(y,t)\right|$.

From the above estimates, returning to (19), it follows that

$$\left|v_m'(t)\right|^2 + a\left(t, v_m(t), v_m(t)\right)$$

$$\leq (k_5 + k_7) + 2k_4 \int_0^t \left(|v'_m(s)|^2 + a(s, v_m(s), v_m(s)) \right) ds.$$

Therefore, by Gronwall's Lemma,

$$|v'_m(t)|^2 + a(t, v_m(t), v_m(t)) \leq (k_5 + k_7) e^{2k_4 T} = k_8,$$

that is,

$$\left(v'_m\right) \text{ is bounded in } L^\infty(0, T; H). \tag{20}$$

Now, from the coercivity of $a(t, v, w)$ and as we have $a(t, v_m, v_m) \leq k_8$ then

$$\left\|v'_m(t)\right\|^2 \leq \frac{a(t, v_m(t), v_m(t))}{a_0} \leq \frac{k_8}{a_0},$$

therefore

$$(v_m) \text{ is bounded in } L^\infty(0, T; V). \tag{21}$$

Second Estimate. Taking the derivative with respect to t of the approximate equation (9) and substituting w_j by $v''_m(t)$ we obtain:

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\left(\left|v''_m(t)\right|^2 + 2a'\left(t, v_m(t), v'_m(t)\right) + a\left(t, v'_m(t), v'_m(t)\right)\right) \\ &= \frac{3}{2}a'\left(t, v'_m(t), v'_m(t)\right) + a''\left(t, v_m(t), v'_m(t)\right) \\ &- \left(C'_0(t)\sum_{k=1}^n \frac{\partial v_m(t)}{\partial y_k} y_k, v''_m(t)\right) - \left(C_0(t)\sum_{k=1}^n \frac{\partial v'_m(t)}{\partial y_k} y_k, v''_m(t)\right) \\ &- \left(C'_1(t)\sum_{k=1}^n \frac{\partial v'_m(t)}{\partial y_k} y_k, v''_m(t)\right) - \left(C_1(t)\sum_{k=1}^n \frac{\partial v''_m(t)}{\partial y_k} y_k, v''_m(t)\right). \end{aligned} \tag{22}$$

Following the arguments used in the first estimate we obtain the limitations below

- $\frac{3}{2}a'\left(t, v'_m, v'_m\right) \leq k_9 a\left(t, v'_m, v'_m\right).$
- $-\left(C_0(t)\sum_{k=1}^n \frac{\partial v'_m}{\partial y_k} y_k, v''_m\right) \leq k_3\left(a(t, v'_m, v'_m) + |v''_m(t)|^2\right).$
- $-\left(C_1(t)\sum_{k=1}^n \frac{\partial v''_m}{\partial y_k} y_k, v''_m\right) \leq k_4 |v''_m(t)|^2.$
- $-\left(C'_1(t)\sum_{k=1}^n \frac{\partial v'_m}{\partial y_k} y_k, v''_m\right) \leq k_{10}\left(a(t, v'_m, v'_m) + |v''_m(t)|^2\right)$

- $-\left(C_0'(t)\sum_{k=1}^{n}\frac{\partial v_m}{\partial y_k}y_k, v_m''\right) \leq k_{11}\left(a\left(t,v_m,v_m\right)+\left|v_m''\left(t\right)\right|^2\right)$
  $\leq k_{12}+k_{11}\left|v_m''\left(t\right)\right|^2$
- $a''\left(t,v_m,v_m'\right) \leq k_{13}+k_{14}a\left(t,v_m',v_m'\right).$

Substituting in (22) the above estimates and integrating from 0 to t we have

$$\begin{gathered}\frac{1}{2}\left(\left|v_m''\left(t\right)\right|^2+a\left(t,v_m',v_m'\right)\right) \leq \frac{1}{2}a\left(0,v_m,v_m\right)-a'\left(t,v_m,v_m'\right)\\ +\frac{1}{2}\left|v_m''\left(0\right)\right|^2+k_{15}T+k_{16}\int_0^t\left(a\left(s,v_m'\left(s\right),v_m'\left(s\right)\right)+\left|v_m''\left(s\right)\right|^2\right)ds.\end{gathered} \tag{23}$$

From the coercivity of $a\left(t,v,w\right)$ it follows that

$$\frac{1}{2}a\left(t,v_m',v_m'\right) \geq \frac{a_0}{2}\left\|v_m'\left(t\right)\right\|^2.$$

Furthermore, simple calculations show us that

- $-a'\left(t,v_m,v_m'\right) \leq k_{17}n^2\left(\left\|v_m\left(t\right)\right\|\left\|v_m'\left(t\right)\right\|\right) \leq$

$\leq k_{18}\left\|v_m\left(t\right)\right\|^2+\frac{a_0}{4}\left\|v_m'\left(t\right)\right\|^2 \leq k_{19}+\frac{a_0}{4}\left\|v_m'\left(t\right)\right\|^2.$

- $\frac{1}{2}a\left(0,v_{1m},v_{1m}\right) \leq \frac{1}{2}k_6n^2\left\|v_{1m}\right\|^2 \leq k_{20}.$
- $a'\left(0,v_{0m},v_{0m}\right) \leq k_{17}n^2\left\|v_{0m}\right\|^2 \leq k_{21}.$
- $a\left(s,v_m'\left(s\right),v_m'\left(s\right)\right) \leq k_{22}\left\|v_m'\left(s\right)\right\|^2.$

Coming back in (23) we have,

$$\begin{gathered}\frac{1}{2}\left|v_m''\left(t\right)\right|^2+\frac{a_0}{4}\left\|v_m'\left(t\right)\right\|^2 \leq \left|v_m''\left(0\right)\right|^2+k_{23}+\\ +k_{24}\int_0^t\left(\frac{1}{2}\left|v_m''\left(s\right)\right|^2+\frac{a_0}{4}\left\|v_m'\left(s\right)\right\|^2\right)ds.\end{gathered} \tag{24}$$

Now we observe that it is necessary to obtain a limitation of $\left|v_m''\left(0\right)\right|^2$. This will be done below.

Calculating (9) at $t=0$ and substituting w_j by $v''_m(0)$, we obtain

$$|v''_m(0)|^2 = -\int_\Omega \sum_{k,l=1}^n a_{kl}(y,0) \frac{\partial v_m(y,0)}{\partial y_k} \frac{\partial v''_m(y,0)}{\partial y_l} dy -$$
$$-C_0(0)\int_\Omega \sum_{k=1}^n \frac{\partial v_m(0)}{\partial y_k} y_k v''_m(y,0)\,dy - C_1(0)\int_\Omega \sum_{k=1}^n \frac{\partial v'_m(0)}{\partial y_k} y_k v''_m(y,0)\,dy. \tag{25}$$

Note that, using the Gauss' Lemma, Cauchy Schwarz inequality and the convergences from (9) we obtain

$$-\int_\Omega \sum_{k,l=1}^n a_{kl}(y,0) \frac{\partial v_m(y,0)}{\partial y_k} \frac{\partial v''_m(y,0)}{\partial y_l} dy \le k_{25} \left|v''_m(0)\right|,$$

$$-C_0(0)\int_\Omega \sum_{k=1}^n \frac{\partial v_m(0)}{\partial y_k} y_k v''_m(y,0)\,dy \le k_{26}\left|v''_m(0)\right|$$

and

$$-C_1(0)\int_\Omega \sum_{k=1}^n \frac{\partial v'_m(0)}{\partial y_k} y_k v''_m(y,0)\,dy \le k_{27}\left|v''_m(0)\right|.$$

Therefore, from (25), we get

$$\left|v''_m(0)\right|^2 \le k_{28}. \tag{26}$$

Substituting (26) in (24) and applying Gronwall inequality, follows that

$$\frac{1}{2}\left|v''_m(t)\right|^2 + \frac{a_0}{4}\left\|v'_m(t)\right\|^2 \le k_{29} e^{k_{24}T} = k_{30}.$$

and then we conclude that

$$\left(v''_m\right) \text{ is bounded in } L^\infty(0,T;H(\Omega)) \tag{27}$$

$$\left(v'_m\right) \text{ is bounded in } L^\infty(0,T;V(\Omega)). \tag{28}$$

Third Estimate. Multiplying the approximate equation by $\lambda_j g_{jm}$ and adding on j we obtain

$$-\int_\Omega \sum_{k,l=1}^n a_{kl}(y,t) \frac{\partial v_m(t)}{\partial y_k} \frac{\partial \Delta v_m(t)}{\partial y_l} dy = \left(v''_m(t), \Delta v_m(t)\right) +$$
$$+C_0(t)\int_\Omega \sum_{k=1}^n \frac{\partial v_m(t)}{\partial y_k} y_k \Delta v_m(t)\,dy + C_1(t)\int_\Omega \sum_{k=1}^n \frac{\partial v'_m(t)}{\partial y_k} y_k \Delta v_m(t)\,dy.$$

From Cauchy Schwarz inequality and previous estimates, we get

$$\left(v_m'', \Delta v_m\right) \leq k_{31} \left|\Delta v_m (t)\right|,$$

$$C_0(t) \int_\Omega \sum_{k=1}^{n} \frac{\partial v_m}{\partial y_k} y_k, \Delta v_m dy \leq k_{32} \left|\Delta v_m (t)\right|$$

and

$$C_1(t) \int_\Omega \sum_{k=1}^{n} \frac{\partial v_m'}{\partial y_k} y_k \Delta v_m dy \leq k_{33} \left|\Delta v_m (t)\right|.$$

Therefore

$$-\int_\Omega \sum_{k,l=1}^{n} a_{kl} (y,t) \frac{\partial v_m (t)}{\partial y_k} \frac{\partial \Delta v_m (t)}{\partial y_l} dy \leq k_{34} \left|\Delta v_m (t)\right|. \tag{29}$$

Furthermore

$$\begin{aligned} &-\int_\Omega \sum_{k,l=1}^{n} a_{kl} (y,t) \frac{\partial v_m}{\partial y_k} \frac{\partial \Delta v_m}{\partial y_l} dy - \int_\Omega \sum_{k,l=1}^{n} \frac{\partial}{\partial y_l} \left(a_{kl} (y,t)\right) \frac{\partial v_m}{\partial y_k} \Delta v_m dy \\ &= \int_\Omega \sum_{k,l=1}^{n} a_{kl} (y,t) \frac{\partial^2 v_m}{\partial y_l \partial y_k} \Delta v_m dy. \end{aligned} \tag{30}$$

Noting that $\dfrac{\partial}{\partial y_l} (a_{kl})$ is bounded and using Cauchy Schwarz inequality and previous estimates, we obtain

$$-\int_\Omega \sum_{k,l=1}^{n} \frac{\partial}{\partial y_l} \left(a_{kl} (y,t)\right) \frac{\partial v_m}{\partial y_k} \Delta v_m dy \leq k_{35} \left|\Delta v_m (t)\right| \tag{31}$$

Therefore, using (29) and (31) in (30) it follows that

$$\int_\Omega \sum_{k,l=1}^{n} a_{kl} (y,t) \frac{\partial^2 v_m}{\partial y_l \partial y_k} \Delta v_m dy \leq k_{36} \left|\Delta v_m (t)\right|. \tag{32}$$

Finally from the coercivity of $a(t,v,w)$ we have

$$\int_\Omega \sum_{k,l=1}^{n} a_{kl} (y,t) \frac{\partial^2 v_m}{\partial y_l \partial y_k} \Delta v_m dy \geq a_0 \left|\Delta v_m (t)\right|^2, \tag{33}$$

then, combining (32) and (33) we conclude that

$$|\Delta v_m(t)| \leq \frac{k_{37}}{a_0},$$

that is,

$$(\Delta v_m) \text{ is bounded in } L^\infty\left(0,T;\left(L^2(\Omega)\right)^n\right), \tag{34}$$

and thus,

$$(v_m) \text{ is bounded in } L^\infty\left(0,T;\left(H^2(\Omega)\right)^n \cap V\right). \tag{35}$$

The limitations obtained in $(21), (27), (28), (34)$ and (35) allow us to pass the limit in the approximate equation and get

$$\begin{aligned}&\int_0^T (v''(t), w)\,\theta(t)\,dt + \int_0^T a(t,v,w)\,\theta(t)\,dt\\&+\int_0^T \left(C_0(t)\sum_{k=1}^n \frac{\partial v}{\partial y_k} y_k, w\right)\theta(t)\,dt\\&+\int_0^T \left(C_1(t)\sum_{k=1}^n \frac{\partial v'}{\partial y_k} y_k, \Delta v_m\right)\theta(t)\,dt = 0,\end{aligned}$$

for all $w \in V(\Omega)$ and $\theta \in \mathcal{D}(0,T)$.

Taking in particular $w \in \mathcal{V}$, we have

$$\left(v''(t), w\right) + a(t,v,w) + \left(C_0(t)\sum_{k=1}^n \frac{\partial v}{\partial y_k} y_k, w\right) + \left(C_1(t)\sum_{k=1}^n \frac{\partial v'}{\partial y_k} y_k, \Delta v_m\right) = 0$$

in $\mathcal{D}'(0,T)$, for all $w \in \mathcal{V}$. Therefore,

$$v'' + A(t)\,v + C_0(t)\nabla v \cdot y + C_1(t)\nabla v' \cdot y = 0 \tag{36}$$

in $\mathcal{D}'(0,T;\mathcal{V}')$.

To recover the term on the left side of (3), we apply the following Lemma (cf. [11]):

Lemma 6. *Let be $g \in (\mathcal{D}'(\Omega))^n$ such that $\langle g, \psi\rangle = 0$ for all $\psi \in \mathcal{V}$. Then*
(i) There exists $q \in \mathcal{D}'(\Omega)$ such that $g = \nabla q K^{-1}$;
(ii) Furthermore, if $g \in \left(L^2(\Omega)\right)^n$ then $q \in H^1(\Omega)$.

The uniqueness is proved by using the standard energy method. □

Proof of Theorem 2. We consider the change of variables $x = K(t)\,y$, and we define

$$\begin{aligned}v_0(y) &= u_0(K(0)y)\\ v_1(y) &= u_1(K(0)y) + \frac{k'(0)}{k(0)}\nabla v_0 \cdot y\end{aligned}$$

We can see that we are in the conditions of Theorem 4. It follows that there exist a unique solution v of problem (3). By considering $u\,(x,t) = v\,(y,t)$ with $x = K\,(t)\,y$ we verify that u is the solution of Theorem 2.

The regularity and uniqueness of u claimed in Theorem 2 are obtained by the regularity and uniqueness of the solution v of Theorem 4. □

References

[1] G.O. Antunes, F.D. Araruna, Equação hiperbólica com um termo de resistência, In: *Tópicos em Equações Diferenciais Parciais* (Ed. Luis Adauto Medeiros), Rio de Janeiro, IM-UFRJ, **I** (2006), 86-98.

[2] G.O. Antunes, F.D. Araruna, L.A. Medeiros, Simultaneous controllability for a system with a resistance term, *Tendências em Matematica Aplicada e Computacional*, **3**, No. 1 (2002).

[3] R.C. Charão, J.C. Oliveira, Stabilization of a locally damped incompressible wave equation, *J. Math. Anal. Appl.*, **303** (2005), 699-725.

[4] J. Cooper, C. Bardos, A nonlinear wave-equation in a time dependent domain, *J. Math. Anal. Applications*, **42** (1973), 29-60.

[5] L.C. Evans, *Partial Differential Equations*, Graduate Studies in Mathematics, Volume 19, AMS, Providence (1997).

[6] A. Inoue, Sur $\Box u + u^3 = f$ dans un ouvert non cylindrique, *J. Math. Anal. Appl.*, **46** (1974), 777-819.

[7] O.A. Ladyzhenskaya, Initial-boundary value problem for Navier-Stokes equations in domains with time-varying boundaries, *Sem. Math. V.A. Steklov Math. Inst.*, Leningrad, **11**, 35-46.

[8] J.L. Lions, On some hyperbolic equations with a pressure term, In: *Proceedings of the Conference Dedicated to Louis Nirenberg Held in Trento*, Italy, September 3-8, 1990, Harlow, Longman Scientific and Technical, Pitman Res. Notes Math. Ser. 269 (1992), 196-208.

[9] J.L. Lions, *Quelques Méthodes de Résolution des Problemes aux Limites Non Linéaires*, Dunod, Paris (1969).

[10] J.L. Lions, Problèmes aux limites dans les équations aux dérivées partielles, Deuxième Édition, *Séminaire de Mathématiques Supérieures*, No. 1 (Été, 1962); Les Presses de l'Université de Montréal, Montreal, 1965, see also Oeuvres Choisies de Jacques Louis Lions, Volume I (2003), 431-476.

[11] M.M. Miranda, J.L. Ferrel, The Navier Stokes equations in non-cylindrical domains, *Comp. Appl. Math.*, **16**, No. 3, (1997), 247-265.

[12] L.A. Medeiros, Non-linear wave equations in domains with variable boundary, *Arch. Rational Mech. Anal.*, **47** (1972), 47-58.

[13] L.A. Medeiros, Y. Ebihara, On the regular solutions for some classes of Navier-Stokes equation, *Annales Fac. Sci. Toulouse*, **9** (1988).

[14] A. Rocha dos Santos, *Exact Controllability in Dynamic Incompressible Materials*, Ph.D. Dissertation, Instituto de Matemática – UFRJ, Rio de Janeiro, RJ, Brasil (1996).

[15] R. Temam, *Navier-Stokes Equations and Nonlinear Functional Analysis*, CBMS-NSF, Regional Conference Series in Applied Mathematics, **41**, SIAM (1983).