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CHARACTERIZING THE SOLUTION SET OF UNCERTAIN POLYNOMIAL SYSTEMS

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Abstract: This paper addresses the estimation of the set of admissible solutions of uncertain polynomial systems with coefficients depending polynomially on an uncertain vector constrained in a polytope. This is a problem relevant to several disciplines, met for instance when investigating the steady states of dynamical systems. It is shown that an outer estimate with fixed shape of this set can be obtained by solving a convex optimization problem with LMI constraints. Then, a necessary and sufficient condition is provided for establishing the tightness of this estimate. Lastly, it is shown how the proposed approach can be extended in order to address the computation of the minimum volume outer estimate in a certain given class.

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1. Introduction

It is well known that solving systems of polynomial equations is important in numerous disciplines, for instance in order to determine the steady states of dynamical systems [8]. Often, such systems are not exactly known due to the presence of uncertainties in their coefficients, which means that the determination of the solutions should be repeated for all admissible values of the uncertainty. However, this is generally not practicable because solving system of polynomial equations is a nontrivial problem, see for instance [9], and because the set of admissible values of the uncertainty is typically not finite. Therefore, it appears necessary to provide concise descriptions of the set of admissible solutions, for instance through outer approximations of simple shape.

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This paper addresses the estimation of the set of admissible solutions of uncertain polynomial systems with coefficients depending polynomially on an uncertain vector constrained in a polytope. First, the determination of outer estimates with fixed shape is considered, showing that an upper bound can be obtained by solving an eigenvalue problem (EVP), which belongs to the class of convex optimization problems with linear matrix inequality (LMI) constraints [1]. Then, a necessary and sufficient condition for establishing the tightness of the found estimate is provided, which typically requires linear algebra operations. Lastly, it is shown how the proposed approach can be extended in order to address the computation of the minimum volume outer estimate among the sublevel sets of polynomials of a given degree. Some numerical examples illustrate the use and potentialities of the proposed approach. This paper extends our methodology for solving polynomial systems via LMIs proposed in [5, 3] to the case of uncertain polynomial systems.

The paper is organized as follows. Section 2 provides the preliminaries. Section 3 addresses the computation of outer estimates with fixed shape. The computation of outer estimates with variable shape is considered in Section 4. Section 5 reports some examples. Lastly, Section 6 concludes the paper.

2. Preliminaries

The notation is as follows: $\mathbb{N}, \mathbb{R}$: natural number set (including 0) and real number set; 0_n: origin of $\mathbb{R}^n$; $\mathbb{R}^n_0$: $\mathbb{R}^n \setminus \{0_n\}$; I_n: $n \times n$ identity matrix; A': transpose of the vector/matrix A; $A > 0$ ($A \geq 0$): symmetric positive definite (semidefinite) matrix A; $\mathrm{conv}(\mathcal{S})$: convex hull of the elements in the set $\mathcal{S}$; $\mathrm{vol}(\mathcal{S})$: volume of the set $\mathcal{S}$; $\|x\| = \sqrt{x'x}$ with $x \in \mathbb{R}^n$; $x^y = x_1^{y_1} \cdots x_n^{y_n}$ with $x, y \in \mathbb{R}^n$; s.t.: with respect to, subject to.

We consider the uncertain polynomial system

$$\begin{cases} f(x,\theta) = 0_n \\ \theta \in \Theta \end{cases} \tag{1}$$

where $x \in \mathbb{R}^n$, $\theta \in \mathbb{R}^q$ represents the uncertainty, $f : \mathbb{R}^n \times \mathbb{R}^q \to \mathbb{R}^n$ is polynomial, and Θ is a bounded convex polytope expressed as

$$\Theta = \mathrm{conv}\left(\left\{\theta^{(1)}, \ldots, \theta^{(r)}\right\}\right) \tag{2}$$

where $\theta^{(1)}, \ldots, \theta^{(r)} \in \mathbb{R}^q$ are given vectors, and $\mathrm{conv}(\cdot)$ denotes the convex hull. The set of admissible solutions of (1) is defined as

$$\mathcal{E} = \{x \in \mathbb{R}^n : \ f(x,\theta) = 0_n \text{ for some } \theta \in \Theta\}. \tag{3}$$

The first problem that we address is the determination of outer estimates of $\mathcal{E}$ of the form

$$\mathcal{G}(\gamma) = \{x \in \mathbb{R}^n : \; g(x) \leq \gamma\} \tag{4}$$

where $g(x)$ is a given polynomial of degree $2d_g$ and $\gamma \in \mathbb{R}$. In particular, the problem consists of estimating the smallest outer estimate of $\mathcal{E}$ with fixed shape defined by $g(x)$, which is denoted by $\mathcal{G}(\gamma^*)$ where γ^* is the solution of

$$\gamma^* = \inf_{\gamma \geq 0} \gamma \;\; \text{s.t.} \;\; \mathcal{E} \subseteq \mathcal{G}(\gamma). \tag{5}$$

The second problem that we address is the estimation of $\mathcal{E}$ with variable shape sets, specifically we consider the determination of the minimum volume outer estimate among the sublevel sets of polynomials of a given degree, i.e.

$$\mathcal{G}^* = \{x \in \mathbb{R}^n : \; g^*(x) \leq 1\} \tag{6}$$

where $g^*(x)$ is the solution of

$$g^*(\cdot) = \arg \inf_{g(\cdot) \in \mathcal{P}(n, 2d_g)} \text{vol}(\mathcal{G}(1)) \;\; \text{s.t.} \;\; \mathcal{E} \subseteq \mathcal{G}(1) \tag{7}$$

and $\mathcal{P}(n, 2d_g)$ is the set of polynomials in n variables of degree $2d$.

2.1. Representation of Polynomials

Before proceeding we briefly introduce a key tool that will be exploited in the sequel. For $x \in \mathbb{R}^n$, let $p(x)$ be a polynomial of degree $2d$. Let $x_{pol}^{\{d\}} \in \mathbb{R}^{\sigma(n,d)}$ be a vector containing all monomials of degree not greater than d in x, where $\sigma(n, d)$ is the number of such monomials:

$$\sigma(n, d) = \frac{(n + d)!}{n!d!}. \tag{8}$$

Then, $p(x)$ can be expressed via the square matrix representation (SMR) also known as Gram matrix method [7, 6] as

$$p(x) = x_{pol}^{\{d\}'} \left(P + L(\alpha)\right) x_{pol}^{\{d\}} \tag{9}$$

where P is a symmetric matrix, $L(\alpha)$ is a linear parametrization of the subspace

$$\mathcal{L} = \left\{ L = L' : \; x_{pol}^{\{d\}'} L x_{pol}^{\{d\}} = 0 \right\}, \tag{10}$$

and α is a free vector with size equal to the dimension of $\mathcal{L}$.

Homogeneous polynomials can be represented with a more compact SMR. Specifically, let $h(x)$ be a homogeneous polynomial of degree $2d$, and let $x_{hom}^{\{d\}} \in \mathbb{R}^{\sigma(n-1,d)}$ be a vector containing all monomials of degree d in x. Then, $h(x)$ can be expressed via the SMR as

$$h(x) = x_{hom}^{\{d\}'} \left(H + L(\alpha)\right) x_{hom}^{\{d\}} \tag{11}$$

where H and $L(\alpha)$ are defined analogously to the previous case.

The SMR is useful because it allows one to investigate positivity of polynomials. Indeed, one can establish whether a polynomial is a sum of squares of polynomials (SOS) by solving a convex optimization problem with LMIs. Specifically, $p(x)$ (resp., $h(x)$) is SOS if and only if there exists α such that

$$P + L(\alpha) \geq 0 \quad (\text{resp., } H + L(\alpha) \geq 0) \tag{12}$$

which is an LMI feasibility test. LMI feasibility tests can be checked by solving a convex optimization problem, see for instance [1]. See also [6, 4] for details on SOS polynomials.

3. Fixed Shape Estimate

Our starting point for establishing whether $\mathcal{G}(\gamma) \in \mathcal{E}$ is Stengle's Positivstellensatz [11]. Specifically, let us suppose first that $f(x,\theta)$ is independent on θ, and denote with $f(x)$ this function. One has that, if there exists a vector of polynomials $u(x)$ such that

$$\gamma - g(x) + u(x)' f(x) \geq 0 \;\; \forall x \in \mathbb{R}^n, \tag{13}$$

then

$$\mathcal{G}(\gamma) \supseteq \left\{ x \in \mathbb{R}^n : \; f(x) = 0_n \right\}. \tag{14}$$

This idea can be extended to the presence of uncertainties by introducing a parameter-dependent version of (13). Specifically, if there exists a vector of polynomials $u(x,\theta)$ such that

$$\gamma - g(x) + u(x,\theta)' f(x,\theta) \geq 0 \;\; \forall x \in \mathbb{R}^n \; \forall \theta \in \Theta \tag{15}$$

then one has that $\mathcal{G}(\gamma) \in \mathcal{E}$.

Let us observe, however, that (15) requires to check the non-negativity of a polynomial in x for all θ in Θ. One way to overcome this problem can be to introduce other unknown multipliers similar to $u(x,\theta)$ for taking into account the structure of Θ. Another way that does not require the introduction of other unknown multipliers is the following.

Let us express a generic θ in Θ as

$$\theta = \sum_{i=1}^{r} \phi_i \theta^{(i)} \tag{16}$$

where $\phi = (\phi_1, \ldots, \phi_r)'$ is a vector in the simplex Φ given by

$$\Phi = \left\{ \phi \in \mathbb{R}^r : \sum_{i=1}^{r} \phi_i = 1, \ \phi_i \geq 0 \right\}. \tag{17}$$

Let us denote with d_1 and d_2 the degree of $f(x, \theta)$ in x and θ, respectively. By substituting the expression of θ in (16) in $f(x, \theta)$ and weighting each monomial by a suitable power of $\phi_1 + \ldots + \phi_r$, we obtain a polynomial $h(x, \phi)$ such that:

- $h(x, \phi) = f(x, \theta)$ for all $\phi \in \Phi$ and θ given by (16);
- $h(x, \phi)$ is polynomial of degree d_1 in x and homogeneous polynomial of degree d_2 in ϕ.

Next, let us introduce the notation

$$\mathrm{sq}(\phi) = \left(\phi_1^2, \ldots, \phi_r^2\right)' \tag{18}$$

and the polynomial

$$a(\phi) = \left(\sum_{i=1}^{r} \phi_i \right)^{d_2}. \tag{19}$$

The following result provides a method to establish whether (15) holds.

Theorem 1. *Suppose that there exists $u(x, \phi)$ polynomial in x and homogeneous polynomial in θ such that*

$$z(x, \mathrm{sq}(\phi)) \geq 0 \ \ \forall x \in \mathbb{R}^n \ \forall \phi \in \mathbb{R}^r \tag{20}$$

where

$$z(x, \phi) = (\gamma - g(x)) a(\phi) + u(x, \phi)' h(x, \phi). \tag{21}$$

Then, $\mathcal{G}(\gamma) \in \mathcal{E}$.

Proof. First, let us observe that (20) holds if and only if

$$z(x,\phi) \geq 0 \quad \forall x \in \mathbb{R}^n \ \forall \phi \in \Phi. \tag{22}$$

In fact, (20) clearly implies (22) since in the former ϕ is free while in the latter ϕ belongs to Φ. Then, in order to show that (22) implies (20), let us suppose for contradiction that (22) holds but (20) does not. This means that there exist x and ϕ such that

$$z(x, \mathrm{sq}(\phi)) < 0. \tag{23}$$

Clearly, $\phi \neq 0_r$ since $z(x, \mathrm{sq}(0_r)) = 0$, and hence one can define $\hat{\phi} = \mathrm{prj}(\phi)$ where

$$\mathrm{prj}(\phi) = \frac{\mathrm{sq}(\phi)}{\|\phi\|^2}. \tag{24}$$

It follows that $\hat{\phi} \in \Phi$ and

$$z(x,\hat{\phi}) = \frac{1}{\|\phi\|^{2d_2}} z(x, \mathrm{sq}(\phi)) < 0 \tag{25}$$

which contradicts (22).

Second, since (22) holds, one has that

$$\gamma - g(x) + u(x,\phi)' h(x,\phi) \geq 0 \quad \forall x \in \mathbb{R}^n \ \forall \phi \in \Phi \tag{26}$$

since $a(\phi) = 1$ for all $\phi \in \Phi$. Moreover, for all $\phi \in \Phi$ there exists $\theta \in \Theta$ such that $h(x,\phi) = f(x,\theta)$ and vice versa, and therefore one has that (26) is equivalent to (15). □

The condition in Theorem 1 can be investigated via LMIs by exploiting the idea of parameter-dependent SOS polynomials introduced in [2]. Specifically, let us define the notation

$$\Delta(A,b,c) = (b \otimes c)' A (b \otimes c) \tag{27}$$

where b and c are vectors and A is a matrix of suitable dimension. Let us express $u(x,\phi)$ as

$$u(x,\phi) = U\left(x_{pol}^{\{d_x\}} \otimes \phi_{hom}^{\{d_\phi\}}\right) \tag{28}$$

for some integers d_x (degree in x) and d_ϕ (degree in ϕ), and for some matrix U. Let $H(U)$, V and W be any symmetric matrices satisfying

$$u(x, \mathrm{sq}(\phi))' h(x, \mathrm{sq}(\phi)) = \Delta\left(H(U), x_{pol}^{\{d_3\}}, \phi_{hom}^{\{d_4\}}\right) \tag{29}$$

$$
\begin{aligned}
a(\mathrm{sq}(\phi))g(x) &= \Delta\left(V, x_{pol}^{\{d_3\}}, \phi_{hom}^{\{d_4\}}\right) &(30)\\
a(\mathrm{sq}(\phi)) &= \Delta\left(W, x_{pol}^{\{d_3\}}, \phi_{hom}^{\{d_4\}}\right) &(31)
\end{aligned}
$$

where the integers d_3 and d_4 are given by

$$
d_3 = \left\lceil \frac{d_1 + d_x}{2} \right\rceil - d_g, \quad d_4 = d_2 + d_\phi. \tag{32}
$$

Lastly, let $N(\alpha)$ be any linear parametrization of

$$
\mathcal{N} = \left\{ N = N' : \Delta\left(N, x_{pol}^{\{d_3\}}, \phi_{hom}^{\{d_4\}}\right) = 0 \right\} \tag{33}
$$

where α is a free vector. The following result shows how the condition in Theorem 1 can be investigated via LMIs, in particular providing an upper bound of γ^* in (5) via an EVP.

Theorem 2. *Define the EVP*

$$
\gamma^\# = \inf_{\gamma, U, \alpha} \gamma \quad s.t. \quad H(U) - V + \gamma W + N(\alpha) \geq 0. \tag{34}
$$

Then, $\gamma^\# \geq \gamma^*$.

Proof. Let us pre- and post- multiply the LMI in (34) by $\left(x_{pol}^{\{d_3\}} \otimes \phi_{hom}^{\{d_4\}}\right)'$ and $x_{pol}^{\{d_3\}} \otimes \phi_{hom}^{\{d_4\}}$, respectively. It follows that

$$
\begin{aligned}
0 &\geq \Delta\left(H(U) - V + \gamma W + N(\alpha), x_{pol}^{\{d_3\}}, \phi_{hom}^{\{d_4\}}\right) \\
&= (\gamma - g(x))a(\mathrm{sq}(\phi)) + u(x, \mathrm{sq}(\phi))'h(x, \mathrm{sq}(\phi))
\end{aligned} \tag{35}
$$

for all $x \in \mathbb{R}^n$ and $\phi \in \mathbb{R}^r$, where it has been taken into account that

$$
\Delta\left(N(\alpha), x_{pol}^{\{d_3\}}, \psi_{hom}^{\{d_4\}}\right) = 0. \tag{36}
$$

Hence, (20) holds, and from Theorem 1 one has that $\mathcal{G}(\gamma) \in \mathcal{E}$ whenever the LMI in (34) holds. Therefore, $\gamma^\# \geq \gamma^*$. □

A question that naturally arises concerns the tightness of the found upper bound: is $\gamma^\# = \gamma^*$? The following result provides a necessary and sufficient condition for answering to this question.

Theorem 3. *Let U and α be optimal values in* (34), *and define*

$$
J = H(U) - V + \gamma^\# W + N(\alpha). \tag{37}
$$

Then, $\gamma^{\#} = \gamma^*$ *if and only if there exist* $x \in \mathbb{R}^n$ *and* $\psi \in \mathbb{R}_0^r$ *such that*

$$\begin{cases} \left(x_{pol}^{\{d_3\}} \otimes \psi_{hom}^{\{d_4\}}\right) \in \ker(J) \\ f(x,\theta) = 0_n \\ g(x) = \gamma^{\#} \end{cases} \tag{38}$$

where θ *is as in* (16) *with* $\phi = \mathrm{prj}(\psi)$*.*

Proof. "$\Leftarrow$" Suppose that (38) holds, and observe that $\mathrm{prj}(\psi) \in \Phi$ which means that $x \in \mathcal{E}$. Moreover, x satisfies $g(x) = \gamma^{\#}$, hence implying that $\gamma^{\#} \leq \gamma^*$. Also, $\gamma^{\#} \geq \gamma^*$ from Theorem 2, and hence one can conclude that $\gamma^{\#} = \gamma^*$.

"$\Rightarrow$" Suppose that $\gamma^{\#} = \gamma^*$. Let $x \in \mathcal{E}$ be a tangent point between $\mathcal{E}$ and $\mathcal{G}(\gamma^*)$, and let $\phi \in \Phi$ be an uncertain vector corresponding to x. Since $\phi_i \geq 0$ for all $i = 1, \ldots, r$, we can define $\psi = \mathrm{jrp}(\phi)$ where

$$\mathrm{jrp}(\phi) = \frac{\left(\sqrt{\phi_1}, \ldots, \sqrt{\phi_r}\right)'}{\sum_{i=1}^{r} \sqrt{\phi_i}}. \tag{39}$$

Let us observe that $\mathrm{sq}(\psi) = c\phi$ for some $c > 0$, and that

$$h(x, \mathrm{sq}(\psi)) = c^{d_2} h(x,\phi) = c^{d_2} f(x,\theta) = 0. \tag{40}$$

Also, $J \geq 0$ since J is the left-hand side of the LMI in (34) evaluated for the optimal values of the EVP. Let us pre- and post- multiply J by $\left(x_{pol}^{\{d_3\}} \otimes \psi_{hom}^{\{d_4\}}\right)'$ and $x_{pol}^{\{d_3\}} \otimes \psi_{hom}^{\{d_4\}}$, respectively. It follows that

$$\begin{array}{rcl} 0 & \leq & \Delta\left(J, x_{pol}^{\{d_3\}}, \psi_{hom}^{\{d_4\}}\right) \\ & = & (\gamma^* - g(x)) a(\mathrm{sq}(\psi)) + u^{\#}(x, \mathrm{sq}(\psi))' h(x, \mathrm{sq}(\psi)) \\ & = & 0 \end{array} \tag{41}$$

since $g(x) = \gamma^*$ and $h(x, \mathrm{sq}(\psi)) = 0$. From $J \geq 0$ one has that $x_{pol}^{\{d_3\}} \otimes \psi_{hom}^{\{d_4\}}$ must belong to the null space of J. Moreover,

$$\mathrm{prj}(\psi) = \mathrm{prj}(\mathrm{jrp}(\phi)) = \phi \tag{42}$$

and hence (38) holds. □

The condition of Theorem 3 requires to check the existence of vectors $x \in \mathbb{R}^n$ and $\psi \in \mathbb{R}_0^r$ satisfying (38). From the first condition in (38), it follows that these vectors have to satisfy

$$x_{pol}^{\{d_3\}} \otimes \psi_{hom}^{\{d_4\}} = J_1 y \tag{43}$$

where J_1 is a matrix whose columns form a base of the null space of J, and y is a vector of suitable dimension.

A way to verify the existence of vectors $x \in \mathbb{R}^n$ and $\psi \in \mathbb{R}_0^r$ fulfilling (43) for some y and determine them is as follows. First, let us observe that the vector $x_{pol}^{\{d_3\}} \otimes \psi_{hom}^{\{d_4\}}$ has the structure

$$x_{pol}^{\{d_3\}} \otimes \psi_{hom}^{\{d_4\}} = \begin{pmatrix} \psi_{hom}^{\{d_4\}} \\ x_1 \psi_{hom}^{\{d_4\}} \\ \vdots \\ x_n^{d_3} \psi_{hom}^{\{d_4\}} \end{pmatrix}. \tag{44}$$

Let $J_1^{(0)}, \ldots, J_1^{(\sigma(n,d_3)}$ be the sub-matrices of J_1, with $J_1^{(i)}$ containing the rows of J_1 from the $(i-1)\sigma(r-1,d_4)+1$-th row to the $i\sigma(r-1,d_4)$-th row. It follows from (43) that

$$\psi_{hom}^{\{d_4\}} = J_1^{(0)} y. \tag{45}$$

The vectors ψ and y satisfying (45) can be found with the approach proposed in [5, 6]. Once ψ and y have been determined, the vector x fulfilling (43) can directly be read from (44) according to

$$x_i \psi_{hom}^{\{d_4\}} = J_1^{(i)} y. \tag{46}$$

4. Variable Shape Estimate

Here we show how the methodology proposed in the previous section can further be elaborated in order to search for the minimum volume estimate of $\mathcal{E}$ defined by (6)–(7).

A candidate of $g^*(x)$ can be found from Theorem 2 by computing $\gamma^\#$ for different $g(x)$ in $\mathcal{P}(n, 2d_g)$ and taking the one that minimizes the volume of $\mathcal{G}(\gamma^\#)$. This amounts to solving

$$\begin{aligned} \mu^\# \quad = \quad & \inf_{g(\cdot)\in\mathcal{P}(n,2d_g)} \operatorname{vol}(\mathcal{G}(\gamma)) \\ & \text{s.t.} \;\; \exists U, \gamma, \alpha : \;\; H(U) - V + \gamma W + N(\alpha) \geq 0. \end{aligned} \tag{47}$$

Let us consider first the case $d_g = 1$, i.e. quadratic $g(x)$. It follows that

$$g(x) = (x - x_0)' G (x - x_0) \tag{48}$$

for some $x_0 \in \mathbb{R}^n$ and symmetric G. Clearly, $G > 0$ since we are looking for a bounded outer estimate of $\mathcal{E}$. The sublevel set $\mathcal{G}(1)$ is hence an ellipsoid, and its volume is given by

$$\mathrm{vol}(\mathcal{G}(1)) = \frac{\kappa_1}{\sqrt{\det(G)}} \tag{49}$$

where $\kappa_1 > 0$ is a constant depending on n. Let us suppose that x_0 is fixed, and let $V(G)$ be any symmetric matrix function of suitable dimension satisfying

$$(x - x_0)'G(x - x_0)a(\mathrm{sq}(\phi)) = \Delta\left(V(G), x_{pol}^{\{d_3\}}, \phi_{hom}^{\{d_4\}}\right) \tag{50}$$

where $a(\phi)$ is defined as in (19). Moreover, let us observe that, with $G > 0$ and $\lambda \in \mathbb{R}$, the condition

$$\det(G) > \lambda^{\kappa_2} \tag{51}$$

(where $\kappa_2 > 0$ is a constant depending on n) can equivalently be written as

$$\exists s:\ M(G, \lambda, s) > 0 \tag{52}$$

where s is a vector of variables, and $M(G, \lambda, s)$ is a suitable symmetric linear matrix function in G, λ and s (see for instance [10]).

Theorem 4. *Define the EVP*

$$\begin{array}{rcl} \lambda^{\#} & = & \sup\limits_{G,U,\alpha,\lambda} \ \lambda \\ & & s.t. \left\{ \begin{array}{l} H(U) - V(G) + \gamma W + N(\alpha) \geq 0 \\ M(G, \lambda, s) > 0 \\ G > 0 \end{array} \right. \end{array} \tag{53}$$

and let $G^{\#}$ be any optimal value of G in (53). Then, $\mathcal{E} \subseteq \mathcal{G}^{\#}$ where

$$\begin{array}{rcl} \mathcal{G}^{\#} & = & \left\{x \in \mathbb{R}^n :\ g^{\#}(x) \leq 1\right\} \\ g^{\#}(x) & = & (x - x_0)'G^{\#}(x - x_0). \end{array} \tag{54}$$

Moreover,

$$\mathrm{vol}(\mathcal{G}^{\#}) = \mu^{\#}. \tag{55}$$

Proof. Let us suppose that the LMIs in (53) are fulfilled. Similarly to the proof of Theorem 2, the first LMI implies that $\mathcal{E} \subseteq \mathcal{G}^{\#}$. Moreover, from the second and third LMIs one has that $\det(G) > \lambda^{\kappa_2}$, and hence for any optimal solution of (53) it follows that

$$\det(G^{\#}) = \left(\lambda^{\#}\right)^{\kappa_2}.$$

Therefore, $\mathrm{vol}(\mathcal{G}^{\#}) = \mu^{\#}$. □

Theorem 4 provides a candidate for the minimum volume estimate of $\mathcal{E}$ with ellipsoidal shape via an EVP. In particular, this candidate is the minimum volume estimate achievable by using Theorem 2 with different shape functions $g(x)$ in $\mathcal{P}(n, 2)$.

Theorem 4 can also be used in the case $d_g > 1$, i.e. with shape functions of degree greater than 2. In this case $g(x)$ can be written as

$$g(x) = (x - x_0)_{pol}^{\{d_g\}'} G(x - x_0)_{pol}^{\{d_g\}} \tag{56}$$

for some $x_0 \in \mathbb{R}^n$ and $G = G' \in \mathbb{R}^{\sigma(n,d_g)\times\sigma(n,d_g)}$ with G having zeros on the first row and on the first column (since for similarity with the case $d_g = 1$ we want that $g(x)$ be a positive definite function shifted in x_0). The solution obtained by solving (53), however, is only an approximation of the solution of the problem (47) in this case since (49) does not hold for $d_g > 1$.

5. Examples

In this section we present some illustrative examples of the proposed approach. The LMI problems were solved by using the toolbox SeDuMi for Matlab. The parameter-dependent polynomial multiplier $u(x, \phi)$ is built as in (28) with $d_x = 1$ and $d_\phi = 0$.

5.1. Example 1

Let us consider (1) with

$$f(x,\theta) = \begin{pmatrix} (2+\theta_1)x_1^2 - 2x_1x_2 + 3x_2^2 - x_2 - 2.5 - 1.5\theta_2 \\ x_1^2 + (2-\theta_1)x_1x_2 + x_2^2 - x_1 - 4.5 + 2.5\theta_2 \end{pmatrix}.$$

where $\Theta = [-1, 1]^2$. For this system, (16) can be written as

$$\theta = \begin{pmatrix} -1 & 1 & -1 & 1 \\ -1 & -1 & 1 & 1 \end{pmatrix} \phi.$$

Let us select $g(x) = \|x\|^2$. We find that an upper bound of γ^* is given by $\gamma^{\#} = 4.158$. Figure 1a shows the boundary of the found estimate $\mathcal{G}(\gamma^{\#})$ and solutions of (1) computed for 289 values of θ equally distributed in $[0, 1]^2$.

In order to establish whether $\gamma^{\#}$ is tight, we use Theorem 3, in particular (38) holds with $\psi^{\#} = (0.648, 0.000, 0.762, 0.000)'$ and $x^{\#} = (2.013, 0.324)'$. This implies that $\gamma^{\#} = \gamma^*$.

Figure 1b shows the smallest outer estimate with ellipsoidal shape $\mathcal{G}^{\#}$ centered in the origin obtained from Theorem 4. In particular, the optimal polynomial $g^{\#}(x)$ is given by

$$g^{\#}(x) = 0.250x_1^2 - 0.127x_1x_2 + 0.526x_2^2.$$

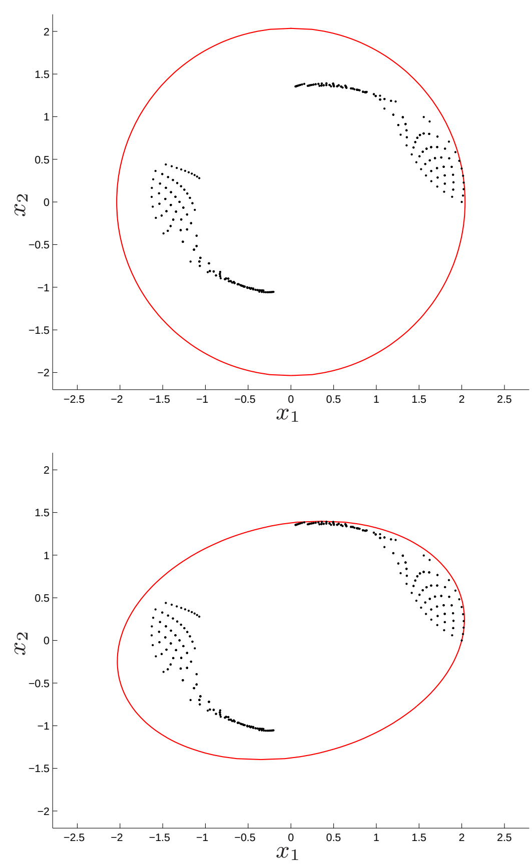

Figure 1: Example 1. (a) Boundary of the estimate $\mathcal{G}(\gamma^{\#})$ for $g(x) = \|x\|^2$ (red disc) and solutions of (1) for 289 values of θ equally distributed in $[0,1]^2$ (black dots). (b) Minimum volume estimate with ellipsoidal shape (red ellipse).

5.2. Example 2

Let us consider (1) with

$$f(x,\theta) = \begin{pmatrix} -x_1 + (1+3\theta)x_1^2 + x_1^3 - x_2^3 - x_3^3 - 1 \\ -x_2 + x_2^2 + x_1^3 + (5-4\theta)x_2^3 + x_3^3 - 1 \\ -(3-2\theta)x_3 - x_3^2 - x_1^3 + x_2^3 + 3x_3^3 - 1 \end{pmatrix}$$

where $\Theta = [0,1]$. For this system, (16) can be simply written as $\theta = \phi_1$. Let us select $g(x) = \|x\|^2$. We find that an upper bound of γ^* is given by $\gamma^\# = 21.863$.

By using Theorem 3 we conclude that $\gamma^\#$ is tight, in particular (38) holds with $\psi^\# = (0.912, 0.411)'$ and $x^\# = (-2.913, 2.752, -2.410)'$.

6. Conclusion

We have shown that an outer estimate with fixed shape of the set of admissible equilibrium points of polynomial systems with coefficients depending polynomially on an uncertain vector constrained in a polytope can be obtained by solving a convex optimization problem with LMI constraints. Moreover, we have provided a necessary and sufficient condition for establishing the tightness of this estimate. Lastly, it has been shown how the proposed approach can be extended in order to address the computation of the minimum volume outer estimate in a certain given class.

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