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ON SHRINKING RETRO BANACH FRAMES

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Abstract: Shrinking retro Banach frames has been defined and studied. Examples have been given to show their existence. Several results related to shrinking retro Banach frames have been obtained. Finally, a stability result for retro Banach frames has been given.

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1. Introduction

Duffin and Schaeffer [6] introduced frames for Hilbert spaces to study some deep problems in non-harmonic Fourier series. In fact, they abstracted the fundamental notion of Gabor [9] for studying signal processing. The ideas of Duffin and Schaeffer did not generate much general interest outside of non-harmonic Fourier series. It took more than 30 years to realize the importance and potential of frames. In 1980, R. Young [16] wrote a book in which he presented frames in the abstract setting and again used them in the context of non-harmonic Fourier series. Daubechies, Grossmann and Meyer [5], in 1986, reintroduced frames and observed that frames can be used to find series expansions of functions in $L^2(R)$ which are similar to the expansions using orthonormal basis, and yet preserves the main features of the signal. Feichtinger and Grochenig [8]

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introduced the notion of atomic decompositions in Banach spaces which was an extension of the notion of atomic decompositions in function spaces introduced by Coifman and Weiss [4]. Grochenig [10] generalized the notion of atomic decompositions and called them Banach frames. For Banach frame literature, one may refer to [1, 2, 3, 7, 13, 14, 15].

Recently, while studying Banach frames in Banach spaces, Jain, Kaushik and Vashisht [11] introduced the notion of retro Banach frames in conjugate Banach spaces. In the present paper, we further study retro Banach frames and defined shrinking retro Banach frames and obtained some results related to shrinking retro Banach frames. Finally, a stability result for retro Banach frames has been given.

2. Preliminaries

Throughout this paper, E will denote a Banach space over the scalar field $\mathbb{K}$ ($\mathbb{R}$ or $\mathbb{C}$), E^* the conjugate space of E, $[x_n]$ the closed linear span of $\{x_n\}$ in the norm topology of E, $[\widetilde{f_n}]$ the closed linear span of $\{f_n\}$ in the $\sigma(E^*, E)$-toplogy of E^*. E_d and $(E^*)_d$, respectively, the associated Banach spaces of the scalar-valued sequences indexed by $\mathbb{N}$. A sequence $\{f_n\}$ in E^* is said to be total over E if $\{x \in E : f_n(x) = 0, n \in \mathbb{N}\} = \{0\}$.

The following results which are referred in this paper are listed in the form of a lemma.

Lemma 2.1. (see [12]) *If E is a Banach space and $\{f_n\} \subset E^*$ is total over E, then E is linearly isometric to the associated Banach space $E_d = \{\{f_n(x)\} : x \in E\}$, where the norm is given by $\|\{f_n(x)\}\|_{E_d} = \|x\|_E$, $x \in E$.*

Next, we give the definition of a retro Banach frame introduced in [11].

Definition 2.2. Let E be a Banach space and E^* be its conjugate space. Let $(E^*)_d$ be a Banach space of scalar-valued sequences associated with E^*, indexed by $\mathbb{N}$. Let $\{x_n\} \subset E$ and $T : (E^*)_d \to E^*$ be given. The pair $(\{x_n\}, T)$ is called a *retro Banach frame* for E^* with respect to $(E^*)_d$ if

(i) $\{f(x_n)\} \in (E^*)_d$, for each $f \in E^*$.

(ii) there exists positive constants A and B with $0 < A \leq B < \infty$ such that

$$A\|f\|_{E^*} \leq \|\{f(x_n)\}\|_{(E^*)_d} \leq B\|f\|_{E^*}, \quad f \in E^* . \tag{2.1}$$

(iii) T is a bounded linear operator such that $T(\{f(x_n)\}) = f$, $f \in E^*$.

The positive constants A and B, respectively, are called lower and upper frame bounds of the retro Banach frame $(\{x_n\}, T)$. The operator $T : (E^*)_d \to E^*$ is called the reconstruction operator (or, the pre-frame operator). The inequality (2.1) is called the retro frame inequality.

A retro Banach frame $(\{x_n\}, T)$ $(\{x_n\} \subset E, T : (E^*)_d \to E^*)$ for E with respect to $(E^*)_d$ and with bounds A, B is said to be *tight*, if it is possible to choose $A = B$, *normalized tight*, if $A = B = 1$ and exact, if there *exists* no reconstruction operator T_0 such that $(\{x_n\}_{n\neq j}, T_0)$ $(j \in \mathbb{N})$ is a retro Banach frame E^*.

Finally, we give the following theorem which will be used in the subsequent results.

Theorem 2.3. (see [11]) *Let $(\{x_n\}, T)$ $(\{x_n\} \subset E, T : (E^*)_d \to E^*)$ be a retro Banach frame for E^* with respect to $(E^*)_d$. Then, $(\{x_n\}, T)$ is exact if and only if if $x_n \notin [x_i]_{i\neq n}$, for all $n \in \mathbb{N}$.*

In view of Theorem 2.3, one may observe that if $(\{x_n\}, T)$ is an exact retro Banach frame for E^*, then there exists a sequence $\{g_n\}$ in E^*, called an admissible sequence to the retro Banach frame $(\{x_n\}, T)$, such that $g_i(x_j) = \delta_{ij}$, for all $i, j \in \mathbb{N}$.

3. Main Results

We being with the definition of shrinking retro Banach frames.

Definition 3.1. Let $(\{x_n\}, T)$ $(\{x_n\} \subset E, T : (E^*)_d \to E^*)$ be an exact retro Banach frame for E^* with admissible sequence $\{f_n\} \subset E^*$. Then $(\{x_n\}, T)$ is called shrinking if there exists an associated Banach space $(E^{**})_d$ and a bounded linear operator $U : (E^{**})_d \to E^{**}$ such that $(\{f_n\}, U)$ is a retro Banach frame for E^{**}.

Towards the existence of shrinking retro Banach frames, we give the following examples

Example 3.2. Let $E = c_0$ and let $\{x_n\}$ be the sequence of unit vectors in E. Then by Lemma 2.1 there exists a bounded linear operator $T : \{\{f(x_n)\} : f \in E^*\} \to E^*$ such that $(\{x_n\}, T)$ is retro Banach frame for E^*. Let $\{f_n\}$ be the sequence of unit vectors in E^*. Then $\{f_n\}$ is an admissible sequence to $\{x_n\}$, such that $[f_n] = E^*$. Therefore, there exists an associated Banach space $(E^{**})_d = \{\{\phi(f_n)\} : \phi \in E^{**}\}$ and a bounded linear operator $U : (E^{**})_d \to E^{**}$ given by $U(\{\phi(f_n)\}) = \phi$, $\phi \in E^{**}$ such that $(\{f_n\}, U)$ is a retro Banach frame for E^{**}. Hence $(\{x_n\}, T)$ is a shrinking retro Banach frame for E^{**}.

Example 3.3. Let $E = \ell^1$ and let $\{x_n\}$ be the sequence of unit vectors in E. Then there is a bounded linear operator $T : \{\{f(x_n)\} : f \in E^*\} \to E^*$ such that $(\{x_n\}, T)$ is retro Banach frame for E^* with respect to the associated Banach space $\{\{f(x_n)\} : f \in E^*\}$. Also, since $x_n \notin [x_i]_{i \neq n}$, $n \in \mathbb{N}$, there is a sequence $\{f_n\}$ in E^* such that $f_i(x_i) = \delta_{ij}$, $(i, j) \in \mathbb{N}$. But $[f_n] \neq E^*$. So, there exists no associated Banach space $(E^{**})_d$ and hence no bounded linear operator $U : (E^{**})_d \to E^{**}$ such that $(\{f_n\}, U)$ is a retro Banach frame for E^{**}. Hence $(\{x_n\}, T)$ is not a shrinking retro Banach frame for E^*.

In the next two results, we construct a new shrinking retro Banach frame with respect to a given shrinking retro Banach frame.

Theorem 3.4. *Let $(\{x_n\}, T)$ $(\{x_n\} \subset E, T : (E^*)_d \to E^*)$ be an exact retro Banach frame for E^*, which is shrinking and let $\{m_n\}$ be a strictly increasing sequence of natural numbers with $m_0 = 0$. Let $\{\xi_n\}$ be a sequence in E defined by*

$$\begin{aligned} \xi_{m_n} &= \sum_{i=1}^{n} x_{m_i}\,, \qquad n = 1, 2, \ldots \\ \xi_n &= x_n, \qquad\quad n \neq m_1, m_2, \ldots \end{aligned}$$

Then there exists an associated Banach space $(E^{**})_{d_0}$ and a bounded linear operator $U : (E^*)_{d_0} \to E^*$ such that $(\{\xi_n\}, U)$ is shrinking retro Banach frame for E^*.

Proof. By Lemma 2.1, there exists an associated Banach space $(E^*)_{d_0} = \{\{f(\xi_n)\} : f \in E^*\}$ and a bounded linear operator $U : (E^*)_{d_0} \to E^*$ such that $(\{\xi_n\}, U)$ is retro Banach frame for E^* with respect to $(E^*)_{d_0}$. Since $(\{x_n\}, T)$ is an exact retro Banach frame for E^*, by Theorem 2.3, there exists a sequence $\{f_n\}$ in E^* such that $f_i(x_j) = \delta_{ij}$, for all $i, j \in \mathbb{N}$. Also, since $(\{x_n\}, T)$ is a shrinking retro Banach frame for E^* and $\{f_n\} \subset E^*$ is an admissible sequence of $\{x_n\}$, it follows that $[f_n] = E^*$. Define a sequence $\{h_n\}$ in E^* by

$$\begin{aligned} h_{m_n} &= f_{m_n} - f_{m_{n+1}}\,, \qquad n = 1, 2, \ldots, \\ h_n &= f_n\,, \qquad\qquad n \neq m_1, m_2, \ldots \end{aligned}$$

Then, $h_i(\xi_j) = \delta_{ij}$, for all $i, j \in \mathbb{N}$ and $[h_n] = E^*$. Therefore, again by Lemma 2.1, there exists an associated Banach $(E^{**})_d = \{\{\phi(h_n)\} : \phi \in E^{**}\}$ and a bounded linear operator $S : (E^{**})_d \to E^{**}$ given by $S(\{\phi(h_n)\}) = \phi$, $\phi \in E^{**}$ such that $(\{h_n\}, S)$ is a retro Banach frame for E^{**}. Hence $(\{\xi_i\}, U)$ is shrinking retro Banach frame for E^*. □

Theorem 3.5. *Let $(\{x_n\}, T)$ $(\{x_n\} \subset E, T : (E^*)_d \to E^*)$ be an exact and shrinking retro Banach frame for E^* and let $\{m_n\}$ be a strictly increasing sequence of natural numbers with $m_0 = 0$. Assume that there is a functional f_0 in E^* such that*

$$f_0(x_{m_n}) = 1, \quad n = 1, 2, \ldots, \quad \text{and} \quad f_0(x_0) = 0, \ n \neq m_1, m_2, \ldots$$

Then, for the sequence $\{\xi\}$ in E, given by

$$\begin{cases} \xi_{m_1} = x_{m_1}, \\ \xi_{m_n} = x_{m_n} - x_{m_{n-1}} & (n = 2, 3, \ldots), \\ \xi_n = x_n & (n \neq m_1, m_2, \ldots), \end{cases}$$

there exists an associated Banach space $(E^*)_{d_0}$ and a bounded linear operator $U : (E^*)_{d_0} \to E^*$ such that $(\{\xi_n\}, U)$ is shrinking retro Banach frame for E^*.

Proof. Note that $[\xi_n] = E$, therefore, by Theorem 2.3 and Lemma 2.1, there is a bounded linear operator $U : \{\{f(\xi_n)\} : f \in E^*\} \to E^*$ such that $(\{\xi\}, U)$ is retro Banach frame for E^*. Since $(\{x_n\}, T)$ is exact and shrinking, there is a sequence $\{f_n\}$ in E^* such that $f_i(x_j) = \delta_{ij}$ $(i, j = 1, 2, \ldots)$ and $[f_n] = E^*$.

Define $\{h_n\}$ in E^* by

$$\begin{aligned} h_{m_1} &= f_0, \\ h_{m_n} &= f_0 - \sum_{i=1}^{n-1} fm_i, \qquad (n = 2, 3, \ldots), \\ h_n &= f_n \qquad\qquad (n \neq m_1, m_2, \ldots). \end{aligned}$$

Then $\{h_n\}$ is an admissible sequence to $\{\xi_n\}$ such that $[h_n] = E^*$. Therefore, there exists an associated Banach space $(E^*)_{d_0} = \{\{f(\xi_n)\} : f \in E^*\}$ and a bounded linear operator $U : (E^*)_{d_0} \to E^*$ such that $(\{\xi_n\}, U)$ is shrinking retro Banach frame for E^* with respect to $(E^*)_{d_0}$. □

In the following result, we prove that the sequence (block sequence in some sense) with respect to a shrinking retro Banach frame is also a shrinking retro Banach frame.

Theorem 3.6. *Let $(\{x_n\}, T)$ $(\{x_n\} \subset E, T : (E^*)_d \to E^*)$ be an exact retro Banach frame for E^* and let $y_n = \sum_{i=1}^{n} \alpha_i x_i$, $\alpha_n \neq 0$ $(n = 1, 2, \ldots)$. Then there is an associated Banach space $(E^*)_d$ and a bounded linear operator $U : (E^*)_d \to E^*$ such that $(\{y_n\}, U)$ is retro Banach frame for E^*. Furthermore, if $(\{x_n\}, T)$ is shrinking, then so is $(\{y_n\}, U)$.*

Proof. By hypothesis, there exists a bounded linear operator $U : \{\{f(y_n)\} : f \in E^*\} \to E^*$ such that $(\{y_n\}, U)$ is retro Banach frame for E^*. Further, by Lemma 2.1 $x_n \notin [x_i]_{\substack{i=1 \\ i \neq n}}^{\infty}$, $n \in \mathbb{N}$. This gives, $y_n \notin [y_i]_{\substack{i=1 \\ i \neq n}}^{\infty}$, $n \in \mathbb{N}$. So, by Theorem 2.3, $(\{y_n\}, U)$ is exact. Furthermore, if $(\{x_n\}, T)$ is shrinking, then there is a sequence $\{f_n\}$ in E^* such that $f_i(x_j) = \delta_{ij}$ $(i, j = 1, 2, \ldots)$ and $[f_n] = E^*$. Consider a sequence $\{h_n\}$ in E^* defined by $h_n = \frac{1}{\alpha_n} f_n$, $n \in \mathbb{N}$. Then $h_i(y_j) = \delta_{ij}$, for all $i, j \in \mathbb{N}$. Also, since $[h_n] = [f_n] = E^*$, by Lemma 2.1, there exists in associated Banach space $(E^{**})_{d_0} = \{\{\phi(h_n)\} : \phi \in E^{**}\}$ and a bounded linear operator $S : (E^{**})_{d_0} \to E^{**}$ such that $(\{h_n\}, S)$ is a retro Banach frame for E^{**}. Hence $(\{y_n\}, U)$ is shrinking. $\square$

Finally, we give the following stability result for retro Banach frames.

Theorem 3.7. *Let E be a Banach space and let $(\{x_n\}, T)$ $(\{x_n\} \subset E, T : (E^*)_d \to E^*)$ be a retro Banach frame for E^*. Then for a give sequence $\{y_n\} \subset E$ there exists a bounded linear operator $U : (E^*)_{d_0} \to E^*$ such that $(\{y_n\}, U)$ is a retro Banach frame for E^* with respect to $(E^*)_{d_0}$ if there exists a constant λ with $0 \leq \lambda < 1$ such that*

$$\left\| \sum_{i=1}^{m} a_i (x_i - y_i) \right\| \leq \lambda \left\| \sum_{i=1}^{m} a_i x_i \right\| \tag{3.1}$$

for all finite sequence of scalars $a_1, a_2, \ldots, a_n$.

Proof. If $[y_n] \neq E$, then by Lemma 2.1, there exists an $x \in E$, $x \notin [y_n]$ such that

$$\|x\| < \frac{1}{\lambda} \operatorname{dist}(x, [y_n]) \tag{3.2}$$

Also, by Hahn-Banach Theorem, there exists an $f \in E^*$ such that

$$f(x) = 1, \quad f(y_i) = 0, \quad i = 1, 2, \ldots \text{ and } \|f\| = \frac{1}{\operatorname{dist}(x, [y_n])}.$$

Now since $(\{x_n\}, T)$ is a retro Banach frame, we may write

$$x = \lim_{n \to \infty} \sum_{i=1}^{m_n} a_i^{(n)} x_i$$

Therefore, we have

$$1 = \lim_{n \to \infty} \left| f \left(\sum_{i=1}^{m_n} a_i^{(n)} x_i \right) \right|$$

$$
\begin{aligned}
&= \lim_{n\to\infty} \left| f\left(\sum_{i=1}^{m_n} a_i^{(n)}(x_i - y_i)\right)\right| \\
&\le \lim_{n\to\infty} \|f\| \left\| \sum_{i=1}^{m_n} a_i^{(n)}(x_i - y_i)\right\| \\
&\le \frac{\lambda\|x\|}{\operatorname{dist}(x,[y_n])} \qquad \text{(by (3.1))}
\end{aligned}
$$

This contradicts (3.2). So $[y_n] = E$. Therefore, by the lemma, there exists an associated Banach space $(E^*)_{d_0} = \{\{f(y_n)\} : f \in E^*\}$ with norm given by $\|\{f(y_n)\}\| = \|f\|$, $f \in E^*$ and a reconstruction operator $U : (E^*)_{d_0} \to E^*$ defined by $U(\{f(y_n)\}) = f$, $f \in E^*$ such that $(\{y_n\}, U)$ is a retro Banach frame for E^* with respect to $(E^*)_{d_0}$. □

Remark 3.8. The constant λ in Theorem 3.7 may not be 1. Indeed, if $(\{x_n\}, T)$ is a retro Banach frame for E and $y_n = 0$, $n \in \mathbb{N}$. Then (3.1) is satisfied. But there exists no associated Banach space $(E^*)_d$ and hence no reconstruction operator $U : (E^*)_d \to E^*$ such that $(\{y_n\}, U)$ is a retro Banach frame for E^*.

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