

EXISTENCE AND UNIQUENESS RESULTS FOR  
FUZZY IMPULSIVE INTEGRODIFFERENTIAL EQUATIONS

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**Abstract:** In this paper, we discuss the existence and uniqueness of fuzzy impulsive solutions for nonlinear fuzzy impulsive integrodifferential equation by using via Banach fixed point analysis approach and the fuzzy number whose values are normal, convex upper semicontinuous and compactly supported interval in  $E_N$ .

**AMS Subject Classification:** 34A10, 26E50, 47E05

**Key Words:** fuzzy impulsive integrodifferential equation, initial value problem, normal fuzzy convex sets, fixed point theorem existence and uniqueness

1. Introduction

There exists an extensive theory for neutral functional integrodifferential equations which includes qualitative behaviour of classes of such equations and applications to biological and engineering processes for details, [1], [5], [6], [8]. However, the concrete example is the radio cardiogram, where the two compartments correspond to the left and right ventricles of the pulmonary and systematic circulation. Pipes coming out from and returning into the same compartment may represent shunts, and the equation representing this model is a nonlinear neutral volterra integrodifferential equations in [2], [3], [4]. For these reasons, there has been an increasing interest in studying equations that can be described in the form

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Received: May 17, 2011

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$$\begin{aligned}
 x'(t) &= a(t)x(t) + \int_0^t k(t,s,x(s))ds + f(t,x(t)), \quad t \in J \quad (1.1) \\
 \Delta x|_{t=t_k} &= I_k(x(t_k)), \quad k = 1, 2, 3, \dots, m \\
 x(0) &= x_0
 \end{aligned}$$

Where  $x_t$  represents the history of  $x$  at  $t$  and  $a : J \rightarrow E_N$  is a fuzzy coefficient.  $E_N$  is set of all upper semi-continuous, convex, normal fuzzy numbers with bounded  $\alpha$ -level intervals,

$$\Delta x(t_k) = x(t_k^+) - x(t_k^-),$$

$x(t_k^+)$  and  $x(t_k^-)$  denote the right limits and the left limits of  $x(t)$  at  $t = t_k$  respectively,  $0 < t_1 < t_2 < \dots < t_k < T$ , and  $f, g : J \times E_N \rightarrow E_N$  are nonlinear continuous functions.

The rest of this paper is organized as follows: In Section 2 we recall briefly some some basic definitions and preliminary facts about multi-fuzzy number valued maps and evolution systems which will be used throughout this this paper. The existence and uniqueness theorem for equation 1.1 and their proof is arranged in Section 3.

Finally, in Section 4 an example is presented to illustrate the applications of the obtained result.

## 2. Preliminaries

A fuzzy subset of  $R^n$  is defined in terms of a membership function which assigns to each point  $x \in R^n$  a grade of membership in the fuzzy set. Such a membership function is denoted by

$$u : R^n \rightarrow [0, 1].$$

Throughout this paper, we assume that  $u$  maps  $R^n$  onto  $[0, 1]$ ,  $[u]^0$  is a bounded subset of  $R^n$ ,  $u$  is upper semicontinuous, and  $u$  is fuzzy convex. We denote by  $E^n$  the space of all fuzzy subsets  $u$  of  $R^n$  which are normal, fuzzy convex, and upper semicontinuous fuzzy sets with bounded supports. In particular,  $E'$  denote the space of all fuzzy subsets  $u$  of  $R$ .

A fuzzy number  $a$  in real line  $R$  is a fuzzy set characterized by a membership function  $\mu_a$  as

$$\mu_a : R \rightarrow [0, 1],$$

A fuzzy number  $a$  is expressed as

$$a = \int_{x \in R} \frac{\mu_a(x)}{x}$$

with the understanding that  $\mu_a(x) \in [0, 1]$  represent the grade of membership of  $x$  in  $a$  and  $\int$  denotes the union of  $\mu_a(x)$

**Definition 2.1.** A fuzzy number  $a$  in  $R$  is said to be convex if for any real numbers  $x, y, z$  in  $R$  with  $x \leq y \leq z$ ,

$$\mu_a(y) \geq \min \{ \mu_a(x), \mu_a(z) \}.$$

**Definition 2.2.** The height of a fuzzy set is the largest membership value attained by any point.

**Definition 2.3.** If the height of a fuzzy set equals one, then the fuzzy set is called a normal fuzzy set.

Thus, a fuzzy number  $a$  in  $R$  is called normal if the following holds:

$$\max_z \mu_a(x) = 1$$

**Result 2.1.** Let  $E_n$  be the set of all upper semicontinuous convex normal fuzzy numbers with bounded  $\alpha$ - level intervals [7] this means that if  $a \in E_N$ , then the  $\alpha$ - level set

$$[a]^\alpha = \{ x \in R : a(x) \geq \alpha, 0 < \alpha \leq 1 \}$$

is a closed bounded interval which we denote by

$$[a]^\alpha = [a_q^\alpha, a_r^\alpha],$$

and there exists a  $t_0 \in R$  such that  $a(t_0) = 1$ .

**Result 2.2.** Two fuzzy numbers  $a$  and  $b$  are called equal  $a = b$ , if  $\mu_a(x) = \mu_b(x)$  for all  $x \in R$ . It follows that

$$a = b \Leftrightarrow [a]^\alpha = [b]^\alpha,$$

for all  $\alpha \in (0, 1]$ .

**Result 2.3.** A fuzzy number  $a$  may be decomposed into its level sets through the resolution identity

$$a = \int_0^1 \alpha [a]^\alpha$$

where  $\alpha [a]^\alpha$  is the product of a scalar  $\alpha$  with the set  $[a]^\alpha$  and  $\int$  is the union of  $[a]^\alpha$  with  $\alpha$  ranging from 0 to 1.

**Definition 2.4.** The support of a fuzzy set  $A$  in the universal set  $U$  is a crisp set that contains all the elements of  $U$  that have nonzero membership values in  $A$ , that is

$$supp(A) = \{x \in U; \mu_a(x) > 0\}$$

where  $supp(A)$  denotes the support of fuzzy set  $A$ .

Hence, the support  $\Gamma_a$  of a fuzzy member  $a$  is defined, as a special case of level set, by the following:

$$\Gamma_a = \{x : \mu_a(x) > 0\}.$$

**Definition 2.5.** A fuzzy number  $a$  in  $R$  is said to be positive if  $0 < a_1 < a_2$  holds for the support  $\Gamma_a = \{a_1, a_2\}$  of  $a$ ; that is,  $\Gamma_a$  is in the positive real line. Similarly,  $a$  is called negative if  $a_1 \leq a_2 \leq 0$  and zero if  $a_1 \leq 0 \leq a_2$ .

**Lemma 2.1.** (see [9]) *If  $a, b \in E_N$ , then for  $\alpha \in (0, 1]$ ,*

$$\begin{aligned} [a + b]^\alpha &= [a_q^\alpha + b_q^\alpha, a_r^\alpha + b_r^\alpha], \\ [a \times b]^\alpha &= [\min \{a_i^\alpha, b_j^\alpha\}, \max \{a_i^\alpha, b_j^\alpha\}], \quad i, j = q, r, \\ [a - b]^\alpha &= [a_q^\alpha - b_r^\alpha, a_r^\alpha - b_q^\alpha]. \end{aligned}$$

**Lemma 2.2.** (see [9]) *Let  $[a_q^\alpha, a_r^\alpha], 0 < \alpha \leq 1$ , be a given family of nonempty intervals. If*

$$[a_q^\beta, a_r^\beta] \subset [a_q^\alpha, a_r^\alpha]. \text{ for all } 0 < \alpha \leq \beta \quad \text{for all } 0 < \alpha \leq \beta. \tag{2.1}$$

and

$$[\lim_{k \rightarrow \infty} a_q^{\alpha_k}, \lim_{k \rightarrow \infty} a_r^{\alpha_k}] = [a_q^\alpha, a_r^\alpha] \tag{2.2}$$

whenever  $(\alpha_k)$  is non decreasing sequence converging to  $\alpha \in (0, 1]$ , then the family  $[a_q^\alpha, a_r^\alpha], 0 < \alpha \leq 1$ , are the  $\alpha$ - level sets of a fuzzy number  $a \in E_N$ ; conversely if  $[a_q^\alpha, a_r^\alpha], 0 < \alpha \leq 1$ , are the  $\alpha$ - level sets of a fuzzy number  $a \in E_N$ , then conditions (2.1) and (2.2) hold true.

Let  $x$  be a point in  $R^n$  and  $A$  be a nonempty subset of  $R^n$ . We define the distance  $d(x, A)$  from  $x$  to  $A$  by

$$d(x, A) = \inf \{ \|x - a\|; a \in A \} \tag{2.3}$$

Now, let  $A$  and  $B$  be nonempty subsets of  $R^n$ . We define the Hausdorff separation of  $B$  from  $A$  by

$$dH^*(B, A) = \sup \{ d(b, A); b \in B \} \tag{2.4}$$

In general  $dH^*(A, B) \neq dH^*(B, A)$  we define the Hausdorff distance between nonempty subsets of  $A$  and  $B$  of  $R^n$  by

$$dH(A, B) = \max \{dH^*(A, B), dH^*(B, A)\} \tag{2.5}$$

This is now symmetric in  $A$  and  $B$ . Consequently,

- (1)  $dH(A, B) \geq 0$  with  $dH(A, B) = 0$  if and only if  $\bar{A} = \bar{B}$ ,
- (2)  $dH(A, B) = dH(B, A)$  and
- (3)  $dH(A, B) \leq dH(A, C) + dH(C, B)$ ,

for any nonempty subsets of  $A, B$  and  $C$  of  $R^n$ . The Hausdorff distance (2.5) is a metric, the Hausdorff metric. The supremum metric  $d_\infty$  on  $E^n$  is defined by

$$d_\infty(u, v) = \sup \{dH([u]^\alpha, [v]^\alpha); \alpha \in (0, 1)\}, \tag{2.6}$$

for all  $u, v \in E^n$ , and is obviously metric on  $E^n$ . The supremum metric  $H_1$  on  $C(J; E^n)$  is defined by

$$H_1(x, y) = \sup \{d_\infty(x(t), y(t)); t \in J\}, \tag{2.7}$$

for all  $x, y \in C(J; E^n)$ .

### 3. Existence and Uniqueness Results for Fuzzy Impulsive Integrodifferential Equations

In this section, we consider the existence and uniqueness of the fuzzy solution for nonlinear fuzzy impulsive integrodifferential equations

$$\begin{aligned} x'(t) &= a(t)x(t) + \int_0^t k(t, s, x(s))ds + f(t, x(t)), \quad t \in J, \quad t \neq t_k, \\ \Delta x|_{t=t_k} &= I_k(x(t_k)), \quad k = 1, 2, 3, \dots, m, \\ x(0) &= x_0, \end{aligned} \tag{3.1}$$

with fuzzy co-efficient  $a : J \rightarrow E_N$ , initial value  $x_0 \in E_N$ , inhomogeneous terms  $f : J \times E_N \rightarrow E_N$ , and  $k : J \times J \times E_N \rightarrow E_N$  are continuous,

$$\Delta x(t_k) = x(t_k^+) - x(t_k^-)$$

$x(t_k^+)$  and  $x(t_k^-)$  denote the right limits and the left limits of  $x(t)$  at  $t = t_k$  respectively,  $0 < t_1 < t_2 < \dots < t_k < T$ . and satisfy a global Lipschitz condition,

i.e., there exists a finite constant  $\eta_1 > 0$  such that

- (H<sub>1</sub>) (i)  $dH([f(s, \xi_1(s))]^\alpha, [f(s, \xi_2(s))]^\alpha) \leq \eta_1 dH([\xi_1(s)]^\alpha, [\xi_2(s)]^\alpha)$   
 and  
 (ii)  $dH([k(t, s, \xi_1(s))]^\alpha, [k(t, s, \xi_2(s))]^\alpha) \leq \eta_1 dH([\xi_1(s)]^\alpha, [\xi_2(s)]^\alpha)$   
 for all  $\xi_1(s), \xi_2(s) \in E_N$ .

(H<sub>2</sub>) There exist two positive constants  $dk$  and  $dk'$ , where  $k = 1, 2, \dots, m$ .

- (i)  $\|I_k(\xi_1(t_k))\| \leq dk$   
 (ii)  $\|I_k(\xi_2(t_k))\| \leq dk'$

Let  $I$  be a real interval. A mapping  $x : \rightarrow E_N$  is called a fuzzy process. we denote

$$[x(t)]^\alpha = [x_q^\alpha(t), x_r^\alpha(t)], \quad t \in I, \quad 0 < \alpha \leq 1$$

The derivative  $x'(t) \in E_N$  of a fuzzy process  $x$ , then

$$[x'(t)]^\alpha = [(x_q^\alpha)'(t), (x_r^\alpha)'(t)], \quad 0 < \alpha \leq 1$$

The fuzzy integral

$$\int_a^b x(t)dt, \quad a, b \in I$$

is defined by

$$\int_a^b [x(t)dt]^\alpha = \left[ \int_a^b x_q^\alpha(t)dt, \int_a^b x_r^{alpha}(t)dt \right]$$

Provided that the Lebesgue integrals on the right exist.

**Theorem 3.1.** *Let  $T > 0$ ,  $f$  and  $k$  satisfy a global Lipschitz condition, for every  $x_0 \in E_N$ , then the fuzzy impulsive integro differential equation has a unique solution  $x \in c(J; E_N)$ .*

*Proof.* For each  $\xi(t) \in E_N, t \in J$ . Define

$$\begin{aligned} (G_0\xi)(t) &= S(t)x_0 + \int_0^t S(t-s) \left( \int_0^s k(s,r, \xi(r))dr \right) ds + \\ &\quad \sum_{t < t_k} S(t-t_k)I_k(\xi(t_k)) + \int_0^t S(t-s)f(s, \xi(s))ds. \end{aligned}$$

where  $S(t)$  is a fuzzy number and

$$\begin{aligned} [S(t)]^\alpha &= [S_q^\alpha(t), S_r^\alpha(t)] \\ &= \left[ \exp \left\{ \int_0^t a_q^\alpha(s)ds \right\}, \exp \left\{ \int_0^t a_r^\alpha(s)ds \right\} \right] \end{aligned}$$

and  $S_i^\alpha(t)(i = q, r)$  is continuous. That is, there exists a constant  $\eta_2 > 0$  such that  $|S_i^\alpha(t)| \leq \eta_2$  for all  $t \in J$ . Thus,  $G_0\xi : J \rightarrow E_N$  is continuous, and  $G_0 : C(J : E_N) \rightarrow C(J : E_N)$ . For  $\xi_1, \xi_2 \in C(J : E_N)$

$$\begin{aligned}
 & dH([\!(G_0\xi_1)(t)\!]^\alpha, [\!(G_0\xi_2)(t)\!]^\alpha) \\
 = & dH\left([\!(S(t)x_0 + \int_0^t S(t-s)\left(\int_0^s K(s,r,\xi_1(r))dr\right)ds + \int_0^t S(t-s)f(s,\xi_1(s))ds\!]^\alpha\right. \\
 & \left. + \sum_{t < t_k} [S(t-t_k)I_k(\xi_1(t_k))]^\alpha\right. \\
 & \left. [\!(S(t)x_0 + \int_0^t S(t-s)\left(\int_0^s k(s,r,\xi_2(r))dr\right)ds + \int_0^t S(t-s)f(s,\xi_2(s))ds\!]^\alpha\right) \\
 & \left. + \sum_{t < t_k} [S(t-t_k)I_k(\xi_2(t_k))]^\alpha\right) \\
 = & dH\left([\!(S(t)x_0\!]^\alpha + [\!\int_0^t S(t-s)\left(\int_0^s K(s,r,\xi_1(r))dr\right)ds + \int_0^t S(t-s)f(s,\xi_1(s))ds\!]^\alpha\right. \\
 & \left. + \sum_{t < t_k} S(t-t_k)I_k(\xi_1(t_k))\!]^\alpha\right. \\
 & \left. [\!(S(t)x_0\!]^\alpha + [\!\int_0^t S(t-s)\left(\int_0^s k(s,r,\xi_2(r))dr\right)ds + \int_0^t S(t-s)f(s,\xi_2(s))ds\!]^\alpha\right. \\
 & \left. + \sum_{t < t_k} S(t-t_k)I_k(\xi_2(t_k))\!]^\alpha\right) \\
 = & dH\left([\!\int_0^t S(t-s)\left(\int_0^s K(s,r,\xi_1(r))dr\right)ds\!]^\alpha + [\!\int_0^t S(t-s)f(s,\xi_1(s))ds\!]^\alpha\right. \\
 & \left. + \sum_{t < t_k} [S(t-t_k)I_k(\xi_1(t_k))\!]^\alpha\right. \\
 & \left. [\!\int_0^t S(t-s)\left(\int_0^s k(s,r,\xi_2(r))dr\right)ds\!]^\alpha + [\!\int_0^t S(t-s)f(s,\xi_2(s))ds\!]^\alpha\right) \\
 & \left. + \sum_{t < t_k} [S(t-t_k)I_k(\xi_2(t_k))\!]^\alpha\right) \\
 = & dH\left(\int_0^t [S(t-s)\left(\int_0^s K(s,r,\xi_1(r))dr\right)]^\alpha ds + \int_0^t [S(t-s)f(s,\xi_1(s))]^\alpha ds\right. \\
 & \left. + \sum_{t < t_k} [S(t-t_k)I_k(\xi_1(t_k))]^\alpha\right. \\
 & \left. \int_0^t [S(t-s)\left(\int_0^s k(s,r,\xi_2(r))dr\right)]^\alpha ds + \int_0^t [S(t-s)f(s,\xi_2(s))]^\alpha ds\right)
 \end{aligned}$$

$$\begin{aligned}
 & + \sum_{t < t_k} [S(t - t_k)I_k(\xi_2(t_k))]^\alpha \\
 \leq & \int_0^t dH \left( [S(t - s) \left( \int_0^s K(s, r, \xi_1(r)) dr \right)]^\alpha + [S(t - s)f(s, \xi_1(s))]^\alpha \right. \\
 & \left. + \sum_{t < t_k} [S(t - t_k)I_k(\xi_1(t_k))]^\alpha \right. \\
 & \left. [S(t - s) \left( \int_0^s k(s, r, \xi_2(r)) dr \right)]^\alpha + [S(t - s)f(s, \xi_2(s))]^\alpha \right) ds \\
 & + \sum_{t < t_k} [S(t - t_k)I_k(\xi_2(t_k))]^\alpha \\
 = & \int_0^t dH \left( [S_q^\alpha(t - s) \left( \int_0^s K_q^\alpha(s, r, \xi_1(r)) dr \right) + S_q^\alpha(t - s)f_q^\alpha(s, \xi_1(s)) \right. \\
 & \left. + \sum_{t < t_k} [S_q^\alpha(t - t_k)I_k(\xi_1(t_k))] \right),
 \end{aligned}$$

$$\begin{aligned}
 & S_r^\alpha(t - s) \left( \int_0^s K_r^\alpha(s, r, \xi_1(r)) dr \right) + S_r^\alpha(t - s)f_r^\alpha(s, \xi_1(s)) \\
 & + [S_q^\alpha(t - s) \left( \int_0^s K_q^\alpha(s, r, \xi_2(r)) dr \right) + S_q^\alpha(t - s)f_q^\alpha(s, \xi_2(s))],
 \end{aligned}$$

$$\begin{aligned}
 & S_r^\alpha(t - s) \left( \int_0^s K_r^\alpha(s, r, \xi_2(r)) dr \right) + S_r^\alpha(t - s)f_r^\alpha(s, \xi_2(s)) \Big] ds \\
 & + \sum_{t < t_k} [S_r^\alpha(t - t_k)I_k(\xi_2(t_k))] \\
 = & \int_0^t \max \left( \left| [S_q^\alpha(t - s) \left( \int_0^s K_q^\alpha(s, r, \xi_2(r)) dr \right) + S_q^\alpha(t - s)f_q^\alpha(s, \xi_2(s)) \right. \right. \\
 & \left. \left. + \sum_{t < t_k} [S_q^\alpha(t - t_k)I_k(\xi_2(t_k))] \right] \right. \\
 & \left. [S_q^\alpha(t - s) \left( \int_0^s K_q^\alpha(s, r, \xi_1(r)) dr \right) + S_q^\alpha(t - s)f_q^\alpha(s, \xi_1(s))] \right. \\
 & \left. + \sum_{t < t_k} [S_q^\alpha(t - t_k)I_k(\xi_1(t_k))] \right|),
 \end{aligned}$$

$$\left| [S_r^\alpha(t - s) \left( \int_0^s K_r^\alpha(s, r, \xi_2(r)) dr \right) + S_r^\alpha(t - s)f_r^\alpha(s, \xi_2(s)) \right]$$



$$\begin{aligned}
 & + \sum_{t < t_k} [S_r^\alpha(t - t_k)I_k(\xi_2(t_k))] \\
 & [S_r^\alpha(t - s)(\int_0^s K_r^\alpha(s, r, \xi_1(r))dr) + S_r^\alpha(t - s)f_r^\alpha(s, \xi_1(s)) \\
 & + \sum_{t < t_k} [S_r^\alpha(t - t_k)I_k(\xi_1(t_k))]] ds \\
 = & \int_0^t \max \left( |S_q^\alpha(t - s)([\int_0^s K_q^\alpha(s, r, \xi_2(r))dr) + f_q^\alpha(s, \xi_2(s))] \right. \\
 & \left. - [(\int_0^s K_q^\alpha(s, r, \xi_1(r))dr) + f_q^\alpha(s, \xi_1(s))] \right) | \\
 & + \sum_{t < t_k} [S_q^\alpha(t - t_k)I_k(\xi_2(t_k))] + \sum_{t < t_k} [S_q^\alpha(t - t_k)I_k(\xi_1(t_k))] | \\
 & | [S_r^\alpha(t - s)([\int_0^s K_r^\alpha(s, r, \xi_2(r))dr) + f_r^\alpha(s, \xi_2(s))] \\
 & - [(\int_0^s K_r^\alpha(s, r, \xi_1(r))dr) + f_r^\alpha(s, \xi_1(s))] | \Big) ds \\
 & + \sum_{t < t_k} [S_q^\alpha(t - t_k)I_k(\xi_2(t_k))] + \sum_{t < t_k} [S_q^\alpha(t - t_k)I_k(\xi_1(t_k))] \\
 & \leq \eta_2 \int_0^t \max \left( | [(\int_0^s K_q^\alpha(s, r, \xi_2(r))dr) + f_q^\alpha(s, \xi_2(s))] \right. \\
 & \left. - [(\int_0^s K_q^\alpha(s, r, \xi_1(r))dr) + f_q^\alpha(s, \xi_1(s))] \right) | \\
 & + \eta_2 \sum_{t < t_k} I_k(\xi_2(t_k)) + \eta_2 \sum_{t < t_k} I_k(\xi_1(t_k)) \\
 & | [(\int_0^s K_r^\alpha(s, r, \xi_2(r))dr) + f_r^\alpha(s, \xi_2(s))] \\
 & - [(\int_0^s K_r^\alpha(s, r, \xi_1(r))dr) + f_r^\alpha(s, \xi_1(s))] | \Big) ds \\
 & + \eta_2 \sum_{t < t_k} I_k(\xi_2(t_k)) + \eta_2 \sum_{t < t_k} I_k(\xi_1(t_k)) \\
 & \leq \eta_2 \int_0^t \max \left( | [(\int_0^s K_q^\alpha(s, r, \xi_2(r))dr) + f_q^\alpha(s, \xi_2(s))] \right. \\
 & \left. - [(\int_0^s K_q^\alpha(s, r, \xi_1(r))dr) + f_q^\alpha(s, \xi_1(s))] \right) | \\
 & + \eta_2 \sum_{t < t_k} dk' + \eta_2 \sum_{t < t_k} dk
 \end{aligned}$$

$$\begin{aligned}
 & \left| \left[ \left( \int_0^s K_r^\alpha(s, r, \xi_2(r)) dr \right) + f_r^\alpha(s, \xi_2(s)) \right] \right. \\
 & \left. - \left[ \left( \int_0^s K_r^\alpha(s, r, \xi_1(r)) dr \right) + f_r^\alpha(s, \xi_1(s)) \right] \right| ds \\
 & \quad + \eta_2 \sum_{t < t_k} dk' + \eta_2 \sum_{t < t_k} dk \\
 = & \eta_2 \int_0^t dH \left( \left[ \left( \int_0^s K_q^\alpha(s, r, \xi_1(r)) dr \right) + f_q^\alpha(s, \xi_1(s)) \right], \right. \\
 & \left. \left[ \left( \int_0^s K_r^\alpha(s, r, \xi_1(r)) dr \right) + f_r^\alpha(s, \xi_1(s)) \right] \right) \\
 & \quad + \eta_2 \sum_{t < t_k} dk' + \eta_2 \sum_{t < t_k} dk \\
 & \left[ \left[ \left( \int_0^s K_q^\alpha(s, r, \xi_2(r)) dr \right) + f_q^\alpha(s, \xi_2(s)) \right] \right. \\
 & \left. \left[ \left( \int_0^s K_r^\alpha(s, r, \xi_2(r)) dr \right) + f_r^\alpha(s, \xi_2(s)) \right] \right] ds \\
 & \quad + \eta_2 \sum_{t < t_k} dk' + \eta_2 \sum_{t < t_k} dk \\
 = & \eta_2 \int_0^t dH \left( \left[ \left( \int_0^s K(s, r, \xi_1(r)) dr \right) + f(s, \xi_1(s)) + \eta_2 \sum_{t < t_k} dk \right]^\alpha, \right. \\
 & \left. \left[ \left( \int_0^s K(s, r, \xi_2(r)) dr \right) + f(s, \xi_2(s)) + \eta_2 \sum_{t < t_k} dk' \right]^\alpha \right) ds \\
 = & \eta_2 \int_0^t dH \left( \left[ \int_0^s K(s, r, \xi_1(r)) dr \right]^\alpha, \left[ \int_0^s K(s, r, \xi_2(r)) dr \right]^\alpha \right) ds \\
 & \quad + \eta_2 \int_0^t dH \left( [f(s, \xi_1(s))]^\alpha, [f(s, \xi_2(s))]^\alpha \right) ds \\
 & \quad + \eta_2 \sum_{t < t_k} dk + \eta_2 \sum_{t < t_k} dk' \\
 \leq & \eta_2 \int_0^t \eta_1 T dH([\xi_1(s)]^\alpha, [\xi_2(s)]^\alpha) ds + \eta_2 \int_0^t \eta_1 dH([\xi_1(s)]^\alpha, [\xi_2(s)]^\alpha) ds \\
 & \quad + \eta_2 \sum_{t < t_k} dk + \eta_2 \sum_{t < t_k} dk' \\
 \leq & \eta_1 \eta_2 (T + 1) \int_0^t dH([\xi_1(s)]^\alpha, [\xi_2(s)]^\alpha) ds + \eta_2 \sum_{t < t_k} dk + \eta_2 \sum_{t < t_k} dk'
 \end{aligned}$$

$$\leq \eta\eta_2 \int_0^t dH([\xi_1(s)]^\alpha, [\xi_2(s)]^\alpha) ds + \eta_2 \sum_{t < t_k} dk + \eta_2 \sum_{t < t_k} dk',$$

where  $\eta = \eta_1(T + 1)$ .

Therefore,

$$\begin{aligned} d\infty((G_0\xi_1)(t), (G_0\xi_2)(t)) &= \sup_{\alpha \in (0,1]} dH([(G_0\xi_1)(t)]^\alpha, [(G_0\xi_2)(t)]^\alpha) \\ &\leq (\eta\eta_2) \int_0^1 \sup_{\alpha \in (0,1]} dH([\xi_1(s)]^\alpha, [\xi_2(s)]^\alpha) ds + \eta_2 \sum_{t < t_k} dk + \eta_2 \sum_{t < t_k} dk' \\ &= (\eta\eta_2) \int_0^1 d\infty(\xi_1(s), \xi_2(s)) ds + \eta_2 \sum_{t < t_k} dk + \eta_2 \sum_{t < t_k} dk' \end{aligned}$$

$$\begin{aligned} H_1(G_0\xi_1, G_0\xi_2) &= \sup_{t \in J} d\infty((G_0\xi_1)(t), (G_0\xi_2)(t)) \\ &\leq (\eta\eta_2) \sup_{t \in J} \int_0^t d\infty(\xi_1(s), \xi_2(s)) ds + \eta_2 \sum_{t < t_k} dk + \eta_2 \sum_{t < t_k} dk' \\ &\leq (\eta\eta_2)TH_1(\xi_1, \xi_2) + \eta_2 \sum_{t < t_k} dk + \eta_2 \sum_{t < t_k} dk' \end{aligned}$$

we take sufficiently small  $T$ ,

$$(\eta\eta_2T) + \eta_2 \sum_{t < t_k} dk + \eta_2 \sum_{t < t_k} dk' < 1$$

Here  $G_0$  is a contraction mapping. By the Banach fixed-point theorem, fuzzy impulsive integrodifferential equation has a unique fixed point  $x \in C(J : E_N)$ . □

**Example 3.1.** Consider the fuzzy impulsive solution of the nonlinear fuzzy integrodifferential equations

$$\begin{aligned} x' &= 2x + 2tx(t)^2 + 2tx^2, & t \in J, \\ \Delta x|_{t=t_k} &= I_k(x(t_k^-)) \\ x(0) &= 2 \in E_N \end{aligned}$$

The  $\alpha$ - level set of fuzzy number 2 is

$$[2]^\alpha = [\alpha + 1, 3 - \alpha], \quad \text{for all } \alpha \in [0, 1].$$

Let

$$\int_0^t K(t, s, x(s))ds = 2tx(t)^2, \quad f(t, x(t)) = 2tx(t)^2,$$

and

$$\eta_2 = 3T[|x_r^\alpha(t) + y_r^\alpha(t)|] > 0.$$

Then the  $\alpha$ - level set of  $\int_0^t K(t, s, x(s))ds$  is

$$\begin{aligned} \left[ \int_0^t K(t, s, x(s))ds \right]^\alpha &= [2t x(t)^2]^\alpha \\ &= t[2]^\alpha [x(t)^2]^\alpha \\ &= t[\alpha + 1, 3 - \alpha][(x_q^\alpha(t))^2, (x_r^\alpha(t))^2] \\ &= t[(\alpha + 1)(x_q^\alpha(t))^2, (3 - \alpha)(x_r^\alpha(t))^2], \end{aligned}$$

where  $[x(t)]^\alpha = [x_q^\alpha(t), x_r^\alpha(t)]$  and  $[2]^\alpha = [\alpha + 1, 3 - \alpha]$  for all  $\alpha \in [0, 1]$  and the  $\alpha$ -level set of  $f(t, x(t))$  is

$$\begin{aligned} [f(t, x(t))]^\alpha &= [2tx(t)^2]^\alpha \\ &= t[2]^\alpha [x(t)^2]^\alpha \\ &= t[\alpha + 1, 3 - \alpha][(x_q^\alpha(t))^2, (x_r^\alpha(t))^2] \\ &= t[(\alpha + 1)(x_q^\alpha(t))^2, (3 - \alpha)(x_r^\alpha(t))^2], \end{aligned}$$

where  $[x(t)]^\alpha = [x_q^\alpha(t), x_r^\alpha(t)]$  and  $[2]^\alpha = [\alpha + 1, 3 - \alpha]$ , for all  $\alpha \in [0, 1]$ . and

(i)  $||I_k(x(t_k^-))|| \leq dk,$

(ii)  $||I_k(y(t_k^-))|| \leq dk', \quad dk, dk' > 0$

Then

$$\begin{aligned} &\left[ \int_0^t K(t, s, x(s))ds \right]^\alpha + [f(t, x(t))]^\alpha + I_k(x(t_k^-)) \\ &= t[(\alpha + 1)(x_q^\alpha(t))^2, (3 - \alpha)(x_r^\alpha(t))^2] + t[(\alpha + 1)(x_q^\alpha(t))^2, (3 - \alpha)(x_r^\alpha(t))^2] + dk \\ &= 2t[(\alpha + 1)(x_q^\alpha(t))^2, (3 - \alpha)(x_r^\alpha(t))^2] + dk. \end{aligned}$$

Therefore,

$$\begin{aligned}
 dH & \left( \left( \left[ \int_0^t K(t, s, x(t)) ds \right]^\alpha + [f(t, x(t))]^\alpha + dk \right), \right. \\
 & \left. \left( \left[ \int_0^t K(t, s, y(s)) ds \right]^\alpha + [f(t, y(t))]^\alpha + dk' \right) \right) \\
 & = dH \left( 2t \left[ (\alpha + 1)(x_q^\alpha(t))^2, (3 - \alpha)(x_r^\alpha(t))^2 + dk \right], \right. \\
 & \left. 2t \left[ (\alpha + 1)(y_q^\alpha(t))^2, (3 - \alpha)(y_r^\alpha(t))^2 + dk' \right] \right) \\
 & = 2t \max \left\{ (\alpha + 1) |(x_q^\alpha(t))^2 - (y_q^\alpha(t))^2| + dk, \right. \\
 & \left. (3 - \alpha) |(x_r^\alpha(t))^2 - (y_r^\alpha(t))^2| + dk' \right\} \\
 & \leq T(3 - \alpha) \max \left\{ |x_q^\alpha(t) - y_q^\alpha(t)| |x_q^\alpha(t) + y_q^\alpha(t)| + dk, \right. \\
 & \left. |x_r^\alpha(t) - y_r^\alpha(t)| |x_r^\alpha(t) + y_r^\alpha(t)| + dk' \right\} \\
 & = 3T |x_r^\alpha(t) - y_r^\alpha(t)| \max \left\{ |x_q^\alpha(t) - y_q^\alpha(t)| + dk, |x_r^\alpha(t) - y_r^\alpha(t)| + dk' \right\} \\
 & = \eta_2 dH \left( [x(t)]^\alpha + dk, [y(t)]^\alpha + dk' \right)
 \end{aligned}$$

Since  $f$  and  $g$  satisfy a global Lipschitz condition, from [Theorem 3.1], the fuzzy impulsive integrodifferential equation has a Unique fuzzy solution.

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