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SUBSCHEMES OF A VERONESE EMBEDDING OF THE PLANE WHOSE LINEAR SPAN MEET

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Abstract: Let $X_{2,d} \subset \mathbb{P}^{\binom{d+2}{2}-1}$ be the order d Veronese embedding of $\mathbb{P}^2$. We give a necessary and sufficient condition for the existence of schemes $Z, W \subset X_{2,d}$ and $P \in \langle Z \rangle \cap \langle W \rangle$ such that $Z \cap W = \emptyset$ and P is not contained in the linear span of a scheme strictly contained in either Z or W.

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Let $X \subseteq \mathbb{P}^n$ be an integral and non-degenerate variety defined over an algebraically closed field $\mathbb{K}$. For any closed subscheme E of a projective space $\mathbb{P}^n$ let $\langle E \rangle$ denote its linear span, i.e. the intersection of all hyperplanes containing E, with the convention $\langle E \rangle = \mathbb{P}^n$ if there is no such hyperplane. If E is zero-dimensional, then $\dim(\langle E \rangle) \le \min\{n, \deg(E) - 1\}$ and the condition " $\dim(\langle E \rangle) = \deg(E) - 1$ " is equivalent to " $\langle F \rangle \ne \langle E \rangle$ for every proper subscheme F of E ", i.e. to the linear independence of E.

For any zero-dimensional scheme $Z \subset \mathbb{P}^n$ let $\ll Z \gg$ denote the set of all $P \in \langle Z \rangle$ such that $P \notin \langle Z' \rangle$ for any subscheme $Z' \subsetneq Z$. If either $Z = \emptyset$ of Z is linearly dependent, then $\ll Z \gg = \emptyset$. The converse is not always true (e.g. take a double point of $\mathbb{P}^n$), but it is true for a curvilinear subscheme, because a curvilinear zero-dimensional scheme has only finitely many subschemes.

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For all positive integers m, d let $j_{m,d} : \mathbb{P}^m \to \mathbb{P}^n$, $n := \binom{m+d}{m} - 1$, denote the d-Veronese embedding of $\mathbb{P}^m$. Set $X_{m,d} := j_{m,d}(\mathbb{P}^m)$. Our starting point was [2]. Here we continue [1] in the case $m = 2$ and show that it is very restrictive to have $\ll Z \gg \cap \ll W \gg \neq \emptyset$ with $Z \neq W$ and $Z \cap W = \emptyset$.

Theorem 1. *Fix positive integers d, s, z, w such that*

$$s^2 \leq z + w \leq sd - s^2 + 3s \tag{1}$$

Fix zero-dimensional subschemes A, B of $\mathbb{P}^2$ such that $\deg(A) = z$, $\deg(B) = w$ and $A \cap B = \emptyset$. Assume $\ll j_{2,d}(A) \gg \cap \ll j_{2,d}(B) \gg \neq \emptyset$. Then either $z + w = sd - s^2 + 3s$ and $A \cup B$ is a complete intersection of a curve of degree s and a curve of degree $d - s + 3$ or there is an integer $t \in \{1, \ldots, s - 1\}$ such that $z + w = t(2d + 3 - t)/2 + 1$ and $A \cup B$ is contained in a curve of degree t.

Proof. Assume $\ll j_{2,d}(A) \gg \cap \ll j_{2,d}(B) \gg \neq \emptyset$. Since $\ll j_{2,d}(A) \gg$ and $\ll j_{2,d}(B) \gg$ are non-empty, the schemes $j_{2,d}(A)$ and $j_{2,d}(B)$ are linearly independent, i.e. $h^1(\mathbb{P}^2, \mathcal{I}_A(d)) = h^1(\mathbb{P}^2, \mathcal{I}_B(d)) = 0$. Since $j_{2,d}(A)$ and $j_{2,d}(B)$ are linearly independent and $A \cap B = \emptyset$, we have $\dim(\langle j_{2,d}(A)\rangle \cap \langle j_{2,d}(B)\rangle) = h^1(\mathbb{P}^2, \mathcal{I}_{A\cup B}(d)) - 1$. Since $\langle j_{2,d}(A)\rangle \cap \langle j_{2,d}(B)\rangle \neq \emptyset$, we get $h^1(\mathbb{P}^2, \mathcal{I}_{A\cup B}(d)) > 0$.

First assume $h^1(\mathbb{P}^2 \mathcal{I}_{A\cup B}(d)) \geq 2$. Since $A \neq B$ and $\ll j_{2,d}(A) \gg \cap \ll j_{2,d}(B) \gg \neq \emptyset$, we have $A \not\supseteq B$. Thus the linear independence of $j_{2,d}(B)$ implies the existence of $W' \subset j_{2,d}(B)$ such that $\deg(W') = \deg(B) - 1$ and $\langle j_{2,d}(A)\rangle \cap \langle W'\rangle = \langle j_{2,d}(A)\rangle \cap \langle j_{2,d}(B)\rangle$. Thus $\langle j_{2,d}(A)\rangle \cap \ll j_{2,d}(B) \gg = \emptyset$, contradiction.

Thus $h^1(\mathbb{P}^2, \mathcal{I}_{A\cup B}(d)) = 1$. Hence d is the maximal integer t such that we have $h^1(\mathbb{P}^2, \mathcal{I}_{A\cup B}(t)) > 0$. Since $\ll Z \gg \cap \ll W \gg \neq \emptyset$, we also get $h^1((\mathbb{P}^2 \mathcal{I}_U(d)) = 0$ for every $U \subsetneq A \cup B$. Since $\deg(A \cup B) = z + w$, we have $s^2 \leq \deg(A \cup B)$ and $d \geq s - 3 + (z + w)/s$. Thus either $z + w \equiv 0 \pmod{s}$ and $A \cup B$ is the complete intersection of a curve of degree s and a curve of degree $(z + w)/s$ or there are an integer t such that $0 < t < s$ and a scheme $F \subseteq A \cup B$ such that $t(d - t + 3) \leq \deg(F) \leq t(d + (5 - t)/2)$ and F is contained in a curve C of degree t ([3], Corollaire 2, with $\tau := d$ and $d := z + w$). In the latter case we take t minimal with this property. First assume $t \leq 3$. Since $h^0(C, \mathcal{O}_C(d)) = (d + 2)(d + 1)/2 - (d - t + 2)(d - t + 1)/2 = t(2d + 3 - t)/2 < \deg(F)$, we have $h^0(C, \mathcal{I}_{C\cap(A\cup B)}(d)) > 0$. Since $h^1(\mathbb{P}^2, \mathcal{I}_C(d - t)) = 0$, we get $h^1(\mathbb{P}^2, \mathcal{I}_{C\cap(A\cup B)}(d)) > 0$. Since $h^1((\mathbb{P}^2 \mathcal{I}_U(d)) = 0$ for every $U \subsetneq A \cup B$, we get $F = A \cup B$ and $A \cup B \subset C$. We also get $\deg(F) = h^0(C, \mathcal{O}_C(d)) + 1$ and hence $z + w = t(2d + 3 - t)/2 + 1$. □

Proposition 1. *Fix positive integer d, z, w, s such that $z + w = t(2d + 3 - t)/2 + 1$. Fix a smooth degree plane curve C and general $A, B \subset C$ such that $\deg(A) = z$, $\deg(B) = w$ and $A \cap B = \emptyset$. Then $\ll j_{2,d}(A) \gg \cap \ll j_{2,d}(B) \gg \neq \emptyset$*

Proof. Notice that $t(2d+3-t) \equiv 0 \pmod 2$. Since $h^0(C, \mathcal{O}_C(d)) = t(2d+3-t)/2$ and $A \cup B$ is general in C, we have $h^1(C, \mathcal{O}_C(d)(-A-B)) = 1$. Thus $h^1(\mathbb{P}^2, \mathcal{I}_{A\cup B}(d)) = 1$. Since $z > 0$, $w > 0$ and A and B are general, we also get $h^1(\mathbb{P}^2, \mathcal{I}_A(d)) = h^1(\mathbb{P}^2, \mathcal{I}_B(d)) = 0$. Since both A and B are curvilinear, we get $\ll j_{2,d}(A) \gg \neq \emptyset$ and $\ll j_{2,d}(B) \gg \neq \emptyset$. Since $h^1(\mathbb{P}^2, \mathcal{I}_{A\cup B}(d)) = 1$, the set $\langle j_{2,d}(A)\rangle \cap \langle j_{2,d}(B)\rangle$ is a unique point, P. Since $A \cup B$ is general inside the irreducible curve C, we have $h^1(C, \mathcal{O}_C(d)(-U)) = 0$ for every $U \subsetneqq A \cup B$. Thus $h^1(\mathbb{P}^2, \mathcal{I}_U(d)) = 0$ for every $U \subsetneqq A \cup B$. Thus $P \in \ll j_{2,d}(A) \gg \cap \ll j_{2,d}(B) \gg$. □

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