International Journal of Pure and Applied Mathematics
Volume 70 No. 4 2011, 595-610

THE UPPER AND LOWER SOLUTION METHOD FOR NONLINEAR FOURTH-ORDER THREE-POINT BOUNDARY VALUE PROBLEM

A. Elhaffaf[1 §], M. Naceri[2]
[1,2]Department of Mathematics
Faculty of Science
Oran University
BP1524, Es-Senia, ALGERIA

Abstract: This paper is concerned with the following fourth-order three-point boundary value problem

$$\begin{cases} u^{(4)}(t) + f(t, u(t), u^{'}(t), u^{''}(t)) = 0, \quad t \in [0,1], \\ u^{'}(0) = 0, u(0) = \lambda u(1), \quad u^{''}(0) = 0, \quad u^{''}(1) = \alpha u^{''}(\eta), \end{cases}$$

where $0 < \eta < 1$, $0 \le \alpha < 1$, $0 \le \lambda < 1$ and $f \in C([0,1] \times \mathbb{R}^3, \mathbb{R})$.

Some existence results are established for this problem via upper and lower solution method and fixed point.

AMS Subject Classification: 34B10, 34B15, 34B14
Key Words: fourth-order ordinary equation, upper and lower solution method, solution existence.

1. Introduction

In this paper, we consider the existence of solution for the fourth-order boundary value problem above due to its importance in physics,some problems of third-order has been studied by many authors.In 2008 Gue,Sun and Zhao [2] established some existence results for a least one positive solution to the third three-point BVP

Received: February 19, 2011

§Correspondence author

$$\begin{cases} u^{'''}(t) + a(t)f(u(t)) = 0, & t \in [0,1], \\ u(0) = u^{'}(0), & u^{'}(1) = \alpha u^{'}(\eta), \end{cases}$$

their main tool was the well-known Guo-Kranoselskii fixed point theorem Recently in [3], Jian-Ping Sun, Qiu-Yan Ren and Ya-Hong Zhao established some existence criteria for the third-order three-point boundary value problem

$$\begin{cases} u^{'''}(t) + a(t)f(t, u(t), u^{'}(t)) = 0, & t \in [0,1], \\ u(0) = u^{'}(0), & u^{'}(1) = \alpha u^{'}(\eta), \end{cases}$$

where $0 < \eta < 1$, $0 \leq \alpha < 1$.

Motiver greatly by [2, 3] in this paper, we will investigate the following nonlinear fourth-order three-point BVP

$$\begin{cases} u^{(4)}(t) + f(t, u(t), u^{'}(t), u^{''}(t)) = 0, & t \in [0,1], \\ u^{'}(0) = 0, u(0) = \lambda u(1), & u^{''}(0) = 0, u^{''}(1) = \alpha u^{''}(\eta), \end{cases} \tag{1}$$

where $0 < \eta < 1$, $0 \leq \alpha < 1$, $0 \leq \lambda < 1$ and $f \in C([0,1] \times \mathbb{R}^3, \mathbb{R})$
Under some conditions of the nonlinear term f, we established some existence criteria are obtained for the BVP (1) by using the upper and lower solution method, and the following fixed point theorem [1].

Theorem 1.1. *Let* ($\mathbf{E}, \mathbf{K}$) *be an ordered Banach space and [a, b] be a nonempty interval in E If* $\mathrm{T} : [a,b] \longrightarrow E$ *is an increasing compact mapping and* $a \leq \mathrm{Ta}$, $\mathrm{Tb} \leq b$ *then* T *has a fixed point in [a, b].*

2. Preliminaries

In this section, we will present some fundamental definition and several important lemmas.

Definition 2.1. Let E be a real Banach space. A nonempty closed convex set $K \subset \mathbb{E}$ is called cone if it satisfies the following two conditions:

1. $(\alpha u + \beta v) \in K$ pour tout $u, v \in K$ et pour $\alpha, \beta \geq 0$
   and

2. $u \in K$ et $-u \in K \Longrightarrow u \equiv 0$

every cone $K \subset \mathbb{E}$ induces an ordoring in E given by $u \leq v$ if and only if $v - u \in K$.

Definition 2.2. if $x \in C^4(I, \mathbb{R})$ satisfies

$$\begin{cases} x^{(4)}(t) + f(t, x(t), x'(t), x''(t)) \geq 0,\ t \in [0,1] \\ x'(0) = 0, x(0) = \lambda u(1),\ x''(0) \leq 0, x''(1) \leq \alpha x''(\eta) \end{cases}$$

then x is called a lower solution of the BVP (1)

Definition 2.3. if $y \in C^4(I, \mathbb{R})$ satisfies

$$\begin{cases} y^{(4)}(t) + f(t, y(t), y'(t), y''(t)) \leq 0,\ t \in [0,1] \\ y'(0) = 0, y(0) = \lambda u(1),\ y''(0) \geq 0, y''(1) \geq \alpha y''(\eta) \end{cases}$$

then y is called a upper solution of the BVP (1)

Let $G_1(t, s)$ be the Green's function of the second-order BVP

$$\begin{cases} u''(t) = v(t),\ t \in [0,1] \\ u'(0) = 0, u(0) = \lambda u(1) \end{cases}$$

then

$$G_1(t, s) = \frac{1}{1-\lambda} \begin{cases} (1-\lambda)(t-s) + \lambda(1-s) & \text{si } 0 \leq s \leq t \leq 1 \\ \lambda(1-s) & \text{si } 0 \leq t \leq s \leq 1 \end{cases}$$

for $G_1(t, s)$, we have the following two lemmas.

Lemma 2.4. $G_1(t, s) \geq 0\ , \forall (t, s) \in [0,1] \times [0,1]$.

Lemma 2.5. $M_1 = \max_{0 \leq t \leq 1} \int_0^1 G_1(t, s)ds$ *then* $M_1 < \frac{1}{2}$.

Proof. Since a simple computation shows that

$$\int_0^1 G_1(t, s)ds = \frac{t^2}{2} + \frac{\lambda}{2(1-\lambda)}$$

it is easy to obtain that

$$M_1 = \begin{cases} \frac{\lambda}{2(1-\lambda)}, & \text{if } \ t = 0. \\ \frac{1}{2(1-\lambda)}, & \text{if } \ t = 1, \end{cases}$$

which implies that $M_1 < \frac{1}{2}$.

Let $G_2(t, s)$ be the Green's function of the second-order BVP

$$\begin{cases} -v''(t) = w(t),\ t \in [0,1] \\ v(0) = 0, u(1) = \alpha u(\eta) \end{cases}$$

then

$$G_2(t,s) = \frac{1}{1-\alpha\eta}\begin{cases} s(1-\alpha\eta)+st(\alpha-1) & \text{si } s \le \min\{\eta,t\} \\ t(1-\alpha\eta)+st(\alpha-1) & \text{si } t \le s \le \eta \\ s(1-\alpha\eta)+t(\alpha\eta-s) & \text{si } \eta \le s \le t \\ t(1-s) & \text{si } \max\{\eta,t\} \le s \end{cases}$$

for $G_2(t,s)$, we have the following two lemmas.

Lemma 2.6. $G_2(t,s) \ge 0 \ ,\forall (t,s) \in [0,1]\times[0,1]$.

Proof. Let be $v''(t) = -w(t)$ then we have

$$\begin{aligned} v'(t) &= -\int_0^t w(s)ds + v'(0) \\ v(t) &= -\int_0^t (t-s)w(s)ds + v'(0)t \\ v(1) &= -\int_0^1 (1-s)w(s)ds + v'(0). \end{aligned}$$

Since

$$\begin{aligned} v(1) &= \alpha v(\eta) \\ \alpha v(\eta) &= -\alpha\int_0^\eta (\eta-s)w(s)ds + \alpha\eta v'(0) \\ v'(0) &= \frac{-\alpha}{1-\alpha\eta}\int_0^\eta (\eta-s)w(s)ds + \frac{1}{1-\alpha\eta}\int_0^1 (1-s)w(s)ds \\ v(t) &= \frac{1}{1-\alpha\eta}[t\int_0^1 (1-s)w(s)ds - \alpha t\int_0^\eta (\eta-s)w(s)ds \\ &\quad -\int_0^t (1-\alpha\eta)(t-s)w(s)ds]. \end{aligned}$$

The first case $s \le \min\{\eta,t\}$ we have

$$\begin{aligned} &t(1-s)-\alpha t(\eta-s)-(1-\alpha\eta)(t-s) \\ &\qquad = t - st - \alpha\eta t + \alpha ts - t + s + \alpha\eta t - \alpha\eta s \\ &\qquad = s(1-\alpha\eta) + st(\eta-1) \end{aligned}$$

and we have:

$$s(1-\alpha\eta)+st(\eta-1) \ge 0 \quad \text{for} \quad s \le \min\{\eta,t\},$$

$(\alpha-1)<0$ and $(1-\alpha\eta)>0$ with $\alpha\eta<\alpha$ because $0<\eta<1$; $0\leq\alpha<1$ then $|\alpha-1|<|1-\alpha\eta|$ with $st\leq s$ then $st|\alpha-1|<s|1-\alpha\eta|$.

Finally $st(\alpha-1)<s(1-\alpha\eta)$.

The same technique is used for the others cases.

Lemma 2.7. *Let* $M_2=\max_{0\leq t\leq 1}\int_0^1 G_2(t,s)ds$, *then* $M_2<\frac{1}{2}$.

Proof. By a simple computation we have

$$\begin{aligned}\int_0^1 G_2(t,s)ds &= \frac{t}{1-\alpha\eta}\int_0^1(1-s)ds-\frac{\alpha t}{1-\alpha\eta}\int_0^\eta(\eta-s)ds-\int_0^t(t-s)ds\\ &= \frac{t}{1-\alpha\eta}[s-\frac{s^2}{2}]_0^1-\frac{\alpha t}{1-\alpha\eta}[\eta s-\frac{s^2}{2}]_0^\eta-[s-\frac{s^2}{2}]_0^t\\ &= \frac{t}{1-\alpha\eta}(1-\frac{1}{2})-\frac{\alpha t}{1-\alpha\eta}(\eta^2-\frac{\eta^2}{2})-(t^2-\frac{t^2}{2})\\ &= -\frac{t^2}{2}+\frac{1-\alpha\eta^2}{2(1-\alpha\eta)}t\end{aligned}$$

it is easy to obtain that

$$M_2=\begin{cases}\frac{1}{8}(\frac{1-\alpha\eta 2}{1-\alpha\eta})^2 & \text{if} \quad \alpha\eta(2-\eta)\leq 1\\ \frac{\alpha\eta(1-\eta)}{2(1-\alpha\eta)} & \text{if} \quad \alpha\eta(2-\eta)\geq 1\end{cases}$$

which implies then $M_2<\frac{1}{2}$.

Lemma 2.8. *Assume that* β_1, β_2 *and* β_3 *are three nonnegative constants with*
$0\leq\beta_1M_1+\beta_2+\beta_3\leq 2$

If $\varphi\in C^2[0,1]$ *satisfies:*

$$\begin{cases}\varphi''(t)\geq\beta_1\int_0^1 G_1(t,s)\varphi(s)ds+\beta_2\int_0^t\varphi(s)ds+\beta_3\varphi(s)\\ \varphi(0)\leq 0,\varphi(1)\leq\alpha\varphi(\eta)\end{cases}\qquad (2)$$

then $\varphi(t)\leq 0$ *for* $t\in[0,1]$

Proof. We consider the following cases

Case. 1 $\beta_1 = \beta_2 = \beta_3 = 0$.
Then $\varphi^{''}(t) \geq 0$ for $t \in [0,1]$ which implies that the graph of $\varphi(t)$ is concave up. Since $\varphi(0) \leq 0$. we only need to prove $\varphi(1) \leq 0$. Suppose on the contrary that $\varphi(1) > 0$ then

$$\varphi(0) \leq 0 < \varphi(1) \leq \alpha\varphi(\eta) < \varphi(\eta)$$

So

$$0 < \frac{\varphi(\eta) - \varphi(0)}{\eta} \leq \frac{\varphi(1) - \varphi(\eta)}{1-\eta} < 0$$

which is a contradiction. Therefore $\varphi(1) \leq 0$.

Case. 2 $\beta_1 = \beta_2 = 0; \beta_3 \neq 0$.
Suppose that there existe t_τ such that $\varphi(t_\tau) > 0$. Then there exists t_0 such that $\varphi(t_0) = \max_{t\in[0,1]} \varphi(t)$ and $\varphi^{'}(t_0) = 0$;$\varphi^{''}(t_0) \leq 0$ It is clear that $t_0 \neq 0$ and $t_0 \neq 1$ since : $\varphi(0) \leq 0$ and $\varphi(t_0) > 0$, if $t_0 = 1$ we have

$$0 < \varphi(1) \leq \alpha\varphi(\eta) < \varphi(\eta) \leq \varphi(1)$$

which is a contradiction. There for $t_0 \neq 1$.

Since $\varphi^{''}(t) \geq \beta_3\varphi(t)$ we have

$$0 \geq \varphi^{''}(t_0) \geq \beta_3\varphi(t_0)$$

then $\varphi(t_0) \leq 0$ which is a contradiction.

Case. 3 $\beta_1 = 0; \beta_2 \neq 0; \beta_3 \neq 0$.
Then $\varphi^{''}(t) \geq \beta_2 \int_0^t \varphi(s)ds + \beta_3\varphi(s)$ Similarly, suppose there exists$\varphi(t_0) = \max_{t\in[0,1]} \varphi(t)$ such that $\varphi^{'}(t_0) = 0$;$\varphi^{''}(t_0) \leq 0$ with $t_0 \neq 0$ et $t_0 \neq 1$ we can obtain that

$$0 \geq \varphi^{''}(t_0) \geq \beta_2 \int_0^{t_0} \varphi(s)ds + \beta_3\varphi(t_0)$$

Then

$$\int_0^{t_0} \varphi(s)ds \leq 0$$

So, there exists $t_1 \in [0, t_0]$ such that $\varphi(t_1) = \min_{t\in[0,t_0]} \varphi(t) < 0$. It follows from Taylor's formula that there exists $\theta \in (t_1, t_0)$ such that:

$$\varphi(t_1) = \varphi(t_0) + \varphi^{'}(t_0)(t_1 - t_0) + \frac{\varphi^{''}(\theta)}{2}(t_1 - t_0)^2$$

Noting $\varphi(t_1) \leq 0$ Then

$$\varphi^{''}(\theta) = \frac{2[\varphi(t_1) - \varphi(t_0)]}{(t_1 - t_0)^2} < \frac{2\varphi(t_1)}{(t_1 - t_0)^2} < 2\varphi(t_1)$$

Then

$$\begin{aligned} 2\varphi(t_1) > \varphi^{''}(\theta) &\geq \beta_2 \int_0^\theta \varphi(s)ds + \beta_3\varphi(\theta) \\ &\geq \beta_2 \int_0^\theta \varphi(t_1)ds + \beta_3\varphi(t_1) \\ &> \beta_2\theta\varphi(t_1) + \beta_3\varphi(t_1) \end{aligned}$$

which implies that $2 < \beta_2\theta + \beta_3 \leq \beta_2 + \beta_3$ this contradiction fact that $\beta_2 + \beta_3 \leq 2$.

Case. 4 $\beta_1 \neq 0; \beta_2 \neq 0; \beta_3 \neq 0$ Then

$$\varphi^{''}(t) \geq \beta_1 \int_0^1 G_1(t,s)\varphi(s)ds + \beta_2 \int_0^t \varphi(s)ds + \beta_3\varphi(s)$$

Similarly, we can obtain that $\varphi(t_0) = \max_{t\in[0,1]} \varphi(t)$ and $\varphi^{'}(t_0) = 0$;$\varphi^{''}(t_0) \leq 0$ with $t_0 \neq 0$ et $t_0 \neq 1$, consequently

$$0 \geq \varphi^{''}(t_0) \geq \beta_1 \int_0^1 G_1(t_0,s)\varphi(s)ds + \beta_2 \int_0^{t_0} \varphi(s)ds + \beta_3\varphi(t_0)$$

which

$$\beta_1 \int_0^1 G_1(t_0,s)\varphi(s)ds + \beta_2 \int_0^{t_0} \varphi(s)ds \leq 0$$

Since $G_1(t,s) \geq 0$so there existe $t_1 \in [0,1]$ such that

$$\varphi(t_1) = \min_{t\in[0,1]} \varphi(t) < 0.$$

It follows from Taylor's formula that exists $\theta \in (t_1, t_0)$ such that:

$$\varphi(t_1) = \varphi(t_0) + \varphi^{'}(t_0)(t_1 - t_0) + \frac{\varphi^{''}(\theta)}{2}(t_1 - t_0)^2$$

$$\varphi^{''}(\theta) = \frac{2[\varphi(t_1) - \varphi(t_0)]}{(t_1 - t_0)^2} < \frac{2\varphi(t_1)}{(t_1 - t_0)^2} < 2\varphi(t_1)$$

Then

$$\begin{aligned} 2\varphi(t_1) > \varphi^{''}(\theta) &\geq \beta_1 \int_0^1 G_1(\theta, s)\varphi(s)ds + \beta_2 \int_0^\theta \varphi(s)ds + \beta_3\varphi(\theta) \\ &\geq \beta_1 \int_0^1 G_1(\theta, s)\varphi(t_1)ds + \beta_2 \int_0^\theta \varphi(t_1)ds + \beta_3\varphi(t_1) \\ 2\varphi(t_1) &> \beta_1 \int_0^1 G_1(\theta, s)\varphi(t_1)ds + \beta_2\theta\varphi(t_1) + \beta_3\varphi(t_1) \\ 2 &< \beta_1 \int_0^1 G_1(\theta, s)ds + \beta_2\theta + \beta_3 \end{aligned}$$

which implies that

$$2 < \beta_1 M_1 + \beta_2\theta + \beta_3 < \beta_1 + \beta_2 + \beta_3.$$

This contradicts the fact that $0 \leq \beta_1 M_1 + \beta_2 + \beta_3 \leq 2$, therefore $\varphi(t) \leq 0$ for $t \in [0, 1]$

3. Main Results

In the remainder of this paper, we always that the following conditions are satisfied:

H. 1 $f \in C(I \times \mathbb{R}^3, \mathbb{R})$ is continuous and there exist three nonnegative constants $\beta_1, \beta_2, \beta_3$ with $0 \leq \beta_1 M_1 + \beta_2 + \beta_3 \leq 2$, $0 \leq M_1(\beta_1 M_1 + \beta_2 + \beta_3) < 1$ where M_1, M_1 are defined as lemmas (2. 5), (2. 7) for $t \in [0, 1]$.

H. 2 $f(t, u_1, v_1, w_1) - f(t, u_2, v_2, w_2) \geq -\beta_1(u_1 - u_2) - \beta_2(v_1 - v_2) - \beta_3(w_1 - w_2)$ for $t \in [0, 1]$ $u_1 \geq u_2$, $v_1 \geq v_2$, $w_1 \geq w_2$.

Theorem 3.1. *If the BVP (1) has a lower solution x and upper solution y with $x^{''} \leq y^{''}$ for $t \in [0, 1]$. The BVP (1) has a solution $u \in C^4(I, \mathbb{R})$ satisfies $x^{''} \leq u^{''} \leq y^{''}$ for $t \in [0, 1]$.*

Proof. Let $u^{''}(t) = v(t)$ then the BVP (1)is equivalent to the following BVP

$$\begin{cases} v^{''}(t) + f(t, (Tv)(t), \int_0^t v(s)ds, v(t)) = 0, \ t \in [0, 1], \\ v(0) = 0, u(1) = \alpha u(\eta) \end{cases} \tag{3}$$

where

$$\begin{cases} (Tv)(t) = \int_0^1 G_1(t,s)ds = u(t),\ t \in [0,1] \\ \\ \int_0^t v(s)ds = v^{'}(t) \end{cases}$$

Let $\mathbb{E} = C[0,1]$ be equipped with the norm $\|v\|_\infty = \max_{t\in[0,1]} |v(t)|$ and $\mathbb{K} = \{v \in \mathbb{E} : v(t) \geq 0 \text{ for } t \in [0,1]\}$ then K is a cone and (E,K) is an ordered Banch space.

Now, if we define operators $L : D \in E \longrightarrow E$ and $N : E \longrightarrow E$ as follows:

$$\begin{cases} (Lv)(t) = v^{''}(t) + \beta_1(Tv)(t) + \beta_2 \int_0^t v(s)ds + \beta_3 v(t) \\ (Nv)(t) = f(t,(Tv)(t), \int_0^t v(s)ds, v(t)) + \beta_1(Tv)(t) + \beta_2 \int_0^t v(s)ds + \beta_3 v(t), \end{cases}$$

where $\mathbb{D} = \{v \in \mathbb{E} : v^{''} \in \mathbb{E},\ v(0) = 0,\ u(1) = \alpha u(\eta)\}$.

Then it is easy to see that the BVP (3) is equivalent to the operator equation

$$Lv = Nv. \tag{4}$$

Now, we shall show that the operator equation (4) is solvable the proof will be given in several steps.

Step. 1 $L : D \in E \longrightarrow E$ inversible.

Suppose $h \in E$. We will find unique $v \in \mathbb{D}$ such that

$$Lv = h$$

Since $Lv = h$ is equivalent to the integral equation

$$v(t) = \int_0^1 G_2(t,s)[h(s) - [\beta_1 \int_0^1 G_1(t,\tau)v(\tau)d\tau + \beta_2 \int_0^s v(\tau)d\tau + \beta_3 v(s)]]ds$$

We define a mapping $A : E \longrightarrow E$ by:

$$(Av)(t) = \int_0^1 G_2(t,s)[h(s) - [\beta_1 \int_0^1 G_1(t,\tau)v(\tau)d\tau + \beta_2 \int_0^s v(\tau)d\tau + \beta_3 v(s)]]ds \tag{5}$$

we shall show that $A : E \longrightarrow E$ is a contraction mapping.

Let $v_1, v_2 \in D$. Then

$$\begin{aligned}
\|(Av_1)(t)-(Av_2)(t)\| &= \| \int_0^1 G_2(t,s)[h(s)-[\beta_1 \int_0^1 G_1(t,\tau)v_1(\tau)d\tau \\
&+ \beta_2 \int_0^s v_1(\tau)d\tau + \beta_3 v_1(s)]]ds \\
&- \int_0^1 G_2(t,s)[h(s)-[\beta_1 \int_0^1 G_1(t,\tau)v_2(\tau)d\tau \\
&+ \beta_2 \int_0^s v_2(\tau)d\tau + \beta_3 v_2(s)]]ds\|
\end{aligned}$$

$$\begin{aligned}
&\leq \max_{t\in[0,1]} \int_0^1 G_2(t,s)[\beta_1 \int_0^1 G_1(t,\tau)|v_1-v_2|(\tau)d\tau \\
&\quad +\beta_2 \int_0^s |v_1-v_2|(\tau)d\tau + \beta_3|v_1-v_2|(s)]ds \\
&\leq \max_{t\in[0,1]} \int_0^1 G_2(t,s)[\beta_1 \int_0^1 G_1(t,\tau) \max_{t\in[0,1]} |v_1-v_2|(\tau)d\tau \\
&\quad +\beta_2 \int_0^s \max_{t\in[0,1]} |v_1-v_2|(\tau)d\tau \\
&+ \beta_3 \max_{t\in[0,1]} |v_1-v_2|(s)]]ds \\
&\leq \max_{t\in[0,1]} \int_0^1 G_2(t,s)[\beta_1 \int_0^1 G_1(t,\tau) \max_{t\in[0,1]} |v_1-v_2|(\tau)d\tau \\
&\quad +\beta_2 \int_0^t \max_{t\in[0,1]} |v_1-v_2|(\tau)d\tau \\
&+ \beta_3 \max_{t\in[0,1]} |v_1-v_2|(s)]]ds \\
&\leq \max_{t\in[0,1]} \int_0^1 G_2(t,s)[\beta_1 M_1\|v_1-v_2\| + \beta_2\|v_1-v_2\| + \beta_3\|v_1-v_2\|] \\
&\leq \max_{t\in[0,1]} \int_0^1 G_2(t,s)[[\beta_1 M_1 + \beta_2 + \beta_3]\|v_1-v_2\|]ds \\
&\leq M_2[\beta_1 M_1 + \beta_2 + \beta_3]\|v_1-v_2\|
\end{aligned}$$

Noting that $0 \leq M_2(\beta_1 M_1 + \beta_2 + \beta_3) < 1$ then there exists unique $v \in D$ such that $Av = v$

Step. 2 $L^{-1} : E \longrightarrow E$ is continuous Assume that $\{h_n\}_{n=1}^{\infty}$, $h \in E$ and $\lim_{n\longrightarrow} h_n = h$. Denote $L^{-1}h_n = v_n$ and $L^{-1}h = v$, then

$$v_n(t) = \int_0^1 G_2(t,s)[h_n(s) - [\beta_1 \int_0^1 G_1(t,\tau)v_n(\tau)d\tau + \beta_2 \int_0^s v_n(\tau)d\tau + \beta_3 v_n(s)]]ds$$

And

$$v(t) = \int_0^1 G_2(t,s)[h(s) - [\beta_1 \int_0^1 G_1(t,\tau)(\tau)d\tau + \beta_2 \int_0^s v(\tau)d\tau + \beta_3 v(s)]]ds$$

So

$$\begin{aligned}
\|v_n(t) - v(t)\| &= \max_{t\in[0,1]} |\int_0^1 G_2(t,s)[(h_n(s) - h(s)) \\
&\quad -[\beta_1 \int_0^1 G_1(t,\tau)(v_n(\tau) - v(\tau))d\tau \\
&+ \beta_2 \int_0^s (v_n(\tau)v_1(\tau))d\tau + \beta_3(v_n(s)v_1(s))]]ds| \\
&\le \max_{t\in[0,1]} \int_0^1 G_2(t,s)[\|h_n(s) - h(s)\| \\
&\quad +(\beta_1 M_1 + \beta_2 + \beta_3)\|v_n - v\|]ds \\
&\le M_2\|h_n(s) - h(s)\| + M_2(\beta_1 M_1 + \beta_2 + \beta_3)\|v_n - v\| \\
\|v_n(t) - v(t)\| &\le \frac{M_2}{1 - M_2(\beta_1 M_1 + \beta_2 + \beta_3)}\|h_n(s) - h(s)\|
\end{aligned}$$

Together with $\lim_{n\longrightarrow} h_n = h$ implies that $\lim_{n\longrightarrow} v_n = v$ this indicates that $L^{-1} : E \longrightarrow E$ is continuous.

Step. 3 $L^{-1}N : E \longrightarrow E$ is completely continuous. Since f et L^{-1} are continuous, we only need to prove that $L^{-1} : E \longrightarrow E$ is compact. Let X be a bounded subset in E then there exists a constant $C > 0$ such that $\|h\| \le C$ for any $h \in X$. $\forall v \in L^{-1}X, \exists h \in X$ such that$v = L^{-1}h$. so

$$v(t) = \int_0^1 G_2(t,s)[h(s) - [\beta_1 \int_0^1 G_1(t,\tau)(\tau)d\tau + \beta_2 \int_0^s v(\tau)d\tau + \beta_3 v(s)]]ds$$

On the one hand for any $v \in L^{-1}X$, we have

$$\begin{aligned}
\|v(t)\| &= \max_{t\in[0,1]} |\int_0^1 G_2(t,s)[h(s) - [\beta_1 \int_0^1 G_1(t,\tau)(\tau)d\tau \\
&\quad +\beta_2 \int_0^s v(\tau)d\tau + \beta_3 v(s)]]ds| \\
&\le \max_{t\in[0,1]} \int_0^1 |G_2(t,s)[h(s) - [\beta_1 \int_0^1 G_1(t,\tau)(\tau)d\tau \\
&\quad +\beta_2 \int_0^s v(\tau)d\tau + \beta_3 v(s)]]|ds \\
&\le \max_{t\in[0,1]} \int_0^1 |G_2(t,s)h(s)| \\
&\quad + \max_{t\in[0,1]} \int_0^1 |G_2(t,s)[\beta_1 \int_0^1 G_1(t,\tau)(\tau)d\tau + \beta_2 \int_0^s v(\tau)d\tau \\
&+ \beta_3 v(s)]]|ds \\
&\le M_2\|h\| + 2M_2\|v\| \\
\|v\| &\le \frac{M_2}{1-2M_2}\|h\| \\
\|v\| &\le \frac{M_2 C}{1-2M_2}
\end{aligned}$$

Which implies that $L^{-1}X$ is uniformly bounded. On the other hand, in view of the uniform continuity of $G_1(t,s)$ and $G_2(t,s)$, we know that for any $s \in [0,1]$

$$\begin{aligned}
\forall \epsilon > 0, \exists \quad & \delta > 0/t_1, t_2 \in [0,1], |t_1 - t_2| < \delta \\
\Rightarrow \quad & |G_1(t_1,s) - G_1(t_2,s)| < \frac{\epsilon}{2\beta_1 M_1} \\
\Rightarrow \quad & |G_2(t_1,s) - G_2(t_2,s)| < \frac{\epsilon 1 - 2M_2}{2C}
\end{aligned}$$

Then for $v \in L^{-1}X$, $t_1, t_2 \in [0,1]$ and $|t_1 - t_2| < \delta$ we have

$$\begin{aligned}
&\|v(t_1) - v(t_2)\| = |\int_0^1 [G_2(t_1,s) - G_2(t_2,s)][h(s) \\
&- [\beta_1 \int_0^1 G_1(t_1,\tau)(\tau)d\tau
\end{aligned}$$

$$
\begin{aligned}
&+\beta_2\int_0^s v(\tau)d\tau+\beta_3 v(s)]] \\
&-[\beta_1\int_0^1 G_2(t_2,s)\int_0^1 (G_1(t_1,\tau)-G_1(t_2,\tau))d\tau]ds| \\
\leq&[\|h\|+(\beta_1 M_1+\beta_2+\beta_3)\|v\|]| \\
&\times\int_0^1 [G_2(t_1,s)-G_2(t_2,s)]|ds \\
&+\beta_1 M_2\int_0^1 |(G_1(t_1,s)-G_1(t_2,s))ds| \\
\leq&\|h\|[1+\frac{2M_2}{1-2M_2}]|\int_0^1 [G_2(t_1,s)-G_2(t_2,s)]|ds \\
&+\beta_1 M_2\int_0^1 |(G_1(t_1,s)-G_1(t_2,s))ds| \\
\leq&\|h\|[1+\frac{C}{1-2M_2}]|\int_0^1 [G_2(t_1,s)-G_2(t_2,s)]|ds \\
&+\beta_1 M_2\int_0^1 |(G_1(t_1,s)-G_1(t_2,s))ds| \\
\|v(t_1)-v(t_2)\|\leq&\frac{C}{1-2M_2}\times\frac{1-2M_2}{C}\times\frac{\epsilon}{2}+\beta_1 M_2\times\frac{1}{\beta_1 M_2}\times\frac{\epsilon}{2} \\
\leq&\frac{\epsilon}{2}+\frac{\epsilon}{2} \\
\leq&\epsilon,
\end{aligned}
$$

which shows that $L^{-1}X$ is equicontinuous. By the Arzela-Ascoli theorem, we know that $L^{-1}: E \longrightarrow E$ is a compact mapping.

Step. 4 $L^{-1}N : E \longrightarrow E$ is increasing Let $h_1, h_2 \in E$ and $h_1 \leq h_2$, then the condition (H_2)implies that $Nh_1 \leq Nh_2$. Denote $v_1 = L^{-1}Nh_1;v_2 = L^{-1}Nh_2$ then $Lv_1 = Nh_1 \leq Nh_2 = Lv_2$ so $Lv_1 \leq Lv_2 \Rightarrow L(v_1 - v_2) \leq 0$.

Setting $q(t) = v_1(t) - v_2(t)$. It follows from lemma(2. 8)

$$
\begin{cases} (Lq)(t)\leq 0 & ,t\in[0,1], \\ q(0)\leq 0, q(1)\leq \alpha q(\eta) \end{cases}
$$

Then $q(t) \leq 0$ for $t \in [0, 1]$.

Therefore $v_1(t) \leq v_2(t)$and $L^{-1}N : E \longrightarrow E$ is increasing.

Step. 5 Let x, y lower and upper solution of the BVP (1) Let $\mu_0 = x^{''}$ and $\omega_0 = y^{''}$, then $\mu_0 \leq L^{-1}N\mu_0$ and $L^{-1}N\omega_0 \leq \omega_0$ and we have

$$\begin{cases} -\mu_0^{''}(t) + \beta_1 \int_0^1 G_1(t,s)\mu_0(s)ds + \beta_2 \int_0^t \mu_0(s)ds \\ \qquad + \beta_3\mu_0(s) \leq (N\mu_0)(t), \quad t \in [0,1], \\ \mu_0(0) \leq 0, \mu_0(1) \leq \alpha\mu_0(\eta) \end{cases} \tag{6}$$

Let $\mu^* = L^{-1}N\mu_0$. Then $L\mu^* = N\mu_0$

$$\begin{cases} -\mu^{*''}(t) + \beta_1 \int_0^1 G_1(t,s)\mu^*(s)ds + \beta_2 \int_0^t \mu^*(s)ds + \beta_3\mu^*(s) \\ \qquad = (N\mu_0)(t), \quad t \in [0,1], \\ \mu^*(0) = 0, \mu^*(1) = \alpha\mu^*(\eta). \end{cases} \tag{7}$$

Denote by $p(t) = \mu_0(t) - \mu^*(t)$. In view of (8) and(9), we know that

$$\begin{cases} -p^{''}(t) + \beta_1 \int_0^1 G_1(t,s)p(s)ds + \beta_2 \int_0^t p(s)ds + \beta_3 p(s) \leq 0, \quad t \in [0,1], \\ p(0) \leq 0, p(1) \leq \alpha p(\eta). \end{cases}$$

By Lemma (2. 8)we get $p(t) \leq 0$ for $t \in [0,1]$, i.e. $\mu_0(t) \leq \mu^*(t) = L^{-1}N\mu_0$ Similarly, we can obtain that $L^{-1}N\omega_0 \leq \omega_0$. It follows from theorem (1. 1) that $L^{-1}N : E \longrightarrow E$ has a fixed point $v^* \in [\mu_0, \omega_0]$ which solves the BVP (1), i.e.

$$u(t) = \int_0^1 G_1(t,s)v^*(s)ds$$

with $\mu_0 \leq v^* \leq \omega_0$ and $x^{''} \leq u^{''} \leq y^{''}$.

4. Examples

Example 4.1. Consider the following BVP

$$\begin{cases} u^{(4)}(t) + \frac{e^{-t}\sin u(t)}{8(t+1)} + \frac{e^{-u^{'}(t)}}{16\sqrt{t+1}} + \frac{t}{10(u^{''}(t)+1)} = 0, \; t \in [0,1] \\ u^{'}(0) = 0, u(0) = \lambda u(1), \; u^{''}(0) = 0, u^{''}(1) = \alpha u^{''}(\eta) \end{cases}$$

since

$$f(t,u,v,w) = \frac{e^{-t}\sin u}{8(t+1)} + \frac{e^{-v}}{16\sqrt{t+1}} + \frac{t}{10(w+1)}$$

It is easy to verify that the condition $(H_2)$is fulfilled with

$$\beta_1 = \frac{1}{8}, \beta_2 = \frac{1}{16}, \beta_3 = \frac{1}{10}$$

And the condition (H_1) is verified as $0 \leq \beta_1 M_1 + \beta_2 + \beta_3 \leq \frac{9}{40}$

$$0 \leq M_1(\beta_1 M_1 + \beta_2 + \beta_3) < \frac{9}{80},$$

therefore $x^{''} = 0$ and $y^{''} = \ln(1+t)$ are lower and upper solution of (1), satisfy the condition

$$\begin{cases} y(t) = \int_0^1 G_1(t,s)\ln(1+t)ds \\ y^{'}(t) = \int_0^1 v(s)ds \\ y^{'}(0) = 0, y(0) = \lambda u(1) \\ y^{''}(0) \geq 0, y^{''}(1) \geq \alpha y^{''}(\eta) \end{cases}$$

$y^{(4)}(t) = -\frac{1}{(1+t)^2}$ we have

$$\begin{aligned} \frac{1}{(1+t)^2} &\geq \frac{1}{8} + \frac{1}{16} + \frac{1}{10} \\ &\geq \frac{e^{-t}\sin y(t)}{8(t+1)} + \frac{e^{-y^{'}(t)}}{16\sqrt{t+1}} + \frac{t}{10(y^{''}(t)+1)} \end{aligned}$$

Then
$y^{(4)}(t) + f(t, y(t), y^{'}(t), y^{''}(t)) \leq 0,\ t \in [0,1]$
By theorem (3. 1), the BVP(4. 1) has a solution such that
$x^{''} \leq u^{''} \leq y^{''}$ where $x^{''} = \mu_0 = 0$ and $y^{''} = \omega_0 = \ln(1+t)$

Example 4.2. Consider the following BVP

$$\begin{cases} u^{(4)}(t) + \frac{t}{4(1+t)}\sin u(t) + \frac{u^{'}(t)}{10(1+u^{'}(t))} - \frac{t^2}{8(t^2+1)}e^{-u^{''}(t)} = 0,\ t \in [0,1], \\ u^{'}(0) = 0, u(0) = \lambda u(1),\ u^{''}(0) = 0, u^{''}(1) = \alpha u^{''}(\eta) \end{cases}$$

Since

$$f(t,u,v,w) = \frac{t}{4(1+t)}\sin u + \frac{v}{10(1+v} - \frac{t^2}{8(t^2+1)}e^{-w} = 0,\ t \in [0,1]$$

It is easy to verify that the condition $(H_2)$is fulfilled with

$$\beta_1 = \frac{1}{4}, \beta_2 = \frac{1}{10}, \beta_3 = \frac{1}{8}$$

And the condition (H_1) is verified as $0 \leq \beta_1 M_1 + \beta_2 + \beta_3 \leq \frac{7}{20}$
$0 \leq M_1(\beta_1 M_1 + \beta_2 + \beta_3) < \frac{7}{40}$. Therefor $x^{''} = 0$ and $y(t) = -\frac{t^5}{60} - \frac{t^4}{24} + \frac{5t^3}{6} + \frac{93}{120(\lambda-1)}$ are lower and upper solution of (1), satisfy the condition

$$\begin{cases} y^{'}(0) = 0, y(0) = \lambda u(1) \\ y^{''}(0) \geq 0, y^{''}(1) = \frac{25}{6} \geq \alpha y^{''}(\eta) = \alpha(\frac{-\eta^3}{3} - \frac{-\eta^2}{2} + 5\eta) \end{cases}$$

Then $y^{(4)}(t) + f(t, y(t), y^{'}(t), y^{''}(t)) \leq 0$, $t \in [0, 1]$
By theorem (3. 1), the BVP(4. 2) has a solution such that
$x^{''} \leq u^{''} \leq y^{''}$ where $x^{''} = \mu_0 = 0$ and $y^{''} = \omega_0 = \frac{-t^3}{3} - \frac{-t^2}{2} + 5t$

Acknowledgments

The authors would like to thank the anonymous referees for their valuable suggestions.

References

[1] H. Ammann, Fixed point equations and nonlinear eigenvalue problems in ordered Banach spaces, *SIAM Rev.*, **18** (1976), 620-709.

[2] L.J. Guo, J.P. Sun, Y.H. Zhao, Existence of positive solution for nonlinear third-order three-point boundary value problem, *Nonlinear Anal.*, **68** (2008), 3151-3158.

[3] Jian-Ping Sun, Qiu-Yan Ren, Ya-Hong Zhao, The upper and lower solution method for nonlinear third-order three-point boundary value problem, *J. Math. Anal. Appl.*, **26** (2010), 1-8.