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AN ITERETED FUNCTION SYSTEM FOR COMMUTING MAPPING

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Abstract: With an open question - is it possible to compress one image containing the other and commuting with each other? We establish the College Theorem for iterated function system (key to image compression) for two such mappings under the contraction condition in the sense of Jungck [1].

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1. Introduction

It is well known that the notion of an iterated function system (IFS) for contraction mapping is one of the most common and most general way to generate fractals. This serves us a framework for extension of an iterated function system for two commuting mappings. First, we introduce the notion of IFS and Collage theorem in a metric space (X,d) and the commuting functions f and g using fixed point theorems for those commuting self mappings. Next, we prove the existence of the attractor for IFS for such commuting mappings.

IFSs were introduced in present form by Hutchinson [4] and further studied by Barnsley in [13]. Recently, IFSs have attracted researchers who work on

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autoregressive time series, engineer sciences, physics, and so forth. For applications of IFSs in image processing theory, in the theory of stochastic growth models, and in the theory of random dynamical systems, one can consult [5]. Here, our effort is to extend Hutchinson's classical framework for fractals for commuting mapping.

2. Iterated Function System

Let X denotes a complete metric space with distance function d and T be a mapping from X into itself. Then T is called a *contraction mapping* if there is a constant $0 \leq s < 1$ such that

$$d(T(x), T(y)) \leq sd(x, y) \tag{1}$$

The constant s is called the contractivity factor for T.

Polish mathematician S. Banach proved a result, regarding contraction mapping in 1922, known as Banach Contraction principle, see [12].

Theorem 2.1. *Let $T : X \to X$ be a contraction mapping, with contractivity factor 'S', on a complete metric space (X, d). Then T possesses exactly one fixed point $x^* \in X$. Moreover, for any point $x \in X$,the sequence $T_n(x) : n = 0, 1, 2, ...$ converges to x. That is $\lim_{n\to\infty} T^n(x) = x^*$, for each $x \in X$.*

IFS generally employ contractive maps over a complete metric space (X, d), where the Banach's celebrated result mentioned above' guarantees the existence and uniqueness of the fixed point known as "attractor". The main property of the contraction mapping which is used in IFS is given by the following lemma:

Lemma 2.2. *Let $T : X \to X$ be a contraction mapping, with contractivity factor 'α', on a complete metric space (X, d). Then T is continuous.*

We now discuss certain definitions required to understand iterated function system. Let (X, d) be a co mplete metric space and H(X) denote the space whose points are the compact subsets of X known as Hausdroff space, other than the empty set. Let $x, y \in X$and let $A, B \in H(X)$. Then:

(1)distancefrom the point x to set B is defined as

$$d(x, B) = min\{d(x, y) : y \in B\},$$

(2) distance from the set A to the set B is defined as

$$d(A,B) = max\{d(x,B) : x \in A\},$$

(3) Housdorff distance from the set A to the set B is defined as

$$h(A,B) = d(A,B) \vee d(B,A),$$

then the function *h(d)* is the metric defined on the space *H(X)*

Note. Throughout this paper the notation $u \vee v$ means the maximum and $u \wedge v$ denotes the minimum of the pair of real numbers u and v.

In IFS, the contractive maps act on the members of Hausdroff space, i.e. the compact subsets of X. Thus, an iterated function system is defined as follows:

A *(hyperbolic)* iterated function system consists of a complete metric space (X,d) together with a finite set of continuous contraction mappings $T_n : X \to X$ with respect to contractivity factor α_n, for $n = 1, 2, 3, ..., N$.

Thus, the following theorem was given by Barnsley [13]:

Theorem 2.3. *Let $X : T_n, n = 1, 2, 3, ..., N$ be a iterated function system with contractivity factor α. Then the transformation $W : H(X) \to H(X)$ defined by $W(B) = \bigcup_{n=1}^{N} Tn(B)$ for all $B \in H(X)$, is a contraction mapping on the complete metric space (H(X), h(d)) with contractivity factor α.*

That is

$$h(W(B), W(C)) \leq \alpha h(B,C).$$

Its unique fixed point, which is also called an attractor, $A \in H(X)$, obeys

$$A = W(A) = \bigcup_{n=1}^{N} T_n(A),$$

and is given by $A = \lim_{n\to\infty} W^{on}(B)$ for any $B \in H(X)$. W^{on} denotes the n-fold composition of W.

The contraction mappings used in IFS are typically affine maps. The iteration dynamics associated with affine maps is not very interesting but when the action of a system of contraction mappings is considered the result is quite remarkable.

3. Iterated Function System for Commuting Mapping

In this section, we shall try to explore the possibility of improvement in IFS by replacing contraction mapping by a more useful mapping known as Commuting mapping. In 1976, Jungck [1] introduced a mapping, which was defined as follows:

Let g be a continuous mapping of a complete metric space (X, d)into itself. Then f has a fixed point in X iff there exist $\alpha \in (0, 1)$ and a mapping $g : X \to X$ which commutes with f and satisfies

$$g(X) \subset f(X)$$

and

$$d(g(x), g(y)) \leq \alpha d(f(x), f(y)) \forall x, y \in X.$$

On the basis of definition of (hyperbolic) iterated function system given by Barnsley [13], we now introduce Iterated function system with Commuting mapping as below:

A Iterated function system with Commuting mapping consists of a complete metric space (X, d) together with mapping $f : X \to X$ and $g : X \to X$ with contractivity factor α.

Proposition 3.1. *Let f be a continuous mapping of a complete metric space (X, d) into itself. Then f has a fixed point in X iff there exists $\alpha \in (0, 1)$ and a mapping $g : X \to X$ which commutes with f and satisfies*

$$g(X) \subset f(X)$$

and

$$d(g(x), g(y)) \leq \alpha d(f(x), f(y)) \forall x, y \in X \qquad (*)$$

Indeed, f and g have a unique common fixed point if the above condition holds.

Proof. To see that the stated condition is necessary, suppose that $f(a) = a$ for some $a \in X$. Define $g : X \to X$ by $g(x) = a$ for all $x \in X$. Then $g(f(x)) = a$ and $f(g(x)) = f(a) = a(x \in X)$, so $g(f(x)) = f(g(x)) \forall x \in X$ and g commutes with f. Moreover, $g(x) = a = f(a) \forall x \in X$ so that $g(X) \subset f(X)$. Finally, for any $\alpha \in (0, 1)$ we have for all $x, y \in X$:

$$d(g(x), g(y)) \quad = \quad d(a, a) = 0 \leq \alpha d(f(x), f(y))$$

Thus the above condition holds.

On the other hand, suppose ther e is a mapping g of X into itself which commutes with f and for which equation [*] holds. We show that this condition is sufficient to ensure that f and g have a unique common fixed point.

To this end, let $x_0 \in X$ and let x_1 be such that $f(x_1) = g(x_0)$. In general, choose x_n so that $(1) f(x_n) = g(x_{n-1})$. We can do this since $g(X) \subset f(X)$. The relation [*] and (1) imply that $d(f(x_{n+1}), f(x_n)) \leq \alpha d(f(x_n), f(x_{n-1}))$ for all n. The lemma yields $t \in X$ such that (2) $f(x_n) \to t$. But then (1) implies that (3) $g(x_n) \to t$.

Now since f is continuous, [*] implies that both f and g are continuous. Hence (2) and (3) demand that $g(f(x_n)) \to g(t)$ and $f(g(x_n)) \to f(t)$. But f and g commute so that $g(f(x_n)) = f(g(x_n))$ for all n. Thus $f(t) = g(t)$, and consequently $f(f(t)) = f(g(t)) = g(g(t))$ by commutativity. We can therefore infer

$$d(g(t), g(g(t))) \quad \leq \quad \alpha d(f(t), g(t)) = \alpha d(g(t), g(g(t))).$$

Hence $d(g(t), g(g(t)))(1 - \alpha) \leq 0$. Since $\alpha \in (0, 1), g(t) = g(g(t))$. We now have $g(t) = g(g(t)) = f(g(t))$, i.e. $g(t)$ is a common fixed point of f and g.

To see that f and g can have only one common fixed point, suppose that $x = f(x) = g(x)$ and $y = g(y) = f(y)$. Then [*] implies $d(x, y)(1 - \alpha) \leq 0$. Since $\alpha < 1, x = y$.

Lemma 3.2. *Let f be a mapping,of a set X into itself. Then f has a fixed point iff there is a constant map $g : X \to X$ which commutes with f (i.e. $g(f(X)) = f(g(x)))$ $\forall x \in X$.*

Then $f : H(X) \to H(X)$ and $g : H(X) \to H(X)$ defined by $f(B) = \{f(x) : x \in B\}$ and $g(B) = \{g(x) : x \in B\} \forall B \in H(X)$ is a commuting mapping on $(H(X), h(d))$with contractivity fctor α.

Proof. Since f and g are continuous mappings such that $g : H(X) \to H(X)$ and $g(X) \subset f(X)$

Let $B, C \in H(X)$, then

$$\begin{aligned} h(g(B), g(C) &= d(g(B), g(C)) \vee d(g(C), g(B)) \\ &\leq \alpha d[(f(B), f(C)) \vee d(f(C), f(B))] \\ &= \alpha d[(f(B), f(C)) \vee d(f(C), f(B))] \\ &\leq \alpha h[(f(B), f(C)) \end{aligned}$$

Therefore,

$$h(g(B), g(C)) \quad \leq \quad \alpha h[(f(B), f(C))$$

This completes the proof.

Lemma 3.3. *Let (X,d) be a metric space. Let g_n, $n = 1, 2, 3, ..., N$ be a continuous Reich mappings on $(H(X), h)$. Let the contractivity factor for g_n be denoted by α_n for each n. Define $g' : H(X) \to H(X)$ by $g'(B) = g_1(B) \cup g_2(B) \cup .... \cup g_n(B) = \bigcup_{n=1}^{N} g_n(B)$ for each $B \in H(X)$. Then g' is a commuting mapping wiith contractivity factor $\alpha = max\{\alpha_n : n = 1, 2, ...N\}$.*

Proof. We shall prove the theorem using mathematical induction method using the properties of metric h. For $N = 1$, the statement is obviously true. Now for $N = 2$, we see that

$$\begin{aligned} h(g'(B), g'(C)) &= h(g_1(B) \cup g_2(B), g_1(C) \cup g_2(C)) \\ &\leq h(g_1(B), g_1(C)) \vee h(g_2(B), g_2(C)) \\ &\leq [\alpha_1 h(f_1(B), f_1(C)) \vee \alpha_2 h(f_2(B), f_2(C))] \\ &\leq (\alpha_1 \vee \alpha_2)[h(f_1(B) \vee f_2(B)), h(f_1(C) \vee f_2(C))] \\ &= (\alpha_1 \cup \alpha_2)[h(f_1(B) \cup f_2(B)), h(f_1(C) \cup f_2(C))] \\ &= \alpha[h(f'(B) \cup f'(C)) \end{aligned}$$

Therefore

$$h(g'(B), g'(C)) \leq \alpha[h(f'(B) \cup f'(C))$$

By the condition of mathematical induction Lemma 3.6 is proved.

Thus, from all the above results and the definition of Iterated function system for Commuting mapping, we are in the position to present the following theorem for Iterated function system for Commuting mapping.

Theorem 3.4. *Let $\{X; (g_0), g_1, g_2, ..., g_N\}$, where g_0 is the condensation mapping be a Iterated function system for commuting mapping with contractivity factor α. Then the transformation $g' : H(X) \to H(X)$ defined by $g'(B) = \bigcup_{N}^{n=1} g_n(B)$ for all $B \in H(X)$ is a continuous Commuting mapping on the complete metric space $(H(X), h(d))$ with contracivity factor $'\alpha'$. Its unique fixed point, which is also called an attractor $A \in H(X)$, obeys*

$$A = g'(A) = \bigcup_{n=1}^{N} A,$$

and is given by $A = \lim_{n\to\infty} g'^{on}(B)$ for any $B \in H(X)$.

Based on above mathematical formulation of Proposition 3.1, we can prove the following *college theorem* for *IFS* due to Commuting mapping.

Theorem 3.5. *Let (X, d) be a complete metric space. Let $L \in H(X)$ be given and $\epsilon \geq 0$ be given. Choose an IFS for commuting mapping $\{x; (g_0, g_1, g_N\}$, where g_0 is the condensation mapping with contractivity factor $'\alpha'$, so that*

$$h(L, \bigcup_{n=0,n=1}^{N} g_n(L)) \leq \epsilon.$$

Then

$$h(L, A) \leq \epsilon \frac{1}{1-\alpha},$$

where A is the attractor of the IFS for commuting mapping.

Proof.

$$h(L, A) \leq \sum_{m=1}^{n} h(g'^{(on)}(L), g'^{(on)}(A))$$

$$\leq \sum_{m=1}^{n} \alpha(h(f'^{(on)}(L), f'^{(on)}(A))$$

$$\leq \sum_{m=1}^{n} \frac{\alpha^m}{1-\alpha}(h(f'(L), f'(A))$$

on taking the limit $n \to \infty$, we obtain

$$h(L, A) \leq \frac{1}{1-\alpha}(h(f'(L), f'(A)))$$

This completes the proof.

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