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DIFFERENTIAL SUBORDINATION AND SUPERORDINATION FOR A NEW LINEAR DERIVATIVE OPERATOR

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Abstract: Having tremendous interests in the study of linear operators, a new generalization of linear derivative operator $D_p^{\alpha,\delta}(\mu,c,\lambda)$ is introduced in this current paper. The aim of the paper is to investigate several subordination and superordination for the aforementioned generalized linear derivative operator. Further, we also consider the sandwich-type result for this operator.

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1. Definition and Preliminaries

Let $H(\mathbb{U})$ denote the class of holomorphic functions in the open unit disk $\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}$ of complex plane.For $p \in \mathbb{N}$ and $a \in \mathbb{C}$ we define:

$$H[a,p] = \{f \in H(\mathbb{U}) : f(z) = a + a_p z^p + a_{p+1} z^{p+1} + \cdots\}, \quad z \in \mathbb{U},$$

$$A(p) = \{f \in H(\mathbb{U}) : f(z) = z^p + \sum_{k=p+1}^{\infty} a_k z^k\}, (p \in \mathbb{N}), \tag{1.1}$$

and set $A \equiv A(1)$.

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For functions $f(z) \in A(p)$, given by (1.1), and $g(z)$ given by

$$g(z) = z^p + \sum_{k=p+1}^{\infty} a_k z^k,$$

the Hadamard product (or convolution) $f * g$ of functions f and g is defined by :

$$(f * g)(z) = z^p + \sum_{k=p+1}^{\infty} a_k b_k z^k, (p \in \mathbb{N}).$$

Let f and g be analytic functions in the unit disk $\mathbb{U}$, we say that a function f is subordinate to a function g if there exists an analytic function ω with

$$\omega(0) = 0, \quad |\omega(z)| < 1 \qquad for \quad all \quad (z \in \mathbb{U}),$$

such that

$$f(z) = g(w(z)), \quad (z \in \mathbb{U}).$$

We denote this subordination by $(f \prec g)$. Furthermore, if a function g is univalent in $(\mathbb{U})$ we have the following equivalence

$$f(z) \prec g(z) \Leftrightarrow f(0) = g(0) \text{ and } f(\mathbb{U}) \subset g(\mathbb{U}).$$

A function f belonging to $A(p)$ is said to be p-valently starlike of order β if it satisfies

$$Re\left\{\frac{zf'(z)}{f(z)}\right\} > \beta, \quad (z \in \mathbb{U}),$$

for some $\beta, \quad (0 \leq \beta < p)$. We denote by $S^*_\beta(p)$, the subclass of $A(p)$ consisting of functions which are p-valently starlike of order β in $\mathbb{U}$.

Further, a function f belonging to $A(p)$ is said to be p-valently convex of order β if it satisfies

$$Re\left\{1 + \frac{zf''(z)}{f'(z)}\right\} > \beta, \quad (z \in \mathbb{U}),$$

for some $\beta, \quad (0 \leq \beta < p)$. We denote by $K_\beta(p)$ the subclass of functions in $A(p)$ which are p-valently convex of order β in $\mathbb{U}$.

The method of differential subordinations (also known as the admissible functions method) was perhaps first introduced by Miller and Mocanu in 1978 [1] and the theory started to develop in 1981 [2]. All the details are captured in a book by Miller and Mocanu [3].

Definition 1.1. (see [3]) Let $\psi(r, s, t; z) : \mathbb{C}^3 \times \mathbb{U} \to \mathbb{C}$ and let $h(z)$ be univalent function in $(\mathbb{U})$. If ξ is analytic function in $\mathbb{U}$ and satisfies the (second-order) differential subordination

$$\textit{Let} \quad \psi(\xi(z), z\xi'(z), z^2\xi''(z); z \in \mathbb{U}) \prec h, \quad (z \in \mathbb{U}), \tag{1.2}$$

then ξ is called a solution of the differential subordination. The univalent function q is called a dominant of the solutions of the differential subordination, or, simply, a dominant, if $\xi \prec q$ for all ξ satisfying (1.2). A dominant $\widetilde{q}$ that satisfies $\widetilde{q} \prec q$ for all dominants q of (1.2) is said to be the best dominant of (1.2). (Note that the best dominant is unique up to a rotation of $\mathbb{U}$).

Definition 1.2. (see [4]) Let $\varphi(r, s, t; z) : \mathbb{C}^3 \times \mathbb{U} \to \mathbb{U}$ and let h be analytic function in $\mathbb{U}$. If ξ and $\varphi(\xi(z), z\xi'(z), z^2\xi''(z); z)$ are univalent in $\mathbb{U}$ and ξ satisfy the (second-order) differential superordination

$$h(z) \prec \varphi(\xi(z), z\xi'(z), z^2\xi''(z); z), \tag{1.3}$$

then ξ is called a solution of the differential superordination. An analytic function q is called a subordinate of the solutions of the differential superordination or, simply, a subordinate if $q \prec \xi$ for all ξ satisfying (1.3). A univalent subordinate $\widetilde{q}$ that satisfies $q \prec \widetilde{q}$ for all subordinates q of (1.3) is said to be the best subordinate. (Note that the best subordinate is unique up to a rotation of $\mathbb{U}$).

Definition 1.3. (see [4]) We denote by Q the set of functions f that are analytic and injective on $\overline{\mathbb{U}} \setminus E(f)$, where

$$E(f) = \{\zeta \in \partial\mathbb{U} : \lim_{z \to \zeta} f(z) = \infty\},$$

and are such that $f'(\zeta) \neq 0, \quad \zeta \in \partial\mathbb{U} \setminus E(f)$. The subclass of Q for which $f(0) = a, \quad (a \in \mathbb{C})$ is denoted by $Q(a)$.

In order to prove new results, we shall use the lemmas below

Lemma 1.4. (see [3]) *Let q be univalent function in $\mathbb{U}$ and let θ and ϕ be analytic functions in a domain D containing $q(\mathbb{U})$, with $\phi(w) \neq 0$, when $w \in q(\mathbb{U})$. Set*

$$Q(z) := zq'(z)\phi[q(z)], \qquad h(z) := \theta[q(z)] + Q(z),$$

and suppose that either:

(i) h is convex, or

(ii) Q is starlike.

In addition, assume that:
(iii) $Re\left\{\frac{zh'(z)}{Q(z)}\right\} = Re\left\{\frac{\theta'[q(z)]}{\phi[q(z)]} + \frac{zQ'(z)}{Q(z)}\right\} > 0.$
If ξ *is analytic in* $\mathbb{U}$, *with* $\xi(0) = q(0), \quad \xi(\mathbb{U}) \subset D$, *and*

$$\theta[\xi(z)] + z\xi'(z)\phi[\xi(z)] \prec \theta[q(z)] + zq'(z)\phi[q(z)],$$

then $\xi \prec q$, *and* q *is the best dominant.*

Lemma 1.5. (see [5]) *Let* q *be univalent in* $\mathbb{U}$ *with* $q(0) = a$, *and* θ *and* φ *be analytic in a domain* D *containing* $q(\mathbb{U})$. *Define*

$$Q(z) = zq'(z)\varphi[q(z)], \qquad h(z) = \theta[q(z)] + Q(z).$$

Suppose that:
(i) $Re\left\{\frac{\theta'[q(z])}{\varphi[q(z)]}\right\} > 0,$
(ii) Q *is starlike univalent in* $\mathbb{U}$.
If $\xi \in H[a, 1] \cap Q$, *with* $\xi(\mathbb{U}) \subset D$, *and* $\theta[\xi(z)] + z\xi'(z)\varphi[\xi(z)]$, *is univalent in* $\mathbb{U}$, *then*

$$\theta[q(z)] + zq'(z)\varphi[q(z)] \prec \theta[\xi(z)] + z\xi'(z)\varphi[\xi(z)],$$

implies $q(z) \prec \xi(z)$, *and* $q(z)$ *is the best subordinant.*

Now, $(x)_k$ denotes the Pochhammer symbol (or the shifted factorial) defined by

$$(x)_k = \begin{cases} 1 & for \quad k = 0, x \in \mathbb{C} - \{0\}, \\ x(x+1)(x+2)...(x+k-1) & for \quad k \in \mathbb{N} = 1, 2, 3, ...and \quad x \in \mathbb{C}. \end{cases}$$

For $f \in A(p)$, Mahzoon and Latha [12] introduced the following operator

$$D_p(\mu, c, \lambda) = z^p + \sum_{k=p+1}^{\infty} (1 + \frac{k-p}{p+c}\lambda)^{\mu} a_k z^k,$$

where $\lambda, \mu, c \in \mathbb{R}$, $\lambda, \mu, c \geq 0$.

Now, we introduce the new linear derivative operator as the following:

Definition 1.6. For $f \in A(p)$ the linear operator $D_p^{\alpha,\delta}(\mu, c, \lambda)$ is defined by $D_P^{\alpha,\delta}(\mu, c, \lambda) : A(p) \to A(p)$ as

$$D_p^{\alpha,\delta}(\mu, c, \lambda) = k^{\alpha} * D_p(\mu, c, \lambda) * R^{\delta} \qquad z \in \mathbb{U}, \tag{1.4}$$

where $\lambda, \mu, c \in \mathbb{R}$, $\lambda, \mu, c \geq 0$, $k, \delta \in \mathbb{N}_0 = \{0, 1, 2...\}$, and R^δ denotes the Ruscheweyh derivative operator and given by

$$R^\delta = z + \sum_{k=2}^{\infty} c(\delta, k) a_k z^k \quad for \quad \delta \in \mathbb{N}_0, (z \in \mathbb{U}),$$

where $c(\delta, k) = \frac{(\delta+1)_{k-1}}{(1)_{k-1}}$.

If f is given by (1.1), then we easily find from the equality (1.4) that

$$D_p^{\alpha,\delta}(\mu, c, \lambda) = z^p + \sum_{k=p+n}^{\infty} k^\alpha \left(1 + \frac{k-p}{p+c}\lambda\right)^\mu c(\delta, k) a_k z^k, \tag{1.5}$$

where $c(\delta, k) = \frac{(\delta+1)_{k-1}}{(1)_{k-1}}$.

Special cases of this operator include:

• $D_1^{0,n}(0, c, \lambda) \equiv D_p^{0,n}(0, c, \lambda) \equiv R^n$ is the Ruscheweyh derivative operator, see [6].

• $D_1^{0,n}(\mu, 0, \lambda)$ for, $(\mu \in \mathbb{N}_0 = 0, 1, 2, ...), \equiv R_\lambda^n$ is the generalized Ruscheweyh derivative operator, see [13].

• $D_1^{\alpha,0}(0, c, \lambda) \equiv D_p^{\alpha,0}(0, c, \lambda) \equiv D_1^{0,0}(\mu, 0, 1) \equiv D_p^{0,0}(\mu, 0, 1) \equiv S^n$ is the Salagean derivative operator, see [7].

• $D_1^{0,0}(\mu, 0, \lambda) \equiv S_\lambda^n$ is the Salagean derivative operator introduced by Al-Oboudi [8].

• $D_1^{0,\delta}(\mu, 0, \lambda) \equiv D_\lambda^n$ is the generalized Al-Shaqsi and Darus derivative operator, see [14].

• $D_p^{0,0}(n, \lambda, 1) \equiv I_p(n, \lambda)$, $(n \in \mathbb{Z})$, is the operator studied by Aghalary et al [9].

• $D_1^{0,0}(n, \lambda, 1) \equiv I_1(n, \lambda)$ is investigated by Cho and Srivastava$(n \in \mathbb{Z})$, see [10], and also Cho and Kim [11].

• $D_p^{0,0}(\mu, c, \lambda) \equiv D_p(\mu, c, \lambda)$ is the operator introduced by Mahzoon and Latha [12].

For $k, \delta \in \mathbb{N}_0 = \{0, 1, 2...\}$, and $\lambda, \mu, c \geq 0$, we need the following equality to prove our results

$$(p+c) D_p^{\alpha,\delta}(\mu+1, c, \lambda) f(z) = \lambda z [D_p^{\alpha,\delta}(\mu, c, \lambda) f(z)]' + (p+c-\lambda p) D_p^{\alpha,\delta}(\mu, c, \lambda) f(z). \tag{1.6}$$

2. Main Results

Theorem 2.1. *Let $\lambda > 0, \mu, c \geq 0$ $, \alpha, k \in \mathbb{N}_0 = \{1, 2, 3, 4 \cdots\}$ and $D_p^{\alpha,\delta}(\mu, c, \lambda)$ linear operator given by (1.5). If $f \in A(p)$, and satisfy the differential subordination*

$$\frac{z(D_p^{\alpha,\delta}(\mu+1,c,\lambda)f(z))'}{D_p^{\alpha,\delta}(\mu+1,c,\lambda)f(z)} + (\frac{\lambda - c - p}{\lambda})\frac{D_p^{\alpha,\delta}(\mu+1,c,\lambda)f(z)}{D_p^{\alpha,\delta}(\mu,c,\lambda)f(z)}$$

$$+\frac{1}{\lambda}(p + c - \lambda p) \prec h(z), \tag{2.1}$$

where

$$h(z) = 1 + z + \frac{z}{(1-z)}, \quad (z \in \mathbb{U}).$$

Then

$$\frac{D_p^{\alpha,\delta}(\mu+1,c,\lambda)f(z)}{D_p^{\alpha,\delta}(\mu,c,\lambda)f(z)} \prec 1 + z,$$

and $1 + z$ is the best dominant.

Proof. Setting

$$\frac{D_p^{\alpha,\delta}(\mu+1,c,\lambda)f(z)}{D_p^{\alpha,\delta}(\mu,c,\lambda)f(z)} = \xi(z), \tag{2.2}$$

where

$$\xi(z) = 1 + A_1 z + A_2 z^2 + \cdots, \xi(0) = 1, \mathit{and} \quad \xi \in H[1,1].$$

Differentiating (2.2), we obtain

$$\frac{z\xi'(z)}{\xi(z)} = \frac{z(D_p^{\alpha,\delta}(\mu+1,c,\lambda)f(z))'}{D_p^{\alpha,\delta}(\mu+1,c,\lambda)f(z)} - \frac{z(D_p^{\alpha,\delta}(\mu,c,\lambda)f(z))'}{D_p^{\alpha,\delta}(\mu,c,\lambda)f(z)}. \tag{2.3}$$

Using $(2.3), (1.6)$ becomes

$$\frac{z\xi'(z)}{\xi(z)} = \frac{z(D_p^{\alpha,\delta}(\mu+1,c,\lambda)f(z))'}{D_p^{\alpha,\delta}(\mu+1,c,\lambda)f(z)} - (\frac{c+p}{\lambda})\frac{D_p^{\alpha,\delta}(\mu+1,c,\lambda)f(z)}{D_p^{\alpha,\delta}(\mu,c,\lambda)f(z)} + \frac{1}{\lambda}(p+c-\lambda p),$$

$$\xi(z) + \frac{z\xi'(z)}{\xi(z)} = \frac{z(D_p^{\alpha,\delta}(\mu+1,c,\lambda)f(z))'}{D_p^{\alpha,\delta}(\mu+1,c,\lambda)f(z)} + (\frac{\lambda - c - p}{\lambda})\frac{D_p^{\alpha,\delta}(\mu+1,c,\lambda)f(z)}{D_p^{\alpha,\delta}(\mu,c,\lambda)f(z)}$$

$$+\frac{1}{\lambda}(p + c - \lambda p). \tag{2.4}$$

Using (2.4), the differential subordination (2.1) becomes

$$\xi(z) + \frac{z\xi'(z)}{\xi(z)} \prec 1 + z + \frac{z}{1+z}.$$

To prove the theorem, we use Lemma 1.4. For that, let $q(z) = 1 + z$, $q(u) = \{w \in \mathbb{C} : Re\,|w-1| < 1\}$. We define the functions

$$\theta : D \supset q(u) \to \mathbb{C}$$

given by $\theta(w) = w$,and

$$\phi : D \supset q(u) \to \mathbb{C} :$$

$\phi(w) = \frac{1}{w}$, with $\phi(w) \neq 0$, it can easily be observed that $\theta(z), \phi(z)$ are analytic in $\mathbb{C}$. Next paragraph, we also let

$$\theta(q(z)) = q(z) = 1 + z,$$

and

$$\phi(q(z)) = \frac{1}{1+z},$$

then

$$Q(z) = zq'(z)\phi(q(z)) = \frac{z}{(1+z)},$$

and

$$h(z) = \theta(q(z)) + Q(z) = 1 + z + \frac{z}{1+z}.$$

We now calculate

$$Re\left\{\frac{zQ'(z)}{Q(z)}\right\} = Re\left\{\frac{1}{1+z}\right\} > 0,$$

hence Q is starlike in $\mathbb{U}$. And

$$Re\left\{\frac{\theta'[q(z)]}{\phi(z)}\right\} = Re\{1+z\} > 0,$$

then

$$Re\left\{\frac{zh'(z)}{Q(z)}\right\} = Re\left\{\frac{\theta'[q(z)]}{\phi[q(z)]} + \frac{zQ'(z)}{Q(z)}\right\} > 0.$$

Since Q is starlike and $Re\left\{\frac{zh'(z)}{Q(z}\right\} > 0, z \in \mathbb{U}$, by Lemma 1.4 we have

$$\frac{D_p^{\alpha,\delta}(\mu+1, c, \lambda)f(z)}{D_p^{\alpha,\delta}(\mu, c, \lambda)f(z)} \prec 1 + z,$$

and $z + 1$ the best dominant.

Theorem 2.2. *Let $\lambda > 0$, $\mu, c \geq 0$, $\alpha, k \in \mathbb{N}_0 = \{1, 2, 3, \cdots\}$, and $D_p^{\alpha,\delta}(\mu, c, \lambda)$ linear operator given by (1.5). If $\xi \in H[1, 1] \cap Q, \xi(\mathbb{U}) \subset D$, and*

$$\frac{z(D_p^{\alpha,\delta}(\mu+1, c, \lambda)f(z))'}{D_p^{\alpha,\delta}(\mu+1, c, \lambda)f(z)} + (\frac{\lambda - c - p}{\lambda})\frac{D_p^{\alpha,\delta}(\mu+1, c, \lambda)f(z)}{D_p^{\alpha,\delta}(\mu, c, \lambda)f(z)}$$
$$+\frac{1}{\lambda}(p + c - \lambda p)),$$

is univalent in $\mathbb{U}$. Then

$$1 + \gamma z + \frac{\gamma z}{1 + \gamma z} \prec \frac{z(D_p^{\alpha,\delta}(\mu+1, c, \lambda)f(z))'}{D_p^{\alpha,\delta}(\mu+1, c, \lambda)f(z)} + (\frac{\lambda - c - p}{\lambda})\frac{D_p^{\alpha,\delta}(\mu+1, c, \lambda)f(z)}{D_p^{\alpha,\delta}(\mu, c, \lambda)f(z)}$$
$$+\frac{1}{\lambda}(p + c - \lambda p), \tag{2.5}$$

implies $1 + \gamma z \prec \xi(z) = \frac{D_p^{\alpha,\delta}(\mu+1,c,\lambda)f(z)}{D_P^{\alpha,\delta}(\mu,c,\lambda)f(z)}$, and qis the best subordinat.

Proof. Setting

$$\frac{D_p^{\alpha,\delta}(\mu+1, c, \lambda)f(z)}{D_P^{\alpha,\delta}(\mu, c, \lambda)f(z)} = \xi(z),$$

then by using (2.4),and (2.5)

$$1 + \gamma z + \frac{\gamma z}{1 + \gamma z} \prec \xi(z) + \frac{z\xi'(z)}{\xi(z)}, \quad (z \in \mathbb{U}).$$

In order to prove the theorem, we use Lemma 1.5 and setting $q(z) = 1 + \gamma z$, $q(u) = \{w \in \mathbb{C} : Re|w - 1| > \gamma\}, (0 \leq \gamma < 1)$, $q(\mathbb{U}) \subset D$.

Define the functions $\theta : D \supset q(u) \to \mathbb{C}$; $\theta(w) = w$, and $\phi : D \supset q(u) \to \mathbb{C}$; $\phi(w) = \frac{1}{w}$, with $\phi(w) \neq 0$.

It can easily be observed that $\theta(z)$, $\varphi(z)$ are analytic in $\mathbb{C}$. Also, we let

$$\theta(q(z)) = q(z) = 1 + \gamma z,$$

and

$$\phi(q(z)) = \frac{1}{1 + \gamma z}.$$

Then

$$Q(z) = zq'(z)\phi(q(z)) = \frac{\gamma z}{1 + \gamma z},$$

and

$$h(z) = \theta(q(z)) + Q(z) = 1 + \gamma z + \frac{\gamma z}{1 + \gamma z}.$$

Then

$$Re\left\{\frac{zQ'(z)}{Q(z)}\right\} = Re\left\{\frac{1}{1+\gamma z}\right\} > 0,$$

hence Q is starlike and univalent in $\mathbb{U}$. And also

$$Re\left\{\frac{\theta'[q(z)]}{\phi(z)}\right\} = Re\left\{\frac{\gamma}{1+\gamma z}\right\} > 0.$$

Since the conditions in Lemma 1.5 are satisfied, by using it we obtain

$$1+\gamma z \prec \frac{D_p^{\alpha,\delta}(\mu+1,c,\lambda)f(z))}{D_p^{\alpha,\delta}(\mu,c,\lambda)f(z)},$$

and $1+\gamma z$ is the best subordinant.

Combining the results of differential subordination and superordination, we state the following (sandwich result).

Theorem 2.3. *Let* $\lambda > 0, \mu, c \geq 0 \quad \alpha, k \in \mathbb{N}_0$ *and* $D_p^{\alpha,\delta}(\mu,c,\lambda)$ *linear operator given by* (1.5). *If* $f \in A(p,n)$ *and If the subordination*

$$1+\gamma z+\frac{\gamma z}{1+\gamma z} \prec \frac{z(D_p^{\alpha,\delta}(\mu+1,c,\lambda)f(z))'}{D_p^{\alpha,\delta}(\mu+1,c,\lambda)f(z)}$$

$$+(\frac{\lambda-c-p}{\lambda})\frac{D_p^{\alpha,\delta}(\mu+1,c,\lambda)f(z)}{D_p^{\alpha,\delta}(\mu,c,\lambda)f(z)}+\frac{1}{\lambda}(p+c-\lambda p)) \prec 1+z+\frac{z}{1+z},$$

holds then

$$1+\gamma z \prec \frac{D_p^{\alpha,\delta}(\mu+1,c,\lambda)f(z)}{D_p^{\alpha,\delta}(\mu,c,\lambda)f(z)} \prec 1+z,$$

and $1+\gamma z$ *and* $1+z$ *are respectively the best subordinant and best dominant.*

We omit the proof as it is the same as in proof of the previous theorem.

Other work regarding differential operators for various problems can be found in [15]-[20].

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