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ANALYSIS OF A DISCRETE MACKEY-GLASS COMPETITION SYSTEM WITH TIME DELAYS

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Abstract: In this paper, we propose a discrete Mackey-Glass competition model with time delays. We establish sufficient conditions for the permanence of the model and global attractivity of positive solution of the model. Finally, an example and its numerical simulation are showed for supporting our results.

AMS Subject Classification: 34K25, 39A11
Key Words: discrete model, Mackey-Glass model, time delay, permanence, global attractivity

1. Introduction

Investigate the dynamic behavior of the following discrete Mackey-Glass competition model with time delays:

$$\begin{cases} x_1(n+1) = x_1(n)\exp\Big[\alpha_1(n) + \dfrac{\beta_1(n)}{1+x_1^p(n-\tau_{10})} \\ \qquad -a_{11}(n)x_1(n-\tau_{11}) - a_{12}(n)x_2(n-\tau_{12})\Big], \\ x_2(n+1) = x_2(n)\exp\Big[\alpha_2(n) + \dfrac{\beta_2(n)}{1+x_2^p(n-\tau_{20})} \\ \qquad -a_{21}(n)x_1(n-\tau_{21}) - a_{22}(n)x_2(n-\tau_{22})\Big], \end{cases} \tag{1.1}$$

under the following assumptions.

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(H1) $\alpha_i(n),\ \beta_i(n),\ a_{ij}(n)\ (i,\ j = 1,\ 2)$ are real positive bounded sequences.

(H2) $\tau_{ij}\ (i = 1,\ 2,\ j = 0,\ 1,\ 2)$ are nonnegative integers.

(H3) p is a real positive constant.

Set $\tau = \max\{\tau_{ij} \mid i = 1, 2,\ j = 0, 1, 2\}$. We assume that system (1.1) satisfy the following initial conditions:

$$x_i(\theta) = \phi_i(\theta), \qquad \theta \in [-\tau,\ 0] \cap \mathbf{Z} \ \text{ and } \ \phi_i(\theta) > 0, \quad i = 1,\ 2. \tag{1.2}$$

It is clear that under assumptions (H1)-(H3) system (1.1) has a unique positive solution which satisfies initial conditions (1.2).

Recently, many discrete models have been proposed and studied [1]-[4]. For example, Kon [1] has investigated the discrete Kolmogorov systems. Liao et al. [2] have proposed a discrete multispecies general Gilpin-Ayala competition predator-prey model. They obtain the sufficient conditions for the permanence and global stability. A kind of nonautonomous discrete model of Plankton Allelopathy are discussed by Huo and Li [3]. They obtain the sufficient conditions for the global stability of the positive periodic solution. Many scholars [5]-[7] have proposed that the discrete time models governed by difference equations are more reasonable than the continuous ones when the populations have non-overlapping generations. Moreover discrete time models can also provide efficient computational models of continuous models for numerical simulations. It is well known that, compared to the continuous time systems, the discrete systems are more difficult to study.

Mackey-Glass's models are considered by many authors [8]-[10]. In [8], Wan and Wei have investigated the stability and occurrence of Hopf bifurcation, and obtained an explicit algorithm for determining the direction of the Hopf bifurcation and the stability of the bifurcating periodic solutions. In literature [9], [10], the authors have used Mackey-Glass electronic circuits with time-delayed feedback and demonstrated the dual synchronization of chaos experimentally and numerically. Our main purpose is to investigate the permanence and global attractivity of positive solutions of the model (1.1). Here, system (1.1) with initial conditions (1.2) is said to be permanent, if there exist positive constants m and M with $M \geq m$ such that for any solution $(x_1(n),\ x_2(n))$ of system (1.1)

$$m \leq \liminf_{n\to+\infty} x_i(n) \leq \limsup_{n\to+\infty} x_i(n) \leq M, \quad i = 1,\ 2.$$

2. Permanence

In this section we study the permanence of the system (1.1). In order to obtain the sufficient conditions for the permanence of system (1.1), we give following lemmas.

Now we set $f^u = \sup\{f(n) : n \in \mathbf{N}\}$ and $f^l = \inf\{f(n) : n \in \mathbf{N}\}$, for any bounded sequence $\{f(n)\}$. Let $X(n) = (x_1(n),\ x_2(n))$ be any solution of system (1.1) with initial conditions (1.2).

Lemma 2.1. (see [11]) *Assume that* $\big\{x(n)\big\}$ *satisfies* $x(n) > 0$ *and*

$$x(n+1) \leq x(n)\exp\big\{r(n)(1-ax(n))\big\} \quad \textit{for} \quad n \in [n_1,\ +\infty),$$

where a *is a positive constant. Then*

$$\limsup_{n\to+\infty} x(n) \leq \frac{1}{ar^u}\exp\big(r^u - 1\big).$$

Lemma 2.2. (see [11]) *Assume that* $\big\{x(n)\big\}$ *satisfies*

$$x(n+1) \geq x(n)\exp\big\{r(n)(1-ax(n))\big\} \quad \textit{for} \quad n \geq N_0,$$

$$\limsup_{n\to+\infty} x(n) \leq x^* \quad \textit{and} \quad x(N_0) > 0,$$

where a *is a constant such that* $ax^* > 1$ *and* $N_0 \in \mathbf{N}$*. Then*

$$\liminf_{n\to+\infty} x(n) \geq \frac{1}{a}\exp\big\{r^u(1-ax^*)\big\}.$$

Theorem 2.1. *Any solution* $X(n) = (x_1(n),\ x_2(n))$ *of system (1.1) with initial conditions (1.2) is positive and ultimately bounded and* $\limsup_{n\to+\infty} x_i(n) \leq M_i$*, where*

$$M_i = \frac{1}{a_{ii}^l}\exp\big\{(\alpha_i^u + \beta_i^u)(\tau_{ii}+1) - 1\big\}, \quad i = 1,\ 2.$$

Proof. It follows that $\phi_i(s) > 0$ ($s \in [-\tau,\ 0] \cap \mathbf{Z}$) that $x_i(n) > 0$ ($i = 1,\ 2$) for $n > 0$. From the first equation of system (1.1), we have $x_1(n+1) \leq x_1(n)\exp(\alpha_1(n)+\beta_1(n))$. Set $u_1(n) = \ln x_1(n)$, then $u_1(n+1) - u_1(n) \leq \alpha_1^u + \beta_1^u$ and therefore

$$\sum_{k=n-\tau_{11}}^{n-1}\big(u_1(k+1) - u_1(k)\big) \leq (\alpha_1^u + \beta_1^u)\tau_{11} \tag{2.1}$$

From (2.1), we have $u_1(n) - u_1(n-\tau_{11}) \le (\alpha_1^u + \beta_1^u)\tau_{11}$, Thus

$$x_1(n-\tau_{11}) \ge x_1(n)\exp\big(-(\alpha_1^u+\beta_1^u)\tau_{11}\big).$$

It follows from the first equation of system (1.1) that

$$\begin{aligned} x_1(n+1) &\le x_1(n)\exp\Big[\alpha_1^u+\beta_1^u - a_{11}^l\exp\big(-(\alpha_1^u+\beta_1^u)\tau_{11}\big)x_1(n)\Big] \\ &= x_1(n)\exp\bigg\{(\alpha_1^u+\beta_1^u)\Big[1-\frac{a_{11}^l\exp\big(-(\alpha_1^u+\beta_1^u)\tau_{11}\big)}{\alpha_1^u+\beta_1^u}x_1(n)\Big]\bigg\}. \end{aligned}$$

According to Lemma 2.1, we have

$$\limsup_{n\to+\infty} x_1(n) \le \frac{1}{a_{11}}\exp\big[(\alpha_1^u+\beta_1^u)(\tau_{11}+1)-1\big] \triangleq M_1.$$

Similarly, we can also prove that

$$\limsup_{n\to+\infty} x_2(n) \le \frac{1}{a_{22}}\exp\big[(\alpha_2^u+\beta_2^u)(\tau_{22}+1)-1\big] \triangleq M_2. \qquad \square$$

For the sake of convenience, we define

$$\begin{array}{rcl} \Theta_i^\epsilon & \triangleq & \alpha_i^l + \frac{\beta_i^l}{1+(M_i+\epsilon)^p} - a_{ii}^u(M_i+\epsilon) - a_{ij}^u(M_j+\epsilon), \\ \Lambda_i^\epsilon & \triangleq & \frac{a_{ii}^u\exp(-\tau_{ii}\Theta_i^\epsilon)}{\alpha_i^l+\frac{\beta_i^l}{1+(M_i+\epsilon)^p}-a_{ij}^u(M_j+\epsilon)}, \\ \Lambda_i & = & \lim_{\epsilon\to 0}\Lambda_i^\epsilon, \quad i\ne j, \quad i,\ j=1,\ 2. \end{array}$$

Theorem 2.2. *Suppose that the following inequality holds,*

$$\min\big\{\Lambda_1 M_1,\ \Lambda_2 M_2\big\} > 1, \tag{2.2}$$

where M_1, M_2 are defined in Theorem 2.1. Then for any solution $X(n) = (x_1(n),\ x_2(n))$ of system (1.1) with initial conditions (1.2), we have

$$\liminf_{n\to+\infty} x_i(n) \ge m_i, \quad i=1,\ 2,$$

where $m_i = \frac{1}{\Lambda_i}\exp\bigg\{\Big[\alpha_i^l + \frac{\beta_i^l}{1+(M_i+\epsilon)^p} - a_{ij}^u(M_j+\epsilon)\Big](1-\Lambda_i M_i)\bigg\}$,
$i\ne j, \quad i,\ j=1,\ 2.$

Proof. From condition (2.2), we have $\min\{\Lambda_1^\epsilon M_1,\ \Lambda_2^\epsilon M_2\} > 1$, where $\epsilon > 0$ is sufficiently small. In view of Theorem 2.1, there exists a integer $n_0 > 0$ such that $x_i(n) \le M_i + \epsilon,\ (i = 1,\ 2)$ for all $n \ge n_0$. It follows from the first equation of (1.1) for $n \ge n_0 + \tau$

$$\begin{aligned} x_1(n+1) &\ge x_1(n)\exp\Big[\alpha_1^l + \frac{\beta_1^l}{1+(M_1+\epsilon)^p} - a_{11}^u(M_1+\epsilon) - a_{12}^u(M_2+\epsilon)\Big] \\ &= x_1(n)\exp\big\{\Theta_1^\epsilon\big\}. \end{aligned}$$

Set $X_1(n) = \ln(x_1(n))$, then $X_1(n+1) - X_1(n) \ge \Theta_1^\epsilon$. Therefore

$$\sum_{k=n-\tau_{11}}^{n-1} \big[X_1(k+1) - X_1(k)\big] \ge \tau_{11}\Theta_1^\epsilon,$$

which implies $X_1(n-\tau_{11}) \le X_1(n) - \tau_{11}\Theta_1^\epsilon$, and thus

$$x_1(n-\tau_{11}) \le x_1(n)\exp(-\tau_{11}\Theta_1^\epsilon).$$

It follows from the first equation of system (1.1), we have

$$\begin{aligned} x_1(n+1) &\ge x_1(n)\exp\Big\{\alpha_1^l + \frac{\beta_1^l}{1+(M_1+\epsilon)^p} - a_{12}^u(M_2+\epsilon) \\ &\quad -a_{11}^u\exp(-\tau_{11}\Theta_1^\epsilon)x_1(n)\Big\} \\ &= x_1(n)\exp\Big\{\Big[\alpha_1^l + \frac{\beta_1^l}{1+(M_1+\epsilon)^p} - a_{12}^u(M_2+\epsilon)\Big] \\ &\quad \times(1-\Lambda_1^\epsilon x_1(n))\Big\} \end{aligned}$$

According to Lemma 2.2, we have

$$\liminf_{n\to+\infty} x_1(n) \ge \frac{1}{\Lambda_1^\epsilon}\exp\Big\{\Big[\alpha_1^l + \frac{\beta_1^l}{1+(M_1+\epsilon)^p} - a_{12}^u(M_2+\epsilon)\Big](1-\Lambda_1^\epsilon M_1)\Big\}.$$

Letting $\epsilon \to 0$, we obtain that $\liminf\limits_{n\to+\infty} x_1(n) \ge m_1$. Similarly, we can prove $\liminf\limits_{n\to+\infty} x_2(n) \ge m_2$. □

From Theorem 2.1 and Theorem 2.2, we can easily obtain the following result.

Theorem 2.3. *System (1.1) with initial conditions (1.2) is permanent provided that inequality* $\min\big\{\Lambda_1 M_1,\ \Lambda_2 M_2\big\} > 1$ *holds.*

3. Global Attractivity

Firstly, we introduce a definition and a lemma which will be useful to prove our main result.

Definition 3.1. Solution $(x_1^*(n),\ x_2^*(n))$ of (1.1) is said to be globally attractive if for any solution $(x_1(n),\ x_2(n))$ of (1.1), we have $\lim\limits_{n\to+\infty}\big[x_i(n) - x_i^*(n)\big] = 0,\ i = 1,\ 2.$

Lemma 3.1. For any two positive solution $(x_1^*(n),\ x_2^*(n))$ and $(x_1(n),\ x_2(n))$ of system (1.1), we have

$$\begin{aligned}
\ln\Big(\frac{x_i(n+1)}{x_i^*(n+1)}\Big) &= \ln\Big(\frac{x_i(n)}{x_i^*(n)}\Big) - a_{ii}(n)\big[x_i(n) - x_i^*(n)\big] - a_{ij}(n)\big[x_j(n-\tau_{ij}) \\
&\quad - x_j^*(n-\tau_{ij})\big] - \frac{\beta_i(n)\big[x_i^p(n-\tau_{i0}) - x_i^{*p}(n-\tau_{i0})\big]}{\big[1+x_i^p(n-\tau_{i0})\big]\big[1+x_i^{*p}(n-\tau_{i0})\big]} \\
&\quad + a_{ii}(n)\sum_{s=n-\tau_{ii}}^{n-1}\Big\{\Phi_i(s)\Big[\alpha_i(s) + \frac{\beta_i(s)}{1+x_i^{*p}(s-\tau_{i0})} \\
&\quad - a_{ii}(s)x_i^*(s-\tau_{ii}) - a_{ij}(s)x_j^*(s-\tau_{ij})\Big]\big(x_i(s) - x_i^*(s)\big) \\
&\quad + \Psi_i(s)x_i(s)\Big[\frac{\beta_i(n)\big[x_i^p(n-\tau_{i0}) - x_i^{*p}(n-\tau_{i0})\big]}{\big[1+x_i^p(n-\tau_{i0})\big]\big[1+x_i^{*p}(n-\tau_{i0})\big]} \\
&\quad - a_{ii}(s)\big(x_i(s-\tau_{ii}) - x_i^*(s-\tau_{ii})\big) \\
&\quad - a_{ij}(s)\big(x_j(s-\tau_{ij}) - x_j^*(s-\tau_{ij})\big)\Big]\Big\},
\end{aligned} \tag{3.1}$$

where

$$\begin{aligned}
\Phi_i(s) &= \exp\Big\{\xi_i(s)\Big[\alpha_i(s) + \frac{\beta_i(s)}{1+x_i^{*p}(s-\tau_{i0})} \\
&\quad - a_{ii}(s)x_i^*(s-\tau_{ii}) - a_{ij}(s)x_j^*(s-\tau_{ij})\Big]\Big\}, \\
\Psi_i(s) &= \exp\Big\{\theta_i(s)\Big[\alpha_i(s) + \frac{\beta_i(s)}{1+x_i^{p}(s-\tau_{i0})} \\
&\quad - a_{ii}(s)x_i(s-\tau_{ii}) - a_{ij}(s)x_j(s-\tau_{ij})\Big] \\
&\quad + \big(1-\theta_i(s)\big)\Big[\alpha_i(s) + \frac{\beta_i(s)}{1+x_i^{*p}(s-\tau_{i0})} \\
&\quad - a_{ii}(s)x_i^*(s-\tau_{ii}) - a_{ij}(s)x_j^*(s-\tau_{ij})\Big]\Big\},
\end{aligned} \tag{3.2}$$

and $\xi_i(s),\ \theta_i(s) \in (0,\ 1),\ i \neq j,\ i,\ j = 1,\ 2.$

Proof. For $i \neq j,\ i,\ j = 1, 2$, it follows from system (1.1) that

$$\begin{aligned}
& \ln\Big(\frac{x_i(n+1)}{x_i^*(n+1)}\Big) - \ln\Big(\frac{x_i(n)}{x_i^*(n)}\Big) \\
= \; & \ln\Big(\frac{x_i(n+1)}{x_i(n)}\Big) - \ln\Big(\frac{x_i^*(n+1)}{x_i^*(n)}\Big) \\
= \; & \Big[\alpha_i(n) + \frac{\beta_i(n)}{1 + x_i^p(n-\tau_{i0})} - a_{ii}(n)x_i(n-\tau_{ii}) \\
& -a_{ij}(n)x_j(n-\tau_{ij})\Big] - \Big[\alpha_i(n) + \frac{\beta_i(n)}{1 + x_i^{*p}(n-\tau_{i0})} \\
& -a_{ii}(n)x_i^*(n-\tau_{ii}) - a_{ij}(n)x_j^*(n-\tau_{ij})\Big] \\
= \; & -\frac{\beta_i(n)\big[x_i^p(n-\tau_{i0}) - x_i^{*p}(n-\tau_{i0})\big]}{\big[1 + x_i^p(n-\tau_{i0})\big]\big[1 + x_i^{*p}(n-\tau_{i0})\big]} - a_{ii}(n)\Big[x_i(n-\tau_{ii}) \\
& -x_i^*(n-\tau_{ii})\Big] - a_{ij}(n)\big[x_j(n-\tau_{ij}) - x_j^*(n-\tau_{ij})\big] \\
= \; & -\frac{\beta_i(n)\big[x_i^p(n-\tau_{i0}) - x_i^{*p}(n-\tau_{i0})\big]}{\big[1 + x_i^p(n-\tau_{i0})\big]\big[1 + x_i^{*p}(n-\tau_{i0})\big]} - a_{ii}(n)\Big[x_i(n) - x_i^*(n)\Big] \\
& -a_{ij}(n)\Big[x_j(n-\tau_{ij}) - x_j^*(n-\tau_{ij})\Big] + a_{ii}(n)\Big[x_i(n) - x_i^*(n)\Big] \\
& -a_{ii}(n)\Big[x_i(n-\tau_{ii}) - x_i^*(n-\tau_{ii})\Big].
\end{aligned}$$

Thus

$$\begin{aligned}
& \ln\Big(\frac{x_i(n+1)}{x_i^*(n+1)}\Big) \\
= \; & \ln\Big(\frac{x_i(n)}{x_i^*(n)}\Big) - a_{ij}(n)\Big[x_j(n-\tau_{ij}) - x_j^*(n-\tau_{ij})\Big] \\
& -\frac{\beta_i(n)\big[x_i^p(n-\tau_{i0}) - x_i^{*p}(n-\tau_{i0})\big]}{\big[1 + x_i^p(n-\tau_{i0})\big]\big[1 + x_i^{*p}(n-\tau_{i0})\big]} - a_{ii}(n)\Big\{\big[x_i(n) \\
& -x_i^*(n)\big] - \big(\big[x_i(n) - x_i(n-\tau_{ii})\big] - \big[x_i^*(n) - x_i^*(n-\tau_{ii})\big]\big)\Big\}.
\end{aligned} \tag{3.3}$$

Obviously

$$\begin{aligned}
& \big[x_i(n) - x_i(n-\tau_{ii})\big] - \big[x_i^*(n) - x_i^*(n-\tau_{ii})\big] \\
= \; & \sum_{s=n-\tau_{ii}}^{n-1} \big[x_i(s+1) - x_i(s)\big] - \sum_{s=n-\tau_{ii}}^{n-1} \big[x_i^*(s+1) - x_i^*(s)\big] \\
= \; & \sum_{s=n-\tau_{ii}}^{n-1} \Big[\big(x_i(s+1) - x_i^*(s+1)\big) - \big(x_i(s) - x_i^*(s)\big)\Big],
\end{aligned} \tag{3.4}$$

and

$$
\begin{aligned}
& \big[x_i(s+1)-x_i^*(s+1)\big]-\big[x_i(s)-x_i^*(s)\big] \\
= & \bigg\{x_i(s)\exp\bigg[\alpha_i(s)+\frac{\beta_i(s)}{1+x_i^p(s-\tau_{i0})}-a_{ii}(s)x_i(s-\tau_{ii}) \\
& -a_{ij}(s)x_j(s-\tau_{ij})\bigg]-x_i^*(s)\exp\bigg[\alpha_i(s)+\frac{\beta_i(s)}{1+x_i^{*p}(s-\tau_{i0})} \\
& -a_{ii}(s)x_i^*(s-\tau_{ii})-a_{ij}(s)x_j^*(s-\tau_{ij})\bigg]\bigg\}-\big[x_i(s)-x_i^*(s)\big] \\
= & x_i(s)\bigg\{\exp\bigg[\alpha_i(s)+\frac{\beta_i(s)}{1+x_i^p(s-\tau_{i0})}-a_{ii}(s)x_i(s-\tau_{ii}) \\
& -a_{ij}(s)x_j(s-\tau_{ij})\bigg]-\exp\bigg[\alpha_i(s)+\frac{\beta_i(s)}{1+x_i^{*p}(s-\tau_{i0})} \\
& -a_{ii}(s)x_i^*(s-\tau_{ii})-a_{ij}(s)x_j^*(s-\tau_{ij})\bigg]\bigg\} \\
& +\big[x_i(s)-x_i^*(s)\big]\bigg\{\exp\bigg[\alpha_i(s)+\frac{\beta_i(s)}{1+x_i^{*p}(s-\tau_{i0})} \\
& -a_{ii}(s)x_i^*(s-\tau_{ii})-a_{ij}(s)x_j^*(s-\tau_{ij})\bigg]-1\bigg\}.
\end{aligned}
$$

According to the mean-value theorem, we have

$$
\begin{aligned}
& \big[x_i(s+1)-x_i^*(s+1)\big]-\big[x_i(s)-x_i^*(s)\big] \\
= & \big[x_i(s)-x_i^*(s)\big]\Phi_i(s)\bigg[\alpha_i(s)+\frac{\beta_i(s)}{1+x_i^{*p}(s-\tau_{i0})} \\
& -a_{ii}(s)x_i^*(s-\tau_{ii})-a_{ij}(s)x_j^*(s-\tau_{ij})\bigg] \\
& +x_i(s)\Psi_i(s)\bigg[-\frac{\beta_i(n)\big[x_i^p(n-\tau_{i0})-x_i^{*p}(n-\tau_{i0})\big]}{\big[1+x_i^p(n-\tau_{i0})\big]\big[1+x_i^{*p}(n-\tau_{i0})\big]} \\
& -a_{ii}(s)\big(x_i(s-\tau_{ii})-x_i^*(s-\tau_{ii})\big) \\
& -a_{ij}(s)\big(x_j(s-\tau_{ij})-x_j^*(s-\tau_{ij})\big)\bigg],
\end{aligned}
\tag{3.5}
$$

where $\Phi_i(s)$, $\Psi_i(s)$ are defined by (3.2). From (3.3)-(3.5), we can yield the equation (3.1). □

Theorem 3.1. *Assume that $(x_1^*(n),\ x_2^*(n))$ is a positive solution of system (1.1) with initial conditions (1.2). If there exist positive constants $c_1,\ c_2$ and δ such that*

$$c_i\Gamma_{ii}-\sum_{j=1,j\neq i}^{2}(c_i\Omega_{ij}+c_j\Delta_{ji})\geq\delta,\quad i=1,\ 2,$$

where

$$\begin{aligned} \Gamma_{ii} &= \min\left(a_{ii}^l,\ \tfrac{2}{M_i} - a_{ii}^u\right) - pM_i^{p-1}\beta_i^u - pa_{ii}^u M_i^p \beta_i^u \Psi_i^u \tau_{ii} \\ &\quad -(a_{ii}^u)^2 M_i \Psi_i^u \tau_{ii}, \\ \Omega_{ij} &= a_{ii}^u \Phi_i^u \tau_{ii}\big(\alpha_i^u + \beta_i^u + a_{ii}^u M_i + a_{ij}^u M_j\big), \\ \Delta_{ji} &= a_{ji}^u + a_{jj}^u M_j \Psi_j^u a_{ji}^u \tau_{jj}. \end{aligned} \tag{3.6}$$

Then $(x_1^*(n),\ x_2^*(n))$ *is globally attractive.*

Proof. Let $(x_1(n),\ x_2(n))$ be any positive solution of system (1.1) with initial conditions (1.2). At first, we define $V_{11}(n) = \big|\ln x_1(n) - \ln x_1^*(n)\big|$. According to (3.1), we have

$$\begin{aligned} & \Big|\ln\Big(\frac{x_1(n+1)}{x_1^*(n+1)}\Big)\Big| \\ \leq\ & \Big|\ln\Big(\frac{x_1(n)}{x_1^*(n)}\Big) - a_{11}(n)\big[x_1(n) - x_1^*(n)\big]\Big| + a_{12}(n)\big|x_2(n-\tau_{12}) \\ & -x_2^*(n-\tau_{12})\big| + \beta_1(n)\big|x_1^p(n-\tau_{10}) - x_1^{*p}(n-\tau_{10})\big| \\ & +a_{11}(n)\sum_{s=n-\tau_{11}}^{n-1}\Big\{\Phi_1(s)\Big[\alpha_1(s) + \beta_1(s) + a_{11}(s)\big|x_1^*(s-\tau_{11})\big| \\ & +a_{12}(s)\big|x_2^*(s-\tau_{12})\big|\Big]\big|x_1(s) - x_1^*(s)\big| + \Psi_1(s)|x_1(s)| \\ & \times\Big[\beta_1(s)\Big|x_1^p(n-\tau_{10}) - x_1^{*p}(n-\tau_{10})\Big| + a_{11}(s)\big|x_1(s-\tau_{11}) \\ & -x_1^*(s-\tau_{11})\big| + a_{12}(s)\big|x_2(s-\tau_{12}) - x_2^*(s-\tau_{12})\big|\Big]\Big\}. \end{aligned} \tag{3.7}$$

Since

$$x_i(n) - x_i^*(n) = e^{\ln x_i(n)} - e^{\ln x_i^*(n)} = \rho_i(n)\ln\Big(\frac{x_i(n)}{x_i^*(n)}\Big), \quad i = 1,\ 2,$$

where $0 < \rho_i(n) < \max\{x_i(n),\ x_i^*(n)\},\ i = 1,\ 2$. Thus

$$\ln\Big(\frac{x_i(n)}{x_i^*(n)}\Big) = \frac{1}{\rho_i(n)}\big[x_i(n) - x_1^*(n)\big],$$

and hence

$$
\begin{aligned}
& \Big|\ln\Big(\frac{x_1(n)}{x_1^*(n)}\Big) - a_{11}(n)\big[x_1(n) - x_1^*(n)\big]\Big| \\
= & \Big|\frac{1}{\rho_1(n)}\big(x_1(n) - x_1^*(n)\big) - a_{11}(n)\big(x_1(n) - x_1^*(n)\big)\Big| \\
= & \Big|\ln\Big(\frac{x_1(n)}{x_1^*(n)}\Big)\Big| - \frac{1}{\rho_1(n)}\big|x_1(n) - x_1^*(n)\big| \\
& +\Big|\frac{1}{\rho_1(n)} - a_{11}(n)\Big|\big|x_1(n) - x_1^*(n)\big| \\
= & \Big|\ln\Big(\frac{x_1(n)}{x_1^*(n)}\Big)\Big| - \Big(\frac{1}{\rho_1(n)} - \Big|\frac{1}{\rho_1(n)} - a_{11}(n)\Big|\Big)|x_1(n) - x_1^*(n)|
\end{aligned}
\tag{3.8}
$$

According to Theorem 2.1, we have that there exist $M_i > 0$ $(i = 1,\ 2)$ and a positive integer N_0 such that $0 < x_i(n),\ x_i^*(n) \le M_i$ $(i = 1,\ 2)$ for $n \ge N_0$. It follows from (3.7) and (3.8) that for $n \ge N_0 + \tau$

$$
\begin{aligned}
& \Delta V_{11} \\
= & \Big|\ln\Big(\frac{x_1(n+1)}{x_1^*(n+1)}\Big)\Big| - \Big|\ln\Big(\frac{x_1(n)}{x_1^*(n)}\Big)\Big| \\
\le & -\Big(\frac{1}{\rho_1(n)} - \Big|\frac{1}{\rho_1(n)} - a_{11}(n)\Big|\Big)\big|x_1(n) - x_1^*(n)\big| \\
& +a_{12}(n)\big|x_2(n-\tau_{12}) - x_2^*(n-\tau_{12})\big| + \beta_1(n)\big|x_1^p(n-\tau_{10}) \\
& -x_1^{*p}(n-\tau_{10})\big| + a_{11}(n)\sum_{s=n-\tau_{11}}^{n-1}\Big\{\Phi_1(s)\big[\alpha_1(s) + \beta_1(s) \\
& +a_{11}(s)M_1 + a_{12}(s)M_2\big]\big|x_1(s) - x_1^*(s)\big| + \Psi_1(s)M_1\beta_1(s) \\
& \times\big|x_1^p(s-\tau_{10}) - x_1^{*p}(s-\tau_{10})\big| + \Psi_1(s)M_1a_{11}(s)\big|x_1(s-\tau_{11}) \\
& -x_1^*(s-\tau_{11})\big| + \Psi_1(s)M_1a_{12}(s)\big|x_2(s-\tau_{12}) - x_2^*(s-\tau_{12})\big|\Big\}
\end{aligned}
\tag{3.9}
$$

Secondly, set

$$
\begin{aligned}
& V_{12}(n) \\
= & \sum_{s=n-\tau_{10}}^{n-1}\beta_1(s+\tau_{10})\big|x_1^p(s) - x_1^{*p}(s)\big| + \sum_{s=n-\tau_{12}}^{n-1}a_{12}(s+\tau_{12})\times \\
& \big|x_2(s) - x_2^*(s)\big| + \sum_{s=n}^{n-1+\tau_{11}}a_{11}(s)\sum_{u=s-\tau_{11}}^{n-1}\Big\{\Phi_1(u)\Big[\alpha_1(u) + \beta_1(u) \\
& +a_{11}(u)M_1 + a_{12}(u)M_2\Big]\big|x_1(u) - x_1^*(u)\big| + \Psi_1(u)M_1\beta_1(u) \\
& \times\big|x_1^p(u-\tau_{10}) - x_1^{*p}(u-\tau_{10})\big| + \Psi_1(u)M_1a_{11}(u)\big|x_1(u-\tau_{11}) \\
& -x_1^*(u-\tau_{11})\big| + \Psi_1(u)M_1a_{12}(u)\big|x_2(u-\tau_{12}) - x_2^*(u-\tau_{12})\big|\Big\}
\end{aligned}
$$

Calculating the difference of V_{12} along the solution of (1.1), we have that for $n \geq N_0 + \tau$

$$\begin{aligned}
& \Delta V_{12} \\
= \ & \beta_1(n+\tau_{10})\big|x_1^p(n) - x_1^{*p}(n)\big| - \beta_1(n)\big|x_1^p(n-\tau_{10}) \\
& -x_1^{*p}(n-\tau_{10})\big| + a_{12}(n+\tau_{12})\big|x_2(n) - x_2^*(n)\big| \\
& -a_{12}(n)\big|x_2(n-\tau_{12}) - x_2^*(n-\tau_{12})\big| \\
& + \sum_{s=n+1}^{n+\tau_{11}} a_{11}(s)\Big\{\Phi_1(n)\Big(\alpha_1(n) + \beta_1(n) + a_{11}(n)M_1 \\
& +a_{12}(n)M_2\Big)\big|x_1(n) - x_1^*(n)\big| + \Psi_1(n)M_1\beta_1(n)\big|x_1^p(n-\tau_{10}) \\
& -x_1^{*p}(n-\tau_{10})\big| + \Psi_1(n)M_1a_{11}(n)\big|x_1(n-\tau_{11}) - x_1^*(n-\tau_{11})\big| \\
& +\Psi_1(n)M_1a_{12}(n)\big|x_2(n-\tau_{12}) - x_2^*(n-\tau_{12})\big|\Big\} \\
& -a_{11}(n)\sum_{u=n-\tau_{11}}^{n-1} \Big\{\Phi_1(u)\Big(\alpha_1(u) + \beta_1(u) + a_{11}(u)M_1 \\
& +a_{12}(u)M_2\Big)\big|x_1(u) - x_1^*(u)\big| + \Psi_1(u)M_1\beta_1(u)\big|x_1^p(u-\tau_{10}) \\
& -x_1^{*p}(u-\tau_{10})\big| + \Psi_1(u)M_1a_{11}(u)\big|x_1(u-\tau_{11}) - x_1^*(u-\tau_{11})\big| \\
& +\Psi_1(u)M_1a_{12}(u)\big|x_2(u-\tau_{12}) - x_2^*(u-\tau_{12})\big|\Big\}
\end{aligned}$$

Thirdly, let

$$\begin{aligned}
& V_{13}(n) \\
= \ & M_1\sum_{k=n-\tau_{10}}^{n-1} \beta_1(k+\tau_{10})\Psi_1(k+\tau_{10})\big|x_1^p(k) - x_1^{*p}(k)\big| \\
& \times\sum_{s=k+\tau_{10}+1}^{k+\tau_{10}+\tau_{11}} a_{11}(s) + M_1\sum_{k=n-\tau_{11}}^{n-1} \Psi_1(k+\tau_{11})a_{11}(k+\tau_{11}) \\
& \times\big|x_1(k) - x_1^*(k)\big|\sum_{s=k+\tau_{11}+1}^{k+2\tau_{11}} a_{11}(s) + M_1\sum_{k=n-\tau_{12}}^{n-1} \Psi_1(k+\tau_{12}) \\
& \times a_{12}(k+\tau_{12})\big|x_2(k) - x_2^*(k)\big|\sum_{s=k+\tau_{12}+1}^{k+\tau_{12}+\tau_{11}} a_{11}(s)
\end{aligned} \tag{3.10}$$

Calculating the difference of V_{13} along the solution of (1.1) and (1.2), we yield that for $n \geq N_0 + \tau$,

$$
\begin{aligned}
&\Delta V_{13} \\
= \ & \sum_{s=n+\tau_{10}+1}^{n+\tau_{10}+\tau_{11}} a_{11}(s) M_1 \beta_1(n+\tau_{10}) \Psi_1(n+\tau_{10}) \big| x_1^p(n) - x_1^{*p}(n) \big| \\
& - \sum_{s=n+1}^{n+\tau_{11}} a_{11}(s) M_1 \beta_1(n) \Psi_1(n) \big| x_1^p(n-\tau_{10}) - x_1^{*p}(n-\tau_{10}) \big| \\
& + \sum_{s=n+\tau_{11}+1}^{n+2\tau_{11}} a_{11}(s) M_1 \Psi_1(n+\tau_{11}) a_{11}(n+\tau_{11}) \big| x_1(n) - x_1^*(n) \big| \\
& - \sum_{s=n+1}^{n+\tau_{11}} a_{11}(s) M_1 \Psi_1(n) a_{11}(n) \big| x_1(n-\tau_{11}) - x_1^*(n-\tau_{11}) \big| \\
& + \sum_{s=n+\tau_{12}+1}^{n+\tau_{12}+\tau_{11}} a_{11}(s) M_1 \Psi_1(n+\tau_{12}) a_{12}(n+\tau_{12}) \big| x_2(n) - x_2^*(n) \big| \\
& - \sum_{s=n+1}^{n+\tau_{11}} a_{11}(s) M_1 \Psi_1(n) a_{12}(n) \big| x_2(n-\tau_{12}) - x_2^*(n-\tau_{12}) \big|
\end{aligned}
\tag{3.11}
$$

We take $V_1(n) = V_{11}(n) + V_{12}(n) + V_{13}(n)$. Then from (3.7)-(3.11), we have that for $n \geq N_0 + \tau$

$$
\begin{aligned}
&\Delta V_1 \\
\leq \ & -\Big(\frac{1}{\rho_1(n)} - \big| \frac{1}{\rho_1(n)} - a_{11}(n) \big| \Big) \big| x_1(n) - x_1^*(n) \big| \\
& + a_{12}(n+\tau_{12}) \big| x_2(n) - x_2^*(n) \big| + \beta_1(n+\tau_{10}) \big| x_1^p(n) - x_1^{*p}(n) \big| \\
& + \sum_{s=n+1}^{n+\tau_{11}} a_{11}(s) \Bigg(\Phi_1(n) \big[\alpha_1(n) + \beta_1(n) + a_{11}(n) M_1 \\
& + a_{12}(n) M_2 \big] \Bigg) \big| x_1(n) - x_1^*(n) \big| \\
& + \sum_{s=n+\tau_{10}+1}^{n+\tau_{10}+\tau_{11}} a_{11}(s) M_1 \beta_1(n+\tau_{10}) \Psi_1(n+\tau_{10}) \big| x_1^p(n) - x_1^{*p}(n) \big| \\
& + \sum_{s=n+\tau_{11}+1}^{n+2\tau_{11}} a_{11}(s) M_1 \Psi_1(n+\tau_{11}) a_{11}(n+\tau_{11}) \big| x_1(n) - x_1^*(n) \big| \\
& + \sum_{s=n+\tau_{12}+1}^{n+\tau_{12}+\tau_{11}} a_{11}(s) M_1 \Psi_1(n+\tau_{12}) a_{12}(n+\tau_{12}) \big| x_2(n) - x_2^*(n) \big|.
\end{aligned}
\tag{3.12}
$$

By the mean-value theorem, we have

$$\begin{aligned}
&\big|x_i^p(n) - x_i^{*p}(n)\big| \\
&= p\big[\zeta_i(n)x_i(n) + (1-\zeta_i(n))x_i^*(n)\big]^{p-1}\big|x_i(n) - x_i^*(n)\big| \\
&\le pM_i^{p-1}\big|x_i(n) - x_i^*(n)\big|,
\end{aligned}$$

where $\zeta_i(n) \in (0,\ 1),\ i = 1,\ 2$. It follows from (3.12) that

$$\begin{aligned}
&\Delta V_1 \\
\le\ & -\Big(\frac{1}{\rho_1(n)} - \big|\frac{1}{\rho_1(n)} - a_{11}(n)\big|\Big)\big|x_1(n) - x_1^*(n)\big| + a_{12}(n+\tau_{12}) \\
&\times\big|x_2(n) - x_2^*(n)\big| + pM_1^{p-1}\beta_1(n+\tau_{10})\big|x_1(n) - x_1^*(n)\big| \\
&+ \sum_{s=n+1}^{n+\tau_{11}} a_{11}(s)\bigg(\Phi_1(n)\big[\alpha_1(n) + \beta_1(n) + a_{11}(n)M_1 \\
&+ a_{12}(n)M_2\big]\bigg)\big|x_1(n) - x_1^*(n)\big| \\
&+ \sum_{s=n+\tau_{10}+1}^{n+\tau_{10}+\tau_{11}} a_{11}(s)pM_1^p\beta_1(n+\tau_{10})\Psi_1(n+\tau_{10})\big|x_1(n) - x_1^*(n)\big| \\
&+ \sum_{s=n+\tau_{11}+1}^{n+2\tau_{11}} a_{11}(s)M_1\Psi_1(n+\tau_{11})a_{11}(n+\tau_{11})\big|x_1(n) - x_1^*(n)\big| \\
&+ \sum_{s=n+\tau_{12}+1}^{n+\tau_{12}+\tau_{11}} a_{11}(s)M_1\Psi_1(n+\tau_{12})a_{12}(n+\tau_{12})\big|x_2(n) - x_2^*(n)\big|.
\end{aligned}$$

Similarly, we let $V_{21}(n) = \big|\ln x_2(n) - \ln x_2^*(n)\big|$.

$$\begin{aligned}
&V_{22}(n) \\
=\ & \sum_{s=n-\tau_{20}}^{n-1} \beta_2(s+\tau_{20})\big|x_2^p(s) - x_2^{*p}(s)\big| + \sum_{s=n-\tau_{21}}^{n-1} a_{21}(s+\tau_{21})\big|x_1(s) \\
&- x_1^*(s)\big| + \sum_{s=n}^{n-1+\tau_{22}} a_{22}(s) \sum_{u=s-\tau_{22}}^{n-1} \Big\{\Phi_2(u)\Big[\alpha_2(u) + \beta_2(u) \\
&+ a_{22}(u)M_2 + a_{21}(u)M_1\Big]\big|x_2(u) - x_2^*(u)\big| \\
&+ \Psi_2(u)M_2\beta_2(u)\big|x_2^p(s-\tau_{20}) - x_2^{*p}(s-\tau_{20})\big| \\
&+ \Psi_2(u)M_2a_{22}(u)\big|x_2(u-\tau_{22}) - x_2^*(u-\tau_{22})\big| \\
&+ \Psi_2(u)M_1a_{21}(u)\big|x_1(u-\tau_{21}) - x_1^*(u-\tau_{21})\big|\Big\}
\end{aligned}$$

and

$$
\begin{aligned}
& V_{23}(n) \\
= \; & M_2 \sum_{k=n-\tau_{20}}^{n-1} \beta_2(k+\tau_{20})\Psi_2(k+\tau_{20})\big|x_2^p(k) - x_2^{*p}(k)\big| \times \\
& \sum_{s=k+\tau_{20}+1}^{k+\tau_{20}+\tau_{22}} a_{22}(s) + M_2 \sum_{k=n-\tau_{22}}^{n-1} \Psi_2(k+\tau_{22})a_{22}(k+\tau_{22})\big|x_2(k) \\
& -x_2^*(k)\big| \textstyle\sum_{s=k+\tau_{22}+1}^{k+2\tau_{22}} a_{22}(s) + M_2 \displaystyle\sum_{k=n-\tau_{21}}^{n-1} \Psi_2(k+\tau_{21}) \times \\
& a_{21}(k+\tau_{21})\big|x_1(k) - x_1^*(k)\big| \textstyle\sum_{s=k+\tau_{21}+1}^{k+\tau_{21}+\tau_{22}} a_{22}(s)
\end{aligned}
$$

We also take $V_2(n) = V_{21}(n) + V_{22}(n) + V_{23}(n)$. By similar calculation as ΔV_1, we get for $n \geq N_0 + \tau$,

$$
\begin{aligned}
& \Delta V_2 \\
\leq \; & -\Big(\frac{1}{\rho_2(n)} - \big|\frac{1}{\rho_2(n)} - a_{22}(n)\big|\Big)\big|x_2(n) - x_2^*(n)\big| + a_{21}(n+\tau_{21}) \\
& \times\big|x_1(n) - x_1^*(n)\big| + pM_2^{p-1}\beta_2(n+\tau_{20})\big|x_2(n) - x_2^*(n)\big| \\
& + \sum_{s=n+1}^{n+\tau_{22}} a_{22}(s)\Bigg(\Phi_2(n)\big[\alpha_2(n) + \beta_2(n) + a_{22}(n)M_2 \\
& +a_{21}(n)M_1\big]\Bigg)\big|x_2(n) - x_2^*(n)\big| \\
& + \sum_{s=n+\tau_{20}+1}^{n+\tau_{20}+\tau_{22}} a_{22}(s)pM_2^p\beta_2(n+\tau_{20})\Psi_2(n+\tau_{20})\big|x_2(n) - x_2^*(n)\big| \\
& + \sum_{s=n+\tau_{22}+1}^{n+2\tau_{22}} a_{22}(s)M_2\Psi_2(n+\tau_{22})a_{22}(n+\tau_{22})\big|x_2(n) - x_2^*(n)\big| \\
& + \sum_{s=n+\tau_{21}+1}^{n+\tau_{21}+\tau_{22}} a_{22}(s)M_2\Psi_2(n+\tau_{21})a_{21}(n+\tau_{21})\big|x_1(n) - x_1^*(n)\big|.
\end{aligned}
$$

Let us define a Lyapunov-like discrete function V by

$$V(n) = c_1V_1(n) + c_2V_2(n).$$

It is easy to obtain that $V(N_0+\tau) < +\infty$. Calculating the difference of V along the solution of system (1.1) with initial conditions (1.2), we have that for

$n \geq N_0 + \tau$

$$\begin{aligned}
& \Delta V \\
\leq\; & -\sum_{i=1}^{2}\Big\{c_i\Big[\Big(\frac{1}{\rho_i(n)} - \big|\frac{1}{\rho_i(n)} - a_{ii}(n)\big|\Big) - pM_i^{p-1}\beta_i(n+\tau_{i0}) \\
& - \sum_{s=n+\tau_{i0}+1}^{n+\tau_{i0}+\tau_{ii}} a_{ii}(s)pM_i^p\beta_i(n+\tau_{i0})\Psi_i(n+\tau_{i0}) \\
& - \sum_{s=n+\tau_{ii}+1}^{n+2\tau_{ii}} a_{ii}(s)M_i\Psi_i(n+\tau_{ii})a_{ii}(n+\tau_{ii})\Big] \\
& - \sum_{j=1,j\neq i}^{2}\Big[c_i\sum_{s=n+1}^{n+\tau_{ii}} a_{ii}(s)\Phi_i(n)\Big(\alpha_i(n)+\beta_i(n)+a_{ii}(n)M_i \\
& + a_{ij}(n)M_j\Big) + c_j\Big(a_{ji}(n+\tau_{ji}) + \sum_{s=n+\tau_{ji}+1}^{n+\tau_{ji}+\tau_{jj}} a_{jj}(s)M_j \\
& \times\Psi_j(n+\tau_{ji})a_{ji}(n+\tau_{ji})\Big)\Big]\Big\}|x_i(n)-x_i^*(n)| \\
\leq\; & -\sum_{i=1}^{2}\Big\{c_i\Big[\min\big(a_{ii}^l,\ \frac{2}{M_i} - a_{ii}^u\big) - pM_i^{p-1}\beta_i^u - pa_{ii}^uM_i^p\beta_i^u\Psi_i^u\tau_{ii} \\
& -(a_{ii}^u)^2M_i\Psi_i^u\tau_{ii}\Big] - \sum_{j=1,j\neq i}^{2}\Big[c_ia_{ii}^u\Phi_i^u\tau_{ii}\big(\alpha_i^u+\beta_i^u+a_{ii}^uM_i \\
& + a_{ij}^uM_j\big) + c_j\big(a_{ji}^u + a_{jj}^uM_j\Psi_j^ua_{ji}^u\tau_{jj}\big)\Big]\Big\}|x_i(n)-x_i^*(n)| \\
=\; & -\sum_{i=1}^{2}\Big\{c_i\Gamma_{ii} - \sum_{j=1,j\neq i}^{2}\big(c_i\Omega_{ij}+c_j\Delta_{ji}\big)\Big\}|x_i(n)-x_i^*(n)| \\
\leq\; & -\delta\sum_{i=1}^{2}|x_i(n)-x_i^*(n)|,
\end{aligned}$$

where $\Gamma_{ii},\ \Omega_{ij},\ \Delta_{ji}$ are defined by (3.6). Therefore,

$$\sum_{u=N_0+\tau}^{n}\big(V(u+1)-V(u)\big) \leq -\delta\sum_{u=N_0+\tau}^{n}\sum_{i=1}^{2}|x_i(u)-x_i^*(u)|.$$

which implies that

$$V(n+1) + \delta\sum_{u=N_0+\tau}^{n}\sum_{i=1}^{2}|x_i(u)-x_i^*(u)| \leq V(N_0+\tau),$$

that is,

$$\sum_{u=N_0+\tau}^{n}\sum_{i=1}^{2}|x_i(u)-x_i^*(u)|\leq \frac{V(N_0+\tau)}{\delta}.$$

Therefore,

$$\sum_{n=N_0+\tau}^{\infty}\sum_{i=1}^{2}|x_i(n)-x_i^*(n)|\leq \frac{V(N_0+\tau)}{\delta}<+\infty,$$

which implies $\lim\limits_{n\to+\infty}\sum\limits_{i=1}^{2}|x_i(n)-x_i^*(n)|=0$. Hence $(x_1^*(n),\ x_2^*(n))$ is globally attractive. □

4. Numerical Simulation

Now we let $\alpha_1(n)=0.02$, $\beta_1(n)=0.02+0.01\sin n$, $a_{11}(n)=2.02+0.02\sin n$, $a_{12}(n)=0.015+0.005\sin n$, $\alpha_2(n)=0.03$, $\beta_2(n)=0.02+0.01\sin n$, $a_{21}(n)=0.012+0.02\sin n$, $a_{22}(n)=1.81+0.01\sin n$, $\tau_{10}=2$,$\tau_{11}=1$, $\tau_{12}=3$,$\tau_{20}=3$, $\tau_{21}=2$, $\tau_{22}=1$, $p=2$. Then system (1.1) becomes

$$\begin{cases} x_1(n+1) & = x_1(n)\exp\Big[0.02+\dfrac{0.02+0.01\sin n}{1+x_1^2(n-2)}-(2.02+ \\ & \quad 0.02\sin n)x_1(n-1)-(0.015+0.005\sin n)x_2(n-3)\Big], \\ x_2(n+1) & = x_2(n)\exp\Big[0.03+\dfrac{0.02+0.01\sin n}{1+x_2^2(n-3)}-(0.012+ \\ & \quad 0.02\sin n)x_1(n-2)-(1.81+0.01\sin n)x_2(n-1)\Big]. \end{cases} \tag{3.13}$$

According to (3.6), we can easily obtain that

$$\begin{aligned} &\Gamma_{11}\approx 1.0931,\quad \Omega_{12}\approx 1.0054,\quad \Delta_{21}\approx 0.0197,\\ &\Gamma_{22}\approx 1.045,\quad \Omega_{21}\approx 0.8616,\quad ,\Delta_{12}\approx 0.0287. \end{aligned}$$

Thus

$$E_{11}-(F_{12}+G_{21})\approx 0.0680,\quad E_{22}-(F_{21}+G_{12})\approx 0.1547$$

Obviously, the conditions in Theorem 3.1 are verified. As it can be seen in Fig. 1, the positive solutions of (3.13) are globally attractive. This numerical computer result is in well accordance with our theoretical analysis.

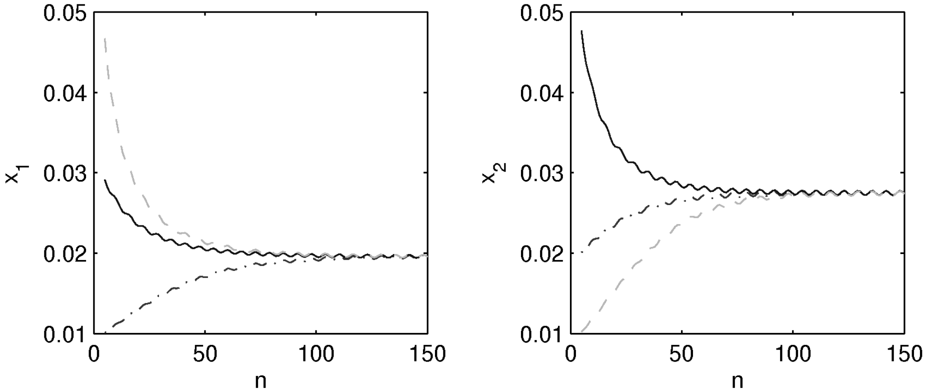

Figure 1: Mathematical simulation of three positive solutions of system (3.13) with initial conditions $(x_1(s),\ x_2(s))$ $(s \in \{-3,\ -2,\ -1,\ 0\}) =$ $(0.01,\ 0.02)$ (dash-dot), $(0.03,\ 0.05)$ (solid) and $(0.05,\ 0.01)$ (dash), respectively

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