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A NOTE ON *G*-FRAME SEQUENCES

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Abstract: We study g-frame sequences in Hilbert spaces and observed with the help of examples that subsequence of a g-frame need not be a g-frame sequence. Finally, we gave a sufficient condition, in terms of g-frame sequences, for a g-frame to be an exact g-frame.

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1. Introduction

In 1952, Duffin and Schaeffer [5] introduced frames for Hilbert spaces while addressing some difficult problems arising from the theory of nonharmonic Fourier series. In particular, they generalized Gabor's method to define frames for Hilbert spaces. Later, in 1986, Daubechies, Grossman and Meyer [4] found a new fundamental application to wavelet and Gabor transforms in which frames played an important role. Today, frames have been widely used in signal processing, data compression, sampling theory and many other fields.

Recently, Sun [9] introduced a g-frame and a g-Riesz bases in a Hilbert space

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and obtained some results for g-frames and g-Riesz bases. He also observed that frame of subspaces (fusion frames) introduced by Casazza and Kutyniok [2] is a particular case of g-frame in Hilbert space. Also, a system of bounded quasi-projectors introduced by Fornasier [6] is a particular case of g-frame in a Hilbert space.

In the present paper, we study g-frame sequences introduced by Najati et al [8] and observed that a subsequence of a g-frame may not be a g-frame sequence. Examples have been given in this direction. Besides other results on g-frame sequences, we give a sufficient condition, in terms of g-frame sequences, for a g-frame to be an exact g-frame.

2. Preliminaries

Throughout this paper, H and K are two Hilbert spaces over $\mathbb{K}$ ($\mathbb{R}$ or $\mathbb{C}$) and $\{H_i\}_{i\in I}$ is a sequence of Hilbert spaces over $\mathbb{K}$, where I is a subset of integers. $B(H,H_i)$ is the collection of all bounded linear operators from H into H_i and I_H is the identity operator on H.

Definition 2.1. A sequence $\{x_i\}_{i\in I} \subset H$ is called a *frame* for H if there exist A, B with $0 < A \leq B < \infty$ such that

$$A\|x\|^2 \leq \sum_{i\in I} |\langle x, x_i\rangle|^2 \leq B\|x\|^2, \qquad x \in H\,.$$

The positive constants A and B, respectively, are called lower and upper frame bounds of the frame $\{x_i\}_{i\in I}$.

Definition 2.2. (see [9]) A sequence $\{\Lambda_i\}_{i\in I} \subseteq B(H,H_i)$ is called a *generalized frame* or simply a *g-frame* for H with respect to $\{H_i\}_{i\in I}$ if there exist A, B with $0 < A \leq B < \infty$ such that

$$A\|x\|^2 \leq \sum_{i\in I} \|\Lambda_i x\|^2 \leq B\|x\|^2, \qquad x \in H\,. \tag{2.1}$$

The positive constants A and B, respectively, are called the lower and upper frame bounds of the g-frame $\{\Lambda_i\}_{i\in I}$. The g-frame $\{\Lambda_i\}_{i\in I}$ is called a *tight g-frame* if $A = B$ and a *Parseval g-frame* if $A = B = 1$. The sequence $\{\Lambda_i\}_{i\in I}$ is called a *g-Bessel sequence* for H with respect to $\{H_i\}_{i\in I}$ with bound B if $\{\Lambda_i\}_{i\in I}$ satisfies the right hand side of the inequality (2.1). The sequence $\{\Lambda_i\}_{i\in I}$ is called an *exact g-frame* if it ceases to be a g-frame whenever any one of its elements is removed. The sequence $\{\Lambda_i\}_{i\in I} \subseteq B(H,H_i)$ is called g-complete if $\{x \in H : \Lambda_i x = 0, \text{ for all } i \in I\} = \{0\}$.

Definition 2.3. (see [9]) A sequence $\{\Lambda_i\}_{i\in I} \subseteq B(H, H_i)$ is called a g-orthonormal basis for H with respect to $\{H_i\}_{i\in I}$ if it satisfies the following

$$\langle \Lambda_i^* g_i, \Lambda_j^* g_j\rangle = \delta_{i,j}\langle g_i, g_j\rangle,\ g_i \in H_i, g_j \in H_j, \quad i, j \in I \tag{2.2}$$

and

$$\sum_{i\in I} \|\Lambda_i x\|^2 = \|x\|^2, \quad x \in H\,. \tag{2.3}$$

Definition 2.4. (see [8]) A sequence $\{\Lambda_i\}_{i\in I} \subseteq B(H, H_i)$ is called a g-frame sequence for H, if it is a g-frame for $\overline{\text{span}}\{\Lambda_i^*(H_i)\}_{i\in I}$.

Notation. For each sequence $\{H_i\}_{i\in I}$, define $\left(\sum_{i\in I} \oplus H_i\right)_{\ell_2}$ by

$$\left(\sum_{i\in I} \oplus H_i\right)_{\ell_2} = \left\{\{a_i\}_{i\in I} : a_i \in H_i, i \in I \text{ and } \sum_{i\in I} \|a_i\|^2 < \infty\right\},$$

with the inner product defined by $\langle\{a_i\}, \{b_i\}\rangle = \sum_{i\in I}\langle a_i, b_i\rangle$.
It is clear that $\left(\sum_{i\in I} \oplus H_i\right)_{\ell_2}$ is a Hilbert space with pointwise operations.

The following results which are referred in this paper are listed in the form of lemmas.

Lemma 2.5. (see [8]) $\{\Lambda_i\}_{i\in I} \subseteq B(H, H_i)$ *is a g-frame for H with respect to $\{H_i\}_{i\in I}$ if and only if*

$$T : \{x_i\}_{i\in I} \to \sum_{i\in I} \Lambda_i^*(x_i)$$

is a well defined and bounded mapping from $\left(\sum_{i\in I} \oplus H_i\right)_{\ell_2}$ onto H.

We call the operator T the *synthesis operator* for $\{\Lambda_i\}_{i\in I}$ and the adjoint T^* of the synthesis operator the *analysis operator* for $\{\Lambda_i\}_{i\in I}$.

Lemma 2.6. (see [8]) *If $\{\Lambda_i\}_{i\in I} \subseteq B(H, H_i)$ is a g-frame for H, then $\overline{\text{span}}\{\Lambda_i^*(H_i)\}_{i\in I} = H$.*

3. Main Result

Let $\{\Lambda_i\}_{i\in I} \subseteq B(H, H_i)$ be a g-frame for H and $\{i_k\}_{k\in I}$ be any infinite increasing sequence in I. Then $\{\Lambda_{i_k}\}_{k\in I}$ may not be a g-frame sequence for H. In this direction, we give the following examples.

Example 3.1. Let $\{e_n\}$ be the orthonormal basis for Hilbert space H with the frame bounds A and B. Define

$$H_n = [e_n], \quad n \in \mathbb{N}\,.$$

For each $i \in \mathbb{N}$, define $\Lambda_i : H \to H_i$ as

$$\Lambda_i(x) = \langle x, e_n\rangle e_n\,, \quad n \in \mathbb{N}\,.$$

Then, $\{\Lambda_i\}_{i\in\mathbb{N}}$ is g-frame for H with respect to $\{H_i\}_{i\in\mathbb{N}}$ with the bounds A and B. Further, $\{\Lambda_{n_k}\}_{k\in\mathbb{N}}$ is a frame sequence for H for every increasing subsequence of $\mathbb{N}$.

Example 3.2. Let $\{e_n\}$ be an orthonormal basis for H. Let $\{x_n\}$ be a sequence in H such that

$$x_1 = e_1$$

and

$$x_{n_k} = x_{n_k+1} = x_{n_k+2} = \cdots = x_{n_{k+1}-1} = \frac{e_k}{\sqrt{k}}\,, \quad k \geq 2,$$

where $n_k = n_{k-1} + (k-1)$, $k \in \mathbb{N}$ and $n_0 = 1$.
Now, for each $i \in \mathbb{N}$, define $\Lambda_i : H \to \mathbb{C}$ as

$$\Lambda_i(x) = \langle x, x_i\rangle, \quad x \in H\,.$$

Then, $\{\Lambda_i\}_{i\in\mathbb{N}}$ is g-frame for H with respect to $\mathbb{C}$. Indeed,

$$\begin{aligned}
\sum_{j=1}^{\infty} \|\Lambda_j(x)\|^2 &= \sum_{j=1}^{\infty} \|\langle x, x_j\rangle\|^2 \\
&= \sum_{j=1}^{\infty}\sum_{i=1}^{j} \left\|\left\langle x, \frac{e_i}{\sqrt{i}}\right\rangle\right\|^2 \\
&= \|x\|^2\,, \quad x \in H\,.
\end{aligned}$$

But $\{\Lambda_{n_k}\}_{k\in\mathbb{N}}$ is not g-frame sequence.

In view of Examples 3.1 and 3.2, the following observations arise naturally

(I) Let $\{\Lambda_i\}_{i\in I} \subseteq B(H, H_i)$ be a g-frame for H and $\{i_k\}_{k\in I}$ be any infinite increasing sequence in I. Then $\{\Lambda_{i_k}\}_{k\in I}$ is a g-frame sequence for H if $\{\Lambda_i\}_{i\in I}$ satisfies (2.2).

(II) Let $\{e_n\}$ be an orthonormal basis for H. Define a sequence $\{x_n\} \subset H$ as

$$x_1 = e_1$$
$$x_n = e_{n-1}, \quad n \geq 2\,.$$

For each $i \in \mathbb{N}$, define $\Lambda_i : H \to \mathbb{C}$ as

$$\Lambda_i(x) = \langle x, x_i\rangle, \quad x \in H\,.$$

Then, for any increasing subsequence $\{i_k\}_{k\in\mathbb{N}}$ of $\mathbb{N}$, $\{\Lambda_{i_k}\}_{k\in\mathbb{N}}$ is a g-frame sequence in H. But $\{\Lambda_i\}_{i\in\mathbb{N}}$ does not satisfies (2.2).

(III) Let $\{\Lambda_i\}_{i\in I} \subseteq B(H, H_i)$ be a g-frame for H with respect to $\{H_i\}_{i\in I}$. Let $\{i_k\}_{k\in I}$ and $\{j_k\}_{k\in I}$ be two infinite increasing sequences in I such that $\{i_k\}_{k\in I} \cup \{j_k\}_{k\in I} = I$. If $\{\Lambda_{i_k}\}_{k\in I}$ is a g-frame sequence for H then $\{\Lambda_{j_k}\}_{k\in I}$ need not be a g-frame sequence for H and vice-versa. Indeed, let $\{e_n\}$ be an orthonormal basis for H. Define a sequence $\{x_n\} \subset H$ as

$$x_{n_k} = x_{n_k+1} = x_{n_k+2} = \cdots = x_{n_{k+1}-1} = \frac{e_{k+1}}{\sqrt{k+1}}, \qquad k \geq 0$$

where, $n_k = n_{k-1} + (k+1)$, $k \in \mathbb{N}$ and $n_0 = 1$.

Then, for each $i = 1, 2, \ldots$, define $\Lambda_i : H \to \mathbb{C}$ as

$$\Lambda_i(x) = \langle x, x_i\rangle, \quad x \in H\,.$$

Let $\{j_k\}_{k\in\mathbb{N}}$ be the infinite increasing sequence such that $\{j_k\}_{k\in\mathbb{N}} = \mathbb{N} \setminus \{n_k\}_{k=0}^{\infty}$. Then, $\{\Lambda_{j_k}\}_{k\in\mathbb{N}}$ is a g-frame sequence for H but $\{\Lambda_{n_k}\}_{k=0}^{\infty}$ is not a g-frame sequence for H.

(IV) Let $\{\Lambda_i\}_{i\in I} \subseteq B(H, H_i)$ be a g-frame for H with bounds A and B, let $\{i_k\}_{k\in I}$ and $\{j_k\}_{k\in I}$ be two infinite increasing sequences with $\{i_k\}_{k\in I} \cup \{j_k\}_{k\in I} = I$ such that $\overline{\text{span}}\{\Lambda^*_{i_k}(H_{i_k})\}_{k\in I} = \overline{\text{span}}\{\Lambda^*_{j_k}(H_{j_k})\}_{k\in I}$. If $\{\Lambda_{i_k}\}_{k\in I}$ is a g-frame sequence for H with bounds A' and B' such that $A' < A$, then $\{\Lambda_{j_k}\}_{k\in I}$ is also a g-frame sequence for H. Indeed, if A' and B' be the frame bounds for $\{\Lambda_{i_k}\}_{k\in I}$. Then

$$A'\|x\|^2 \leq \sum_{k\in I} \|\Lambda_{i_k}(x)\|^2 \leq B'\|x\|^2, \quad x \in \overline{\text{span}}\{\Lambda^*_{i_k}(H_{i_k})\}_{k\in I}.$$

Therefore, by frame inequality for the g-frame $\{\Lambda_i\}_{i\in I}$

$$(A-A')\|x\|^2 \leq \sum_{k\in I}\|\Lambda_{j_k}(x)\|^2 \leq B\|x\|^2, \quad x\in \overline{\text{span}}\ \{\Lambda^*_{j_k}(H_{j_k})\}_{k\in I}\,.$$

In view of observation (IV), we have the following result.

Theorem 3.3. *Let $\{\Lambda_i\}_{i\in I}\subseteq B(H,H_i)$ be a g-frame for H and let $\{i_k\}_{k\in I}$ and $\{j_k\}_{k\in I}$ be two infinite increasing sequences with $\{i_k\}_{k\in I}\cup\{j_k\}_{k\in I}=I$. Let $\{\Lambda_{i_k}\}_{k\in I}$ be a g-frame for H with respect to $\{H_{i_k}\}_{k\in I}$. Then, $\{\Lambda_{j_k}\}_{k\in I}$ is a g-frame for H with respect to $\{H_{j_k}\}_{k\in I}$ if and only if there exits a bounded linear operator $T:\left(\sum_{k\in I}\oplus H_{j_k}\right)_{\ell_2}\to\left(\sum_{k\in I}\oplus H_{i_k}\right)_{\ell_2}$ such that*

$$T(\{\Lambda_{j_k}(x)\}_{k\in I})=\{\Lambda_{i_k}(x)\}_{k\in I},\quad x\in H\,.$$

Proof. Let $\{\Lambda_{i_k}\}_{k\in I}$ be a g-frame for H with respect to $\{H_{i_k}\}_{k\in I}$ with bounds A and B. Since for each $x\in H$

$$\begin{aligned}\sum_{k\in I}\|\Lambda_{i_k}(x)\|^2 &= \sum_{k\in I}\|T(\Lambda_{j_k}(x))\|^2\\ &\leq \|T\|^2\sum_{k\in I}\|\Lambda_{j_k}(x)\|^2\end{aligned}$$

we have

$$\begin{aligned}\sum_{k\in I}\|\Lambda_{j_k}(x)\|^2 &\geq \frac{\sum_{k\in I}\|\Lambda_{i_k}(x)\|^2}{\|T\|^2}\\ &\geq \frac{A\|x\|^2}{\|T\|^2}\end{aligned}$$

Hence $\{\Lambda_{j_k}\}_{k\in I}$ is also a g-frame for H with respect to $\{H_{j_k}\}_{k\in I}$.

Conversely, Let $\{\Lambda_{j_k}\}_{k\in I}$ be a g-frame for H with respect to $\{H_{j_k}\}_{k\in I}$. Then, by Lemma 2.5, $\{\Lambda_{j_k}\}_{k\in I}$ has the synthesis operator

$$T_1:\left(\sum_{k\in I}\oplus H_{j_k}\right)_{\ell_2}\to H \text{ given by}$$
$$T_1(\{\Lambda_{j_k}(x)\}_{k\in I})\to x$$

with the adjoint operator

$$T_1^* : H \to \left(\sum_{k\in I} \oplus H_{j_k}\right)_{\ell_2} \text{ given by}$$
$$T_1^*(x) \to (\{\Lambda_{j_k}(x)\}_{k\in I}), \quad x \in H.$$

Similarly, as $\{\Lambda_{i_k}\}_{k\in I}$ is also a g-frame for H with respect to $\{H_{i_k}\}_{k\in I}$. Then, again by Lemma 2.5, $\{\Lambda_{i_k}\}_{k\in I}$ has the synthesis operator

$$T_2 : \left(\sum_{k\in I} \oplus H_{i_k}\right)_{\ell_2} \to H \text{ given by}$$
$$T_2(\{\Lambda_{i_k}(x)\}_{k\in I}) \to x$$

with the adjoint operator

$$T_2^* : H \to \left(\sum_{k\in I} \oplus H_{i_k}\right)_{\ell_2} \text{ given by}$$
$$T_2^*(x) \to \{\Lambda_{i_k}(x)\}_{k\in I}, \quad x \in H.$$

Then $T = T_2^* T_1 : \left(\sum_{k\in I} \oplus H_{j_k}\right)_{\ell_2} \to \left(\sum_{k\in I} \oplus H_{i_k}\right)_{\ell_2}$ is a bounded linear operator such that

$$T(\{\Lambda_{j_k}(x)\}_{k\in I}) = \{\Lambda_{i_k}(x)\}_{k\in I}, \quad x \in H. \qquad \square$$

Finally, we give a sufficient condition, in terms of g-frame sequences, for a g-frame to be an exact g-frame.

Theorem 3.4. *Let $\{\Lambda_i\}_{i\in I} \subseteq B(H, H_i)$ be a g-frame for H and with optimal bound A and B such that $\{\Lambda_i\}_{i\in I}$ is a sequence of non-zero operators. If for every infinite increasing sequence $\{i_k\}_{k\in I}$ in I, $\{\Lambda_{i_k}\}_{k\in I}$ is a g-frame sequence with bounds A and B, then $\{\Lambda_i\}_{i\in I}$ is an exact g-frame.*

Proof. Suppose that $\{\Lambda_i\}_{i\in I}$ is not an exact. Then, there exists $m \in I$ such that $\{\Lambda_i\}_{i\neq m}$ is a g-frame for H. Therefore, by Lemma 2.6,

$$\Lambda_m^*(x_m) \in \overline{\text{span}}\{\Lambda_i^*(H_i)\}_{i\neq m} = H, \quad x_m \in H_m\,.$$

Let $\{i_k\}_{k\in I}$ be an infinite increasing sequence given by $i_k = k$, $k = 1, 2, \ldots, m-1$ and $i_k = k+1, k = m, m+1, \ldots$. Then $\{\Lambda_{i_k}\}_{k\in I}$ is a g-frame sequence for H with bounds A and B. So,

$$A\|x\|^2 \le \sum_{\substack{i_k\neq m\\ k\in I}} \|\Lambda_{i_k}(x)\|^2 \le B\|x\|^2, \quad x \in \overline{\text{span}}\{\Lambda_{i_k}^*(x_{i_k})\}_{k\in I}.$$

Also, $\{\Lambda_i\}_{i\in I}$ is a g-frame for H with bounds A and B, we have

$$A\|x\|^2 \leq \sum_{i\in I} \|\Lambda_i(x)\|^2 \leq B\|x\|^2, \quad x \in H.$$

This gives $\Lambda_i x = 0$, for $i = m$, a contradiction. Hence $\{\Lambda_i\}_{i\in I}$ is an exact g-frame for H. □

Remark 3.5. The condition in Theorem 3.4 is not necessary. In view of this we have following example

Example 3.6. Let $\{e_n\}$ be an orthonormal basis for H. For each $i \in \mathbb{N}$, define $\Lambda_i : H \to \mathbb{C}$ as

$$\Lambda_i(x) = \langle x, x_i\rangle, \quad x \in H.$$

where $x_1 = e_1$, $x_2 = \dfrac{e_2}{2}$ and $x_i = e_i$, for all $i \geq 3,\ i \in \mathbb{N}$. Let $i_k = k+1$, for all $k \geq 2$ and $i_1 = 1$. Then, $\{i_k\}_{k\in\mathbb{N}}$ is an infinite increasing sequence such that $\{\Lambda_{i_k}\}_{k\in\mathbb{N}}$ is a g-frame sequence for H, which do not have same optimal bounds, but $\{\Lambda_i\}_{i\in\mathbb{N}}$ is an exact g-frame.

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