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DEGREE OF APPROXIMATION OF A FUNCTION BELONGING TO $Lip\,(\xi(t), r)$ CLASS BY $(E,1)(C,1)$ PRODUCT MEANS

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Abstract: In this paper, a new theorem on degree of approximation of a function belonging to $Lip\,(\xi\,(t)\,, r)$ class by $(E,1)(C,1)$ product summability means of its Fourier series has been obtained.

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1. Introduction

Alexits [1], Sahney and Goel [11], Chandra [2], Qureshi and Neha [9], Leindler [5] and Rhoades [10] have determined the degree of approximation of a function belonging to $Lip\alpha$ class by Cesàro, Nörlund and generalized Nörlund single summability methods. Working in the same direction Sahney and Rao [12], Khan [4], Quershi [7,8] have studied the degree of approximation of a function belonging to $Lip(\alpha, r)$ class by Nörlund and generalized Nörlund single summability methods. But nothing seems to have been done so far to obtain the degree of approximation of a function belonging to $Lip\,(\xi\,(t)\,, r)$ class by $(E,1)\,(C,1)$ product summability method. The $Lip\,(\xi\,(t)\,, r)$ class is a gen-

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eralization of $Lip\alpha$ class and $Lip(\alpha, r)$ class. Therefore, in present paper, a theorem on degree of approximation of a function belonging to $Lip\,(\xi\,(t)\,, r)$ class by $(E, 1)\,(C, 1)$ product summability means of its Fourier series has been established.

2. Definitions and Notations

Let $f(x)$ be periodic with period 2π and integrable in the sense of Lebesgue. The Fourier series associated with f at a point x is defined as

$$f\,(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx) \tag{2.1}$$

with n^{th} partial sums $s_n(f; x)$.
L_r- norm is defined by

$$\|f\|_r = \left(\int_0^{2\pi} |f\,(x)|^r \, dx \right)^{\frac{1}{r}}, \; r \geq 1 \tag{2.2}$$

and let the degree of approximation of a function be given by

$$E_n(f) = \min \|t_n - f\|_r \qquad \text{(see [14])}, \tag{2.3}$$

where $t_n(x)$ is some n^{th} degree trigonometric polynomial.

This method of approximation is called trigonometric Fourier approximation (TFA).

A function $f \in Lip\alpha$ if

$$f\,(x + t) - f\,(x) = O\,(|t|^{\alpha}) \quad \text{for} 0 < \alpha \leq 1. \tag{2.4}$$

$f \in Lip\,(\alpha, r\;)$ for $0 \leq x \leq 2\pi$, if

$$\left(\int_0^{2\pi} |f\,(x + t) - f\,(x)|^r \, dx \right)^{\frac{1}{r}} = O\,(|t|^{\alpha})\,, 0 < \alpha \leq 1, \; r \geq 1. \tag{2.5}$$

(see Definition 5.38 of Mc Fadden [6]).

Given a positive increasing function $\xi\,(t)$ and an integer $r \geq 1$, $f \in Lip(\xi(t), r)$ if

$$\left(\int_0^{2\pi} |f\,(x + t) - f\,(x)|^r \, dx \right)^{\frac{1}{r}} = O\,(\xi\,(t)) \tag{2.6}$$

If $\xi(t) = t^{\alpha}$ then $Lip(\xi(t), r)$ class reduces to the $Lip(\alpha, r)$ and if $r \to \infty$ then $Lip(\alpha, r)$ class reduces to the $Lip\alpha$ class.

Let $\sum_{n=0}^{\infty} u_n$ be a given infinite series with sequence of its n^{th} partial sum $\{s_n\}$.

The (C,1) transform is defined as the n^{th} partial sum of (C,1) summability and is given by

$$t_n = \frac{s_0 + s_1 + s_2 + ...s_n}{n+1} = \frac{1}{n+1} \sum_{k=0}^{n} s_k \to s \quad \text{as } n \to \infty,$$

then the infinite series $\sum_{n=0}^{\infty} u_n$ is summable to a definite number s by (C,1) method. If

$$(E, 1) = E_n^1 = \frac{1}{2^n} \sum_{k=0}^{n} \binom{n}{k} s_k \to s \quad \text{as} \quad n \to \infty, \tag{2.7}$$

then the infinite series $\sum_{n=0}^{\infty} u_n$ is said to be summable $(E, 1)$ to the definite number s (see [3]).

The $(E, 1)$ transform of the $(C, 1)$ transform defines $(E, 1)(C, 1)$ transform and we denote it by $(EC)_n^1$.

Thus if

$$(EC)_n^1 = \frac{1}{2^n} \sum_{k=0}^{n} \binom{n}{k} C_k^1 \to s. \tag{2.8}$$

Then the series $\sum_{n=0}^{\infty} u_n$ is said to be summable by $(E, 1)(C, 1)$ means or summable $(E, 1)(C, 1)$ to the definite number s.

We use the following notations:

$$\phi(t) = f(x+t) + f(x-t) - 2f(x),$$

$$K_n(t) = \frac{1}{\pi\ 2^{n+1}} \sum_{k=0}^{n} \left\{ \binom{n}{k} \frac{1}{(1+k)} \sum_{\nu=0}^{k} \frac{\sin\left(\nu + \frac{1}{2}\right) t}{\sin \frac{t}{2}} \right\}.$$

3. Main Theorem

If a function f, 2π-periodic, belongs to $Lip(\xi(t), r)$ class then its degree of approximation by $(E, 1)(C, 1)$ summability means of its Fourier series is given by

$$\left\| (EC)_n^1 - f \right\|_r = O\left[(n+1)^{\frac{1}{r}} \xi\left(\frac{1}{n+1} \right) \right] \tag{3.1}$$

provided $\xi(t)$ *satisfies the following conditions:*

$$\left\{\int_0^{\frac{1}{n+1}} \left(\frac{t\ |\phi(t)\ |}{\xi(t)}\right)^r dt\right\}^{\frac{1}{r}} = O\left(\frac{1}{n+1}\right) \tag{3.2}$$

and

$$\left\{\int_{\frac{1}{n+1}}^{\pi} \left(\frac{t^{-\delta}\,|\phi(t)|}{\xi(t)}\right)^r dt\right\}^{\frac{1}{r}} = O\left\{(n+1)^{\delta}\right\} \tag{3.3}$$

where δ *is an arbitrary number such that* $s(1-\delta)-1>0$, $\frac{1}{r}+\frac{1}{s}=1$, $1\le r\le\infty$, *conditions (3.2) and (3.3) hold uniformly in x and* $(EC)_n^1$ *is* $(E,1)(C,1)$ *means of the Fourier series (2.1).*

4. Two Lemmas

The following lemmas are esential for the proof of our theorem.

Lemma 1.

$$|K_n(t)| = O(n+1) \quad \textit{for} \quad 0\le t\le \frac{1}{n+1}.$$

Proof. For $0\le t\le\frac{1}{n+1}$, $\sin nt \le n\sin t$

$$\begin{aligned}
|K_n(t)| &\le \frac{1}{\pi\ 2^{n+1}}\left|\sum_{k=0}^{n}\left[\binom{n}{k}\frac{1}{(1+k)}\sum_{\nu=0}^{k}\frac{\sin\left(\nu+\frac{1}{2}\right)t}{\sin\frac{t}{2}}\right]\right| \\
&\le \frac{1}{\pi\ 2^{n+1}}\left|\sum_{k=0}^{n}\left[\binom{n}{k}\frac{1}{(1+k)}\sum_{\nu=0}^{k}\frac{(2\nu+1)\sin\frac{t}{2}}{\sin\frac{t}{2}}\right]\right| \\
&= \frac{1}{\pi\ 2^{n+1}}\left|\sum_{k=0}^{n}\left[\binom{n}{k}(k+1)\right]\right| \\
&= O\left[\frac{(n+1)}{2^n}\sum_{k=0}^{n}\left\{\binom{n}{k}\right\}\right] \\
&= O(n+1) \qquad \text{since} \sum_{k=0}^{n}\binom{n}{k} = 2^n.
\end{aligned}$$

□

Lemma 2.

$$|K_n(t)| = O\left(\frac{1}{t}\right), \quad \textit{for} \quad \frac{1}{n+1} \le t \le \pi.$$

Proof. For $\frac{1}{n+1} \le t \le \pi$, by applying Jordan's lemma, $\sin\left(\frac{t}{2}\right) \ge \frac{t}{\pi}$ and $\sin nt \le 1$,

$$\begin{aligned}
|K_n(t)| &\le \frac{1}{\pi\ 2^{n+1}} \left| \sum_{k=0}^{n} \left[\binom{n}{k} \left(\frac{1}{1+k}\right) \sum_{\nu=0}^{k} \frac{\sin\left(\nu+\frac{1}{2}\right)t}{\sin\frac{t}{2}} \right] \right| \\
&\le \frac{1}{\pi\ 2^{n+1}} \left| \sum_{k=0}^{n} \left[\binom{n}{k} \left(\frac{1}{1+k}\right) \sum_{\nu=0}^{k} \left(\frac{1}{t/\pi}\right) \right] \right| \\
&= \frac{1}{2^{n+1}\ t} \left| \sum_{k=0}^{n} \left[\binom{n}{k} \left(\frac{1}{1+k}\right) \sum_{\nu=0}^{k} (1) \right] \right| \\
&= \frac{1}{2^{n+1}\ t} \left| \sum_{k=0}^{n} \left[\binom{n}{k} \right] \right| \\
&= O\left(\frac{1}{t}\right) \qquad \text{since} \sum_{k=0}^{n} \binom{n}{k} = 2^n.
\end{aligned}$$

□

5. Proof of Main Theorem

Following Titchmarsh [13] and using Riemann-Lebesgue theorem, $s_n(f;x)$ of the series (2.1) is given by

$$s_n(f;x) - f(x) = \frac{1}{2\pi} \int_0^{\pi} \phi(t) \frac{\sin\left(n+\frac{1}{2}\right)t}{\sin\frac{t}{2}} dt$$

Therefore, using (2.1), the $(C,1)$ transform $\left(C_n^1\right)$ of $s_n(f;x)$ is given by

$$C_n^1 - f(x) = \frac{1}{2\pi\ (n+1)} \int_0^{\pi} \phi(t) \sum_{k=0}^{n} \frac{\sin\left(k+\frac{1}{2}\right)t}{\sin\frac{t}{2}} dt.$$

Now denoting $(E,1)(C,1)$ transform of $s_n(f;x)$ as $(EC)_n^1$, we may write

$$(EC)_n^1 - f(x) = \frac{1}{2^{n+1}\ \pi}$$

$$\times \sum_{k=0}^{n} \left[\binom{n}{k} \int_0^{\pi} \frac{\phi(t)}{\sin\left(\frac{t}{2}\right)} \left(\frac{1}{k+1}\right) \left\{ \sum_{\nu=0}^{k} \sin\left(\nu + \frac{1}{2}\right) t \right\} dt \right]$$

$$\begin{aligned} &= \int_0^{\pi} \phi(t)\, K_n(t)\, dt \\ &= \left[\int_0^{\frac{1}{n+1}} + \int_{\frac{1}{n+1}}^{\pi} \right] \phi(t)\, K_n(t)\, dt \\ &= I_1 + I_2 \text{ (say)} \end{aligned} \tag{5.1}$$

We consider,

$$|I_1| \le \int_0^{\frac{1}{n+1}} |\phi(t)|\, |K_n(t)|\, dt$$

Applying Hölder's inequality and the fact that $\phi(t) \in Lip(\xi(t), r)$,

$$\begin{aligned} |I_1| &\le \left[\int_0^{\frac{1}{n+1}} \left\{ \frac{t\ |\phi(t)|}{\xi(t)} \right\}^r dt \right]^{\frac{1}{r}} \left[\int_0^{\frac{1}{n+1}} \left\{ \frac{\xi(t)\, |K_n(t)|}{t} \right\}^s dt \right]^{\frac{1}{s}} \\ &= O\left(\frac{1}{n+1}\right) \left[\int_0^{\frac{1}{n+1}} \left\{ \frac{\xi(t)\, |K_n(t)|}{t} \right\}^s dt \right]^{\frac{1}{s}} \quad \text{by (3.2)} \\ &= O\left(\frac{1}{n+1}\right) \left[\int_0^{\frac{1}{n+1}} \left\{ \frac{(n+1)\, \xi(t)}{t} \right\}^s dt \right]^{\frac{1}{s}} \quad \text{by Lemma 1.} \end{aligned}$$

Since $\xi(t)$ is a positive increasing function and using second mean value theorem for integrals,

$$\begin{aligned} I_1 &= O\left\{ \xi\left(\frac{1}{n+1}\right) \right\} \left[\int_{\in}^{\frac{1}{n+1}} \frac{dt}{t^s} \right]^{\frac{1}{s}} \quad \text{for some } 0 < \in < \frac{1}{n+1} \\ &= O\left\{ \xi\left(\frac{1}{n+1}\right) \right\} \left[\left\{ \frac{t^{-s+1}}{-s+1} \right\}_{\in}^{\frac{1}{n+1}} \right]^{\frac{1}{s}} \\ &= O\left\{ \xi\left(\frac{1}{n+1}\right) \right\} \left\{ (n+1)^{1-\frac{1}{s}} \right\} \\ &= O\left\{ (n+1)^{\frac{1}{r}}\, \xi\left(\frac{1}{n+1}\right) \right\} \quad \text{since } \frac{1}{r} + \frac{1}{s} = 1. \end{aligned} \tag{5.2}$$

Now we consider

$$|I_2| \leq \int_{\frac{1}{n+1}}^{\pi} |\phi(t)| \, |K_n(t)| \, dt$$

Using Hölder's inequality,

$$\begin{aligned}
|I_2| &\leq \left[\int_{\frac{1}{n+1}}^{\pi} \left\{\frac{t^{-\delta} \, |\phi(t)|}{\xi(t)}\right\}^r dt\right]^{\frac{1}{r}} \left[\int_{\frac{1}{n+1}}^{\pi} \left\{\frac{\xi(t) \, |K_n(t)|}{t^{-\delta}}\right\}^s dt\right]^{\frac{1}{s}} \\
&= O\left\{(n+1)^{\delta}\right\} \left[\int_{\frac{1}{n+1}}^{\pi} \left\{\frac{\xi(t) \, |K_n(t)|}{t^{-\delta}}\right\}^s dt\right]^{\frac{1}{s}} \quad \text{by (3.3)} \\
&= O\left\{(n+1)^{\delta}\right\} \left[\int_{\frac{1}{n+1}}^{\pi} \left\{\frac{\xi(t)}{t^{1-\delta}}\right\}^s dt\right]^{\frac{1}{s}} \quad \text{by Lemma 2.}
\end{aligned}$$

Now putting $t = \frac{1}{y}$,

$$I_2 = O\left\{(n+1)^{\delta}\right\} \left[\int_{\frac{1}{\pi}}^{n+1} \left\{\frac{\xi\left(\frac{1}{y}\right)}{y^{\delta-1}}\right\}^s \frac{dy}{y^2}\right]^{\frac{1}{s}}$$

Since $\xi(t)$ is a positive increasing function and using second mean value theorem for integrals,

$$\begin{aligned}
I_2 &= O\left\{(n+1)^{\delta} \, \xi\left(\frac{1}{n+1}\right)\right\} \left[\int_{\eta}^{n+1} \frac{dy}{y^{s(\delta-1)+2}}\right]^{\frac{1}{s}}, \text{ for some } \frac{1}{\pi} \leq \eta \leq n+1 \\
&= O\left\{(n+1)^{\delta} \, \xi\left(\frac{1}{n+1}\right)\right\} \left[\int_{1}^{n+1} \frac{dy}{y^{s(\delta-1)+2}}\right]^{\frac{1}{s}}, \text{ for some } \frac{1}{\pi} \leq 1 \leq n+1 \\
&= O\left\{(n+1)^{\delta} \, \xi\left(\frac{1}{n+1}\right)\right\} \left[\left\{\frac{y^{s(1-\delta)-1}}{s(1-\delta)-1}\right\}_1^{n+1}\right]^{\frac{1}{s}} \\
&= O\left\{(n+1)^{\delta} \, \xi\left(\frac{1}{n+1}\right)\right\} \left[(n+1)^{(1-\delta)-\frac{1}{s}}\right] \\
&= O\left\{\xi\left(\frac{1}{n+1}\right)\right\} \left\{(n+1)^{1-\frac{1}{s}}\right\} \\
&= O\left\{(n+1)^{\frac{1}{r}} \, \xi\left(\frac{1}{n+1}\right)\right\} \quad \text{since } \frac{1}{r} + \frac{1}{s} = 1. \qquad (5.3)
\end{aligned}$$

Now combining (5.1), (5.2) and (5.3), we get

$$\left|(EC)_n^1 - f(x)\right| = O\left\{(n+1)^{\frac{1}{r}}\,\xi\left(\frac{1}{n+1}\right)\right\}.$$

Now, using L_r- norm, we get

$$\begin{aligned}\left\|(EC)_n^1 - f\right\|_r &= \left\{\int_0^{2\pi}\left|(EC)_n^1 - f\right|^r dx\right\}^{\frac{1}{r}}\\ &= \left[\left\{\int_0^{2\pi}\left\{(n+1)^{\frac{1}{r}}\,\xi\left(\frac{1}{n+1}\right)\right\}^r dx\right\}^{\frac{1}{r}}\right]\\ &= \left\{(n+1)^{\frac{1}{r}}\,\xi\left(\frac{1}{n+1}\right)\right\}\left[\left\{\int_0^{2\pi} dx\right\}^{\frac{1}{r}}\right]\\ &= \left\{(n+1)^{\frac{1}{r}}\,\xi\left(\frac{1}{n+1}\right)\right\}.\end{aligned}$$

This completes the proof of the main theorem.

6. Applications

Following corollaries can be derived from our main theorem:

Corollary 1. *If* $\xi(t) = t^\alpha, 0 < \alpha \le 1$ *then the class* $Lip(\xi(t), r)$, $r \ge 1$, *reduces to the class* $Lip(\alpha, r)$ *and the degree of approximation of a function* $f \in Lip(\alpha, r), \frac{1}{r} < \alpha < 1$, *is given by*

$$\left|(EC)_n^1 - f\right| = O\left\{\frac{1}{(n+1)^{\alpha - \frac{1}{r}}}\right\}$$

Proof. We have

$$\left\|(EC)_n^1 - f\right\|_r = O\left\{\int_0^{2\pi}\left|(EC)_n^1 - f\right|^r dx\right\}^{\frac{1}{r}}$$

or

$$\left\{(n+1)^{\frac{1}{r}}\,\xi\left(\frac{1}{n+1}\right)\right\} = O\left\{\int_0^{2\pi}\left|(EC)_n^1 - f\right|^r dx\right\}^{\frac{1}{r}}$$

or

$$O(1) = O\left\{ \int_0^{2\pi} \left| (EC)_n^1 - f \right|^r dx \right\}^{\frac{1}{r}} .O\left\{ \frac{1}{(n+1)^{\frac{1}{r}} \xi\left(\frac{1}{n+1}\right)} \right\}$$

Hence

$$\left| (EC)_n^1 - f \right| = O\left\{ (n+1)^{\frac{1}{r}} \xi\left(\frac{1}{n+1}\right) \right\}$$

for if not the right-hand side will be O(1), therefore

$$\left| (EC)_n^1 - f \right| = O\left(\frac{1}{(n+1)^{\alpha - \frac{1}{r}}} \right)$$

□

Corollary 2. *If* $r \to \infty$ *in corollary 1 then the class* $f \in Lip\,(\alpha, r)$ *reduces to the class* $f \in Lip\alpha$ *and the degree of approximation of a function* $f \in \; Lip\alpha$, $0 < \alpha < 1$, *is given by*

$$\left\| (EC)_n^1 - f \right\|_\infty = O\left(\frac{1}{(n+1)^{\alpha}} \right)$$

Remark. An independent proof of corollary 1 can be obtained along the same lines of our theorem.

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