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FINITE DIFFERENCE SCHEMES WHITH EXACT SPECTRUM FOR SOLVING DIFFERENTIAL EQUATIONS WITH BOUNDARY CONDITIONS OF THE FIRST KIND

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Abstract: In this paper, we introduce the notion of finite difference schemes with exact spectrum (FDSES). We use these shemes for numerical solving ordinary differential equations (ODEs) and partial differential equations (PDEs) with boundary conditions of the first kind. We compare the results obtained by FDSES with the results obtained by other finite difference schemes (FDS). We prove several exact scheme propositions and theorems.

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1. Introduction

FDS are numerical methods for an approximation of the solution of differential equations, using finite differences to approximate derivatives. Hence one has to deal with difference equations (systems of linear equations).

Both analytical and numerical solutions of ODEs can be expressed in Fourier series (infinite and finite respectively). In these series appear eigenvalues of the corresponding spectral problems. If in the finite Fourier series eigenvalues of the corresponding spectral problem (discrete problem) are replaced with eigenvalues from the infinite Fourier series, then we obtain so-called **FDSES**. Similarly,

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FDSES can be obtained by using a similarity transformation in order to represent a three-diagonal matrix (matrix of the corresponding difference equation) in the canonical form, a diagonal matrix form with the matrix eigenvalues on the main diagonal.

Solving partial differential equations using the method of lines (MOL), an independent space variable x becomes discrete and derivatives with respect to x are replaced with finite differences, but a time variable t remains continuous. Hence we obtain a system of ODEs that approximates the original PDE. A solution of PDE also can expressed in Fourier series and, similarly to ODEs, FDSES can be obtained.

In this paper, FDSES will be compared with other classical FDS solving second-order linear ODEs, the heat and wave equations.

2. Matrix Spectral Problem

Since a discrete problem of ODEs can be represented as a system of linear equations, we begin with an investigation of a solution of the system using its spectrum.

The system of linear equations can be represented in the following form:

$$Ay = f, \tag{1}$$

where A is an invertible square matrix of size M with real entries, y and f are vectors of size M with real entries. If a full eigenvector set solutions $\{w^{(k)}\}$, $\{\overline{w}^{(k)}\}$ of the corresponding spectral problems

$$Aw = \mu w, A^T\overline{w} = \mu\overline{w}, \tag{2}$$

where A^T is transpose of A ,are known, then eigenvectors of these systems can be chosen as an biorthonormal basis in Euclidean space $\mathbb{R}^M$ and the solution of the system (1) can be obtained in the following form:

$$y = \sum_{k=1}^{M} \alpha_k w^{(k)}, \tag{3}$$

where $\alpha_k \in \mathbb{R}$ are linear combination coefficients.

Assuming that the following series

$$f = \sum_{k=1}^{M} \beta_k w^{(k)} (\beta_k \in \mathbb{R}), \tag{4}$$

exist and taking into account the orthogonal condition $(w^{(j)}, \overline{w}^{(k)}) = \delta_{j,k}$, where $\delta_{j,k}$ is the Kronecker delta, $(w^{(j)}, \overline{w}^{(k)})$ is a scalar product of two vectors, the scalar product of (4) and fixed eigenvector $\overline{w}^{(k)}$ gives $(f, \overline{w}^{(k)}) = \beta_k$ and $\beta_k = (f, \overline{w}^{(k)}), k = \overline{1, M}$. Similarly, $(\overline{w}^{(k)}, Ay) = \mu_k \alpha_k$, it means that $\alpha_k = \frac{\beta_k}{\mu_k}$ and the solution of the problem (1) is as follows:

$$y = \sum_{k=1}^{M} \frac{\beta_k}{\mu_k} w^{(k)}. \tag{5}$$

Spectral problem (2) can be represented in the matrix form

$$AW = WD, A^T \overline{W} = \overline{W} D,$$

where W and $\overline{W}$ are matrices with the corresponding eigenvectors $w_j^{(k)}$, $\overline{w}_j^{(k)}$, $k, j = \overline{1, M}$ on columns, $D = diag(\mu_k)$, $k = \overline{1, M}$ is a diagonal matrix with the eigenvalues μ_k on the main diagonal. It implies that the matrix A can be represented in the canonical form $A = WDW^{-1}$. Taking into account that for an arbitrary matrix function $F(A)$ the equality $F(A) = WF(D)W^{-1}$ holds, we obtain the solution of the system (1) in the following form:

$$y = A^{-1} f = WD^{-1} W^{-1} f.$$

From the biorthonormal condition it follows that $W = \overline{W}^T = E$ or $W^{-1} = \overline{W}^T$, where E is the identity matrix. So $y = WD^{-1}\overline{W}^T$.

Remark 1. Let A be a symmetric matrix ($A = A^T$), then the eigenvector system $\{w^{(k)}\}$ is an orthonormal system for which the condition $(w^{(j)}, w^{(k)}) = \delta_{j,k}$ holds, and $y = WD^{-1}Wf$.

Further, we shall consider discrete (M-finite) and continuous ($M = \infty$) spectrums for the second-order differential $(-\frac{d^2}{dx^2})$ operator with the homogeneous boundary conditions of the first kind in the segment $[0, L]$ and its approximation with the second order central difference in the uniform grid $\overline{\omega}_h = \{x_i = ih, i = \overline{0, N}, Nh = L\}$ respectively.

A discrete problem is in the form (2) ($A = A^T$), where A is a matrix of the size $M = N - 1$

$$A = \frac{1}{h^2} \begin{pmatrix} 2 & -1 & 0 & \dots & 0 & 0 \\ -1 & 2 & -1 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & -1 & 2 \end{pmatrix}$$

with the eigenvalues $\mu_k = \frac{4}{h^2}\sin^2\left(\frac{k\pi h}{2L}\right)$, $k = \overline{1, M}$ and the eigenvectors $w^{(k)}(x_j) = C_k \sin\left(\frac{k\pi x_j}{L}\right) = C_k \sin\left(\frac{k\pi j}{N}\right)$, $k, j = \overline{1, M}$. We determine constants C_k from the orthonormal condition $(w^{(k)}, w^{(j)})_h \equiv h\sum_{i=1}^{M} w^{(k)}(x_i)w^{(j)}(x_i) = \delta_{k,j}$, hence $C_k = \sqrt{\frac{2}{L}}$. Hence the eigenvectors are in the form:

$$w^{(k)}(x) = \sqrt{\frac{2}{L}} \sin\left(\frac{k\pi x}{L}\right), x \in \overline{\omega}_h, k = \overline{1, M}[1]. \tag{6}$$

Each discrete function $f(x)$, $x \in \overline{\omega}_h$ can be expressed as a finite sum of the eigenvectors (6) (discrete Fourier series) similar to (4), $f(x) = \sum_{k=1}^{M} \beta_k w^{(k)}(x)$, where $\beta_k = (f, w^{(k)})$.

In case when $h \to \infty$ we obtain the continuous problem

$$\begin{cases} -X''(x) = \lambda X(x) \\ X(0) = X(L) = 0, \end{cases} \tag{7}$$

with the eigenvalues $\lambda_k = (\frac{k\pi}{L})^2, k = 1, 2, ...$ and the eigenfunctions $X_k(x) = C_k \sin\left(\frac{k\pi x}{L}\right)$, $k = 1, 2, ...$ We determine constantes C_k from the orthonormal condition $(X_k(x), X_j(x)) \equiv \int_0^L X_k(x)X_j(x)dx = \delta_{k,j}$, hence $C_k = \sqrt{\frac{2}{L}}$. Hence the eigenvectors are in the form:

$$X_k(x) = \sqrt{\frac{2}{L}} \sin\left(\frac{k\pi x}{L}\right), k = 1, 2, ... \tag{8}$$

Each continuous function $f(x)$ ($f(0) = f(L) = 0$) can be expressed as a sum of the eigenvectors (8) $f(x) = \sum_{k=1}^{\infty} f_k X_k(x)$, where $f_k = (f(x), X_k(x))$. In case when $f(0) \neq 0$ or $f(L) \neq 0$ the Fourier series convergence can be very slow.

3. FDSES for Solving ODEs

The solution of the nonhomogeneous problem

$$\begin{cases} -u''(x) = f(x), x \in (0, L) \\ u(0) = u(L) = 0, \end{cases} \tag{9}$$

where $f(0) = f(L) = 0$, similarly to (3), can be represented as follows

$$u(x) = \sum_{k=1}^{\infty} \alpha_k X_k(x), \tag{10}$$

where $\alpha_k = \frac{f_k}{\lambda_k}$.

Remark 2. In case when $f(x)$ does not satisfy homogeneuos boundary conditions we can reduce the problem (9) to the two bondary value problems

$$\begin{cases} -u_1''(x) = f(x) + \frac{f(0)-f(L)}{L}x - f(0) \\ u_1(0) = u_1(L) = 0 \end{cases} \tag{11}$$

and

$$\begin{cases} -u_2''(x) = -\frac{f(0)-f(L)}{L}x + f(0) \\ u_2(0) = u_2(L) = 0, \end{cases} \tag{12}$$

where the right hand side of the problem (11) satisfy homogeneuos boundary conditions. The solution of the problem (9) is $u(x) = u_1(x) + u_2(x)$, where $u_2(x) = \frac{a_0}{6}\left(x^3 - xL^2\right) - \frac{b_0}{2}\left(x^2 - xL\right)$, $a_0 = \frac{f(0)-f(L)}{L}$, $b_0 = f(0)$.

FDS approximation for the problem (9) is in the following form

$$\begin{cases} -y_{x\overline{x}}(x) = f(x), x \in \omega_h \\ y(0) = y(L) = 0, \end{cases} \tag{13}$$

where $y_{x\overline{x}_i} = \frac{1}{h^2}\left(y(x_{i+1}) - 2y(x_i) + y(x_{i-1})\right)$, or in the matrix form $Ay = f$. The solution of the problem (13) can be written in the finite series form

$$y(x) = \sum_{k=1}^{M} \alpha^{(k)} w^{(k)}(x), x \in \omega_h, \tag{14}$$

where $\alpha^{(k)} = \frac{f^{(k)}}{\mu_k}$, $f^{(k)} = (f, w^{(k)})_h$.

In case when in the spectral representation $A = WDW$ or in the corresponding series the eigenvalues μ are replaced with λ one obtains FDSES [2].

Proposition 1. *Let* $f(x) = \sum_{i=1}^{M} A_i X_i(x)$, *where* $A_i \in \mathbb{R}$ *and* $X_i(x) = w^{(i)}(x), i = \overline{1,M}, x \in \omega_h$ *are the eigenfunctions of the problem* (7), *then FDSES for the problem* (9) *is exact, it means that values of the analytical solution and values of the numerical solution are equal at the grid points.*

Proof. The analytical solution is in the following form

$$u(x) = \sum_{k=1}^{M} \frac{A_k}{\lambda_k} X_k(x),$$

where $A_k = (X_k, \sum_{i=1}^{M} A_i X_i(x))$ and the numerical solution is

$$y(x) = \sum_{k=1}^{M} \frac{A_k}{\lambda_k} w^{(k)}(x),$$

$x \in \omega_h$, where $A_k = (w^{(k)}, \sum_{i=1}^{M} A_i w^{(i)}(x))_h$, which proves the proposition. □

Example 1. The analytical solution of (9), if $f(x) = \sin(\pi x)$, $L = 1$ is $u(x) = \pi^{-2}\sin(\pi x)$. The following numerical results are obtained using MATLAB: Δu(FDS)=0.0032, Δu(FDSES)=$1.3878 \cdot 10^{-17}$, by $M = 4$. Here $\Delta u = \max_{1\le j\le M} |u(x_j) - y(x_j)|$.

Example 2. If $f(x) = x(x-1)$, $L = 1$, then the analytical solution of (9) is $u(x) = -\frac{x^4}{12} + \frac{x^2}{6} - \frac{x}{12}$. The numerical results using MATLAB are: Δu(FDS)=0.0008, Δu(FDSES)=$1.5936 \cdot 10^{-5}$, by $M = 4$, Δu(FDS)=$2.0833 \cdot 10^{-4}$, Δu(FDSES)=$1.005 \cdot 10^{-6}$, by $M = 9$.

Now, we shall consider the boundary value problem

$$\begin{cases} -u''(x) - bu(x) = f(x) \\ u(0) = u(L) = 0, \end{cases} \tag{15}$$

where $f(0) = f(L) = 0$, $b \in \mathbb{R}$, $b \neq 0$. Eigenfunctions and eigenvectors of the corresponding spectral problems for (15) are the same as for (9), but the corresponding eigenvalues are $\lambda_k = \left(\frac{k\pi}{L}\right)^2 - b, k = 1, 2...$ and $\mu_k = \frac{4}{h^2}\sin^2\left(\frac{k\pi h}{2L}\right) - b, k = \overline{1, M}$ respectively. We define FDSES for (15) as it was done for (9). In some cases, in order to refine numerical results with FDS one uses so-called Bahvalov's method, namely, multiplying $y_{x\bar{x}}$ by the parameter γ. If $b > 0$, $\gamma = (b_1/\sin(b_1))^2$, where $b_1 = \sqrt{b}h/2$ and if $b < 0$, $\gamma = (b_1/\sinh(b_1))^2$, where $b_1 = \sqrt{-b}h/2$ [3].

Proposition 2. *Let* $f(x) = \sum_{i=1}^{M} A_i X_i(x)$, *where* $A_i \in \mathbb{R}$ *and* $X_i(x), i = \overline{1, M}$ *are the eigenfunctions of the problem* (7), *then FDSES for the problem* (15) *is exact.*

Example 3. If $f(x) = 10\sin(3\pi x)$, $b = 50$, $L = 1$, then the analytical solution of (15) is $u(x) = \frac{10\sin(3\pi x)}{-50+9\pi^2}$. The numerical results using MATLAB are: Δu(GDS)=0.3706, Δu(Bah. GDS)=0.1003, Δu(FDSES)=$5.5511 \cdot 10^{-17}$, by $M = 4$.

Example 4. If $f(x) = x(x-1)e^{-x}$, $L = 1$, then

$$u(x) = \frac{4\sin(\sqrt{b}x)(2e^{-1} + \cos(\sqrt{b})b - \cos(\sqrt{b}))}{\sin(\sqrt{b})(b^3 + 3b^2 + 3b + 1)} - \frac{4\cos(\sqrt{b}x)(b-1)}{b^3 + 3b^2 + 3b + 1} -$$

$$\frac{((x^2-x)b^2+(2x-4+2x^2)b+x^2+4+3x)e^{-x}}{(b+1)^3}.$$

The numerical results are: Δu(FDS)=0.0019, Δu(Bah. FDS)=0.001, Δu(FDSES)=$1.6904\cdot 10^{-4}$, by $b=100$, $M=4$, Δu(FDS)=$8.4343\cdot 10^{-4}$, Δu(Bah. FDS)=$2.2147\cdot 10^{-4}$, Δu(FDSES)=$9.2617\cdot 10^{-6}$, by $b=100$, $M=9$.

Now, we shall consider the corresponding spectral problems for the boundary value problem

$$\begin{cases} -u''(x)-au'(x)=f(x) \\ u(0)=u(L)=0, \end{cases} \tag{16}$$

where $f(0)=f(L)=0$, $a\in\mathbb{R}$. The continuous spectral problem is in the following form:

$$\begin{cases} X''(x)+aX'(x)+\lambda X(x)=0 \\ X(0)=X(L)=0. \end{cases} \tag{17}$$

The eigenfunctions in this case are $X_k(x)=\sqrt{\frac{2}{L}}\exp\left(-\frac{ax}{2}\right)\sin\left(\frac{k\pi x}{L}\right)$, $k=1,2,...$ and the eigenvalues are $\lambda_k=\frac{a^2}{4}+\left(\frac{k\pi}{L}\right)^2$, $k=1,2,....$

For the corresponding conjugate problem

$$\begin{cases} \overline{X}''(x)-a\overline{X}'(x)+\lambda\overline{X}(x)=0 \\ \overline{X}(0)=\overline{X}(L)=0, \end{cases}$$

the eigenfunctions are $\overline{X}_k(x)=\sqrt{\frac{2}{L}}\exp\left(\frac{ax}{2}\right)\sin\left(\frac{k\pi x}{L}\right)$, besides the biorthonormal condition $(X_j(x),\overline{X_k}(x))\equiv\int_0^L X_j(x)\overline{X_k}(x)dx=\delta_{j,k}$ holds automatically. The analytical solution of (16) is $u(x)=\sum_{k=1}^{\infty}\alpha_k X_k(x), \alpha_k=\frac{f_k}{\lambda_k}, f_k=(f(x),\overline{X_k}(x))$.

The discrete spectral problem for (16) is

$$\begin{cases} \gamma w_{x\overline{x}}+aw_{\dot{x}}+\mu w=0, x\in\omega_h \\ w(0)=w(L)=0, \end{cases} \tag{18}$$

where $w_{\dot{x}_i}=(y_{i+1}-y_{i-1})/2h$, $\gamma=a_1\coth(a_1)$, $a_1=ah/2$ (Iljin's scheme) [4]. The solution of the discrete spectral problem is $w^{(k)}(x)=\sqrt{\frac{2}{L}}\exp\left(-\frac{ax}{2}\right)\sin\left(\frac{k\pi x}{L}\right)$, $x\in\omega_h, k=\overline{1,M}$, $\mu_k=\frac{2a_1}{h^2\sinh(a_1)}(\cosh(a_1)-\cos\left(\frac{k\pi h}{L}\right)), k=\overline{1,M}$. The solution of the corresponding conjugate problem (take -a instead of a) is $\overline{w}^{(k)}(x)=\sqrt{\frac{2}{L}}\exp\left(\frac{ax}{2}\right)\sin\left(\frac{k\pi x}{L}\right)$.

The numerical solution of the nonhomogeneous problem

$$\begin{cases} -\gamma y_{x\overline{x}} - a y_{\dot{x}} = f(x), x \in \omega_h \\ y(0) = y(L) = 0 \end{cases} \tag{19}$$

is

$$y(x) = \sum_{j=k}^{M} \alpha^{(k)} w^{(k)}(x), x \in \omega_h, \tag{20}$$

where $\alpha^{(k)} = \frac{f^{(k)}}{\mu_k}, f^{(k)} = (f, \overline{w}^{(k)})_h$.

We define FDSES for the problem (16) as it was done previously. In the numerical solution (20) we replace the eigenvalues μ with the eigenvalues λ.

Proposition 3. *Let* $f(x) = \sum_{i=1}^{M} A_i X_i(x)$, *where* $A_i \in \mathbb{R}$ *and* $X_i(x), i = \overline{1, M}$ *are the eigenfunctions of the problem* (17), *then FDSES for the problem* (16) *is exact.*

Example 5. If $f(x) = e^{-\frac{5x}{2}} \sin(3\pi x)$, $a = 5$, $L = 1$, then $u(x) = \frac{20e^{-\frac{5x}{2}} \sin(3\pi x)}{125+180\pi^2}$. The numerical results are: Δu(Il. FDS)=0.0038, Δu(Il.FDSES) $=1.3878 \cdot 10^{-18}$, by $M = 4$.

Example 6. If $f(x) = x(x-1)$, $L = 1$, then $u(x) = \frac{x^2}{2a} + \frac{x^2}{a^2} - \frac{x^3}{3a} - \frac{(a^2-12)e^{-at}}{6a^3(e^{-a}-1)} - \frac{x}{a^2} - \frac{2x}{a^3} + \frac{a^2-12}{6a^3(e^{-a}-1)}$. The numerical results are:

a	M	Δu(Il. FDS)	Δu(Il. FDSES)
1	4	$7.9795 \cdot 10^{-4}$	$2.2690 \cdot 10^{-5}$
1	9	$2.0820 \cdot 10^{-4}$	$1.4678 \cdot 10^{-6}$
20	4	$5.8104 \cdot 10^{-4}$	$6.5513 \cdot 10^{-4}$
20	9	$1.8847 \cdot 10^{-4}$	$3.5775 \cdot 10^{-5}$
50	9	$1.4336 \cdot 10^{-4}$	6.3506
50	99	$2.0726 \cdot 10^{-6}$	$1.6695 \cdot 10^{-5}$
50	199	$5.2016 \cdot 10^{-7}$	$1.3392 \cdot 10^{-7}$

Table 1: Maximal errors

If the parameter a grows, also Δu(Il. FDSES) grows and Il. FDSES is better than Il. FDS, if the number of grid points is large. It happens because of boundary layers with a large amplitude by large a values.

4. MOL and FDSES for an Initial-Boundary Value Problem for the Heat Equation

An initial-boundary value problem for the heat equation with homogenous boundary conditions of the first kind is as follows:

$$\begin{cases} \frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + f(x,t), x \in (0,L), t \geq 0 \\ u|_{t=0} = u_0(x), x \in [0,L] \\ u|_{x=0} = 0, u|_{x=L} = 0, t \geq 0, \end{cases} \tag{21}$$

where f, u_0 are continuous functions, x is a space coordinate, t is a time coordinate.

4.1. Analytical Solution

We are looking for an analytical solution of (21) in the series form

$$u(x,t) = \sum_{k=1}^{\infty} a_k(t) X_k(x), \tag{22}$$

where, $X_k(x) = \sqrt{\frac{2}{L}} \sin\left(\frac{k\pi x}{L}\right), k = 1, 2, ...$ are the eigenfunctions of the differential operator $(-\frac{\partial^2}{\partial x^2})$, $a_k(t)$ are unknown functions. We insert (22) into (21) and assume that $f(x,t)$ can be expressed in the series $f(x,t) = \sum_{k=1}^{\infty} f_k(t) X_k(x)$, where $f_k(t) = (f, X_k)$. Taking into account the previous assumptions we obtain the Cauchy problem for second-order linear ODEs:

$$\begin{cases} \dot{a_k}(t) + \lambda_k a_k(t) = f_k(t) \\ a_k(0) = (u_0, X_k), \end{cases} \tag{23}$$

where $\dot{a_k}(t) = \frac{da_k(t)}{dt}$, $\lambda_k = \left(\frac{k\pi}{L}\right)^2$. The solution of (23) is $a_k(t) = a_k(0)e^{-\lambda_k t} + \int_0^t f_k(\xi) e^{-\lambda_k (t-\xi)} d\xi$.

4.2. MOL and FDS

Solving PDEs with the MOL, we approximate the differential operator $(-\frac{\partial^2}{\partial x^2})$ with the second order central difference (approximation of the second order) in the uniform grid. Hence, we obtain the M-th order system of ODEs. The system looks as follows:

$$\begin{cases} \frac{dy}{dt} = \Lambda y + f(x,t), x \in \omega_h, t \geq 0 \\ y|_{t=0} = u_0(x), x \in \overline{\omega}_h \\ y|_{x=0} = y|_{x=L} = 0, t \geq 0, \end{cases} \quad (24)$$

where $y = y(x,t)$, $x \in \overline{\omega}_h$, $t \geq 0$, $-\Lambda y \equiv -y_{x\overline{x}}$.

We are looking for a solution of (24) in the series form

$$y(x,t) = \sum_{k=1}^{M} a_k(t) w^{(k)}(x), x \in \omega_h \quad (25)$$

where $w^{(k)}(x)$ are the eigenvectors of the difference operator $(-\Lambda y)$, $a_k(t)$ are unknown functions. Assuming that $f(x,t) = \sum_{k=1}^{M} f_k(t) w^{(k)}(x), x \in \omega_h$, where $f_k(t) = (f, w^{(k)})_h$, we obtain the Cauchy problem for second-order linear ODEs:

$$\begin{cases} \dot{a}_k(t) + \mu_k a_k(t) = f_k(t) \\ a_k(0) = (u_0, w^{(k)})_h, \end{cases} \quad (26)$$

where $\mu_k = \frac{4}{h^2} \sin^2\left(\frac{k\pi h}{2L}\right)$. The solution of (25) is

$$a_k(t) = a_k(0) e^{-\mu_k t} + \int_0^t f_k(\xi) e^{-\mu_k (t-\xi)} d\xi.$$

The MOL (24) can be represented in the matrix form:

$$\begin{cases} \frac{dY}{dt} + AY = F(t), t \geq 0 \\ Y(0) = U_0, \end{cases}$$

where $Y(t)$, $F(t)$, U_0 are M-th order column vectors, A is the diagonal matrix. Taking into account the condition $A = WDW$ and using the transformations $\tilde{Y} = WY$, $\tilde{F} = WF$, we obtain the following system:

$$\begin{cases} \frac{d\tilde{Y}}{dt} + D\tilde{Y} = \tilde{F}(t), t \geq 0 \\ \tilde{Y}(0) = WY(0) = WU_0. \end{cases}$$

4.3. FDSES

In case when in the diagonal matrix D the eigenvalues μ_k are replaced with the eigenvalues λ_k, namely, the first M eigenvalues of the differential operator

$(-\frac{\partial^2}{\partial x^2})$, one obtains FDSES. FDSES series representation for (21) is in the following form:

$$y(x,t) = \sum_{k=1}^{M} \tilde{a}_k(t) w^{(k)}(x), x \in \omega_h, \tag{27}$$

where $\tilde{a}_k(t) = \tilde{a}_k(0)e^{-\lambda_k t} + \int_0^t \tilde{f}_k(\xi)e^{-\lambda_k(t-\xi)}d\xi$, $\tilde{a}_k(0) = (u_0, w^{(k)})_h$, $\tilde{f}_k(t) = (f, w^{(k)})_h$.

Theorem 1. *Let* $f(x,t) = \sum_{i=1}^{M_1} b_i(t)X_i(x)$ *and* $u_0(x) = \sum_{i=1}^{M_2} B_i X_i(x)$, *where* $\max\{M_1, M_2\} \le M$,$B_i \in \mathbb{R}$, $b_i(t)$ *are integrable functions and* $X_i(x)$ *are the eigenfunctions* (8), *then the formula* (27) *is exact.*

Proof. Since the eigenfunction $X_i(x)$, $x \in [0, L]$ and the eigenvectors $w^{(i)}(x)$, $x \in \omega_h$ are equal at the grid points, the orthonormal condition implies that

$$a_k(0) = (\sum_{i=1}^{M_2} B_i X_i(x), X_k(x)) = B_k,\ k \le M_2,\ \tilde{a}_k(0) = (\sum_{i=1}^{M_2} B_i X_i(x),$$

$w^{(k)}(x))_h = B_k, k \le M_2$, and $\tilde{a}_k(0) = 0, a_k(0) = 0,\ k > M_2$,

$$f_k(t) = (\sum_{i=1}^{M_1} b_i(t)X_i(x), X_k(x)) = b_k(t), k \le M_1,$$

$\tilde{f}_k(t) = (\sum_{i=1}^{M_1} b_i(t)X_i(x), w^{(k)}(x))_h = b_k(t), k \le M_1$ and $\tilde{f}_k(t) = 0,\ f_k(t) = 0,$ $k > M_1$. It implies that the functions $a_k(t)$ and $\tilde{a}_k(t)$, $k = \overline{1, M}$ are equal and $a_k(t) = 0,\ k > M$. It means that the solutions (analytical and numerical) are equal at the grid points. □

Example 7. If $f(x,t) = \sqrt{2}e^{-\pi t}\sin(\pi x)$, $u_0(x) = \sqrt{2}\sin(\pi x) + \sqrt{2}\sin(4\pi x)$, $L = 1$, then, if $M \ge 4$, the conditions of the Theorem 1. are satisfied. The exact solution is $u(x,t) = (e^{-\pi^2 t} + te^{-\pi^2 t})\sqrt{2}\sin(\pi x) + \frac{16\pi^2 - 1 + e^{-16\pi^2 t}}{(16\pi^2)^2}\sqrt{2}\sin(4\pi x) + e^{-16\pi^2 t}\sqrt{2}\sin(4\pi x)$. The corresponding numerical results are: Δu(GDS)=0.0124, Δu(GDSPS)=$8.3267 \cdot 10^{-17}$, by $M = 4$.

5. MOL and FDSES for an Initial-Boundary Value Problem for the Wave Equation

An initial-boundary value problem for the wave equation is as follows:

$$\begin{cases} \frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2} + f(x,t), x \in (0,L), t \geq 0 \\ u|_{t=0} = u_0(x), x \in [0,L] \\ \frac{\partial u}{\partial t}|_{t=0} = \tilde{u}_0(x), x \in [0,L] \\ u|_{x=0} = 0, u|_{x=L} = 0, t \geq 0, \end{cases} \tag{28}$$

where f, u_0, $\tilde{u}_0$ are continuous functions.

5.1. Analytical Solution

We are looking for an analytical solution of (28) in the series form (22). We determine functions $a_k(t)$ from the following Cauchy problem:

$$\begin{cases} \ddot{a}_k(t) + \lambda_k a_k(t) = f_k(t) \\ a_k(0) = (u_0, X_k), \dot{a}_k = (\tilde{u}_0, X_k), \end{cases} \tag{29}$$

where $f_k(t) = (f, X_k)$, $\lambda_k = \left(\frac{k\pi}{L}\right)^2$. The solution of (29) is

$$a_k(t) = a_k(0)\cos(\sqrt{\lambda_k}t) + \frac{\dot{a}_k(0)}{\sqrt{\lambda_k}}\sin(\sqrt{\lambda_k}t) + \frac{1}{\sqrt{\lambda_k}}\int_0^t f_k(\xi)\sin(\sqrt{\lambda_k}(t-\xi))d\xi.$$

5.2. MOL and FDS

Similarly to 4.2., we approximate the differential operator $(-\frac{\partial^2}{\partial x^2})$ with the second order central difference in the uniform grid and obtain the M-th order system of ODEs:

$$\begin{cases} \frac{d^2 y}{dt^2} = \Lambda y + f(x,t), x \in \omega_h, t \geq 0 \\ y|_{t=0} = u_0(x), \frac{dy}{dt}|_{t=0} = \tilde{u}_0(x), x \in \overline{\omega}_h \\ y|_{x=0} = y|_{x=L} = 0, t \geq 0, \end{cases} \tag{30}$$

where $y = y(x,t)$, $x \in \overline{\omega}_h$, $t \geq 0$.

We are looking for a solution of (30) in the series form (25). We determine functions $a_k(t)$ from the following, similar to (29), Cauchy problem:

$$\begin{cases} \ddot{a}_k(t) + \mu_k a_k(t) = f_k(t) \\ a_k(0) = (u_0, w^{(k)})_h, \dot{a}_k = (\tilde{u}_0, w^{(k)})_h, \end{cases} \tag{31}$$

where $f_k(t) = (f, w^{(k)})_h$, $\mu_k = \frac{4}{h^2}\sin^2\left(\frac{k\pi h}{2L}\right)$. The solution of (31) is $a_k(t) = a_k(0)\cos(\sqrt{\mu_k}t) + \frac{\dot{a}_k(0)}{\sqrt{\mu_k}}\sin(\sqrt{\mu_k}t) + \frac{1}{\sqrt{\mu_k}}\int_0^t f_k(\xi)\sin(\sqrt{\mu_k}(t-\xi))d\xi$.

The MOL (30) can be represented in the matrix form:

$$\begin{cases} \frac{d^2Y}{dt^2} + AY = F(t), t \geq 0 \\ Y(0) = U_0 \\ \tilde{Y}(0) = \tilde{U}_0, \end{cases}$$

where $\tilde{U}_0$ is s M-th order column vector.

5.3. FDSES

We define FDSES as usual, by replacing the eigenvalues μ_k with λ_k in the diagonal matrix A. FDSES series representation for (28) is in the following form:

$$y(x,t) = \sum_{k=1}^{M} \tilde{a}_k(t)w^{(k)}(x), x \in \omega_h, \tag{32}$$

where $\tilde{a}_k(t) = \tilde{a}_k(0)\cos(\sqrt{\lambda_k}t) + \frac{\dot{\tilde{a}}_k(0)}{\sqrt{\lambda_k}}\sin(\sqrt{\lambda_k}t) + \frac{1}{\sqrt{\lambda_k}}\int_0^t \tilde{f}_k(\xi)\sin(\sqrt{\lambda_k}(t-\xi))d\xi$, $\tilde{a}_k(0) = (u_0, w^{(k)})_h$, $\dot{\tilde{a}}_k(0) = (\tilde{u}_0, w^{(k)})_h$, $\tilde{f}_k(t) = (f, w^{(k)})_h$.

Theorem 2. *Let* $f(x,t) = \sum_{i=1}^{M_1} b_i(t)X_i(x)$, $u_0(x) = \sum_{i=1}^{M_2} B_iX_i(x)$ *and* $\tilde{u}_0(x) = \sum_{i=1}^{M_3} \tilde{B}_iX_i(x)$, $\max\{M_1, M_2, M_3\} \leq M$, $B_i, \tilde{B}_i \in \mathbb{R}$, $b_i(t)$ *are integrable functions and* $X_i(x)$ *are the eigenfunctions* (8), *then the formula* (32) *is exact.*

Proof. Since the eigenfunction and eigenvectors are equal at the grid points, as in the first proof we have: $a_k(0) = \tilde{a}_k(0) = B_k, k \leq M_2$ and $a_k(0) = 0$, $\tilde{a}_k(0) = 0$, $k > M_2$, $f_k(t) = \tilde{f}_k(t) = b_k(t), k \leq M_1$ and $\tilde{f}_k(t) = 0$, $f_k(t) = 0$, $k > M_1$, $\dot{a}_k(0) = (\sum_{i=1}^{M_3} \tilde{B}_iX_i(x), X_k(x)) = \tilde{B}_k, k \leq M_2, \dot{\tilde{a}}_k(0) = (\sum_{i=1}^{M_3} \tilde{B}_iX_i(x)$, $w^{(k)}(x))_h = \tilde{B}_k, k \leq M_2$ and $\dot{a}_k(0) = 0$, $\dot{\tilde{a}}_k(0) = 0$, $k > M_3$. It implies that the functions $a_k(t)$ and $\tilde{a}_k(t)$, $k = \overline{1, M}$ are equal and $a_k(t) = 0$, $k > M$. It means that the solutions (analytical and numerical) are equal at the grid points. □

Example 8. If $f(x,t) = 0$, $u_0(x) = \sqrt{2}\sin(\pi x)$, $\tilde{u}_0(x) = \sqrt{2}\sin(2\pi x)$, $L = 1$, then, if $M \geq 2$, the conditions of the Theorem 2. are satisfied. The exact solution is $u(x,t) = \sqrt{2}\cos(\pi t)\sin(\pi x) + \frac{\sqrt{2}}{2\pi}\sin(2\pi t)\sin(2\pi x)$. The corresponding numerical results are: Δu(GDS)=0.0644, Δu(GDSPS)=$6.6613 \cdot 10^{-16}$, by $M = 4$.

6. Conclusion

Numerical results, obtained using MATLAB, show that FDSES is more precise than FDS for solving differential equations, which are in the form $-u''(x) = f(x)$. For equations, which are in the form $-u''(x) - bu(x) = f(x)$, FDSES is more precise than FDSES or Bahvalov's FDSES. For equations in the form $-u''(x) - a'u(x) = f(x)$ FDSES gives more accurate results for small parameter a values. In case when parameter a values are considerably large, FDSES is more precise than FDS only if the number of grid points is large.

For PDEs, namely, the heat and wave equations, exact scheme theorems are proved.

In this paper, only continuous functions $f(x)$ and differential equations with the boundary conditions of the first kind were considered, however it is possible to develop the theory for equations with boundary conditions of the second and third kind, periodic boundary conditions.

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