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ON POINTWISE BINOMIAL APPROXIMATION BY w-FUNCTIONS

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Abstract: This paper, we use Stein's method and w-functions to give a result in the binomial approximation to the distribution of a non-negative integer-valued random variable, in terms of the point metric between two such distributions together with its non uniform bound. Furthermore, for applications, we use the obtained result to approximate some distributions such as hypergeometric, negative hypergeometric and Pólya distributions.

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1. Introduction

Many studies of binomial approximation via Stein's method have yielded useful results in applications of probability and statistics. The first study of binomial approximation by Stein's method, for approximating the number of ones in the binary expansion of a random integer and for problem of counting Latin rectangles, was presented by Stein [6]. Ehm [3] gave lower and upper bounds of the error in the binomial approximation of a sum of n independent indicator random variables, and he applied the result to sampling with and without replacement. Barbour et al. [1] showed that Stein's method could be applied as well in the binomial context as in the Poisson. Soon [5] considered this approximation in connection with a sum of dependent indicator random variables,

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and he applied the result to hypergeometric distribution, random graphs problem and the classical occupancy problem. Wongkasem et al. [8] used Stein's method and w-functions to give an error bound on binomial approximation to a generalized binomial distribution, and Teerapabolarn [7] used the same tools as in Wongkasem et al. [8] to give an error bound on binomial approximation to the beta binomial distribution in the recent paper. However, all bounds as mentioned above are the total variation distance bounds. In this paper, we use Steins method and w-functions to give a non uniform bound in the binomial approximation of a non-negative integer-valued random variable for the point metric between the two distributions.

Let X be a non-negative integer-valued random variable with probability function $p(x) = P(X = x) > 0$ for every x in the support of X, $\mathcal{S}(x)$, and have mean μ and finite variance σ^2 $(0 < \sigma^2 < \infty)$. It is well-known that the distributions of some types of X's can be approximated by a binomial distribution with parameters n and p provided their parameters are satisfied under certain conditions. For example, a hypergeometric distribution can be approximated by a binomial distribution provided that the certain conditions on their parameters are satisfied. Let $b(x_0; n, p) = \binom{n}{x_0} p^{x_0} q^{n-x_0}$ denote a binomial probability function with parameters $n \in \mathbb{N}$ and $p = 1 - q \in (0, 1)$ at $x_0 \in \{0, ..., n\}$. If we expect $p(x_0)$ to be closer to $b(x_0; n, p)$, then it is reasonable to estimate $p(x_0)$ by $b(x_0; n, p)$. For approximating $p(x_0)$ by $b(x_0; n, p)$, a bound for the point metric between $p(x_0)$ and $b(x_0; n, p)$ is a criterion for measuring the accuracy of the approximation.

In this study, we derive a non-uniform bound of the error on the metric between $p(x_0)$ and $b(x_0; n, p)$. The tools for giving our result consist of the so-called w-functions and Steins method for the binomial distribution, which are in Section 2. In Section 3, we use Stein's method and w-functions to give the result in terms of the point metric between $p(x_0)$ and $b(x_0; n, p)$, and we give some applications of the result of this approximation by using the result to approximate some distributions such as hypergeometric, negative hypergeometric and Pólya distributions, which are in the last section.

2. Method

In order to give the result for this approximation, we use the same methodology as in Teerapabolarn [7], which consists of Stein's method and w-functions. For w-functions, Cacoullos and Papathanasiou [2] defined a function w associated with non-negative integer-valued random variable X in the relation

$$w(x)p(x) = \frac{1}{\sigma^2}\sum_{i=0}^{x}(\mu - i)p(i), \ x \in \mathcal{S}(x) \tag{2.1}$$

and, afterwards, Majsnerowska [4] expressed the relation (2.1) as the form

$$w(0) = \frac{\mu}{\sigma^2}, \ w(x) = \frac{1}{\sigma^2}\left\{\mu + \frac{\sigma^2 w(x-1)p(x-1)}{p(x)} - x\right\}, \ x \in \mathcal{S}(x) \setminus \{0\} \tag{2.2}$$

and

$$w(x) \geq 0, \ x \in \mathcal{S}(x), \tag{2.3}$$

where $p(x) > 0$ for every $x \in \mathcal{S}(x)$. The following relation is an important property for proving the result, which was stated by Cacoullos and Papathanasiou [2].

If a non-negative integer-valued random variable X is defined as in Section 1, then

$$E[(X - \mu)g(X)] = \sigma^2 E[w(X)\Delta g(X)], \tag{2.4}$$

for any function $g : \mathbb{N} \cup \{0\} \to \mathbb{R}$ for which $E|w(X)\Delta g(X)| < \infty$, where $\Delta g(x) = g(x+1) - g(x)$. For $g(x) = x$, we have that $E[w(X)] = 1$.

For Stein's method, we start it by using Stein's equation in Barbour et al. [1]. Stein's equation for the binomial distribution with parameters $n \in \mathbb{N}$ and $p \in (0,1)$ is, for given h, of the form

$$(n - x)pg(x+1) - qxg(x) = h(x) - \mathcal{B}_{n,p}(h), \tag{2.5}$$

where $\mathcal{B}_{n,p}(h) = \sum_{k=0}^{n} h(k)\binom{n}{k}p^k q^{n-k}$ and g and h are bounded real-valued functions defined on $\{0, 1, ..., n\}$.

For $A \subseteq \{0, 1, ..., n\}$, let $h_A : \{0, 1, ..., n\} \to \mathbb{R}$ be defined by

$$h_A(x) = \begin{cases} 1 & \text{if } x \in A, \\ 0 & \text{if } x \notin A. \end{cases} \tag{2.6}$$

By following Barbour et al. [1] on pp. 189, let $g_A : \mathbb{N} \cup \{0\} \to \mathbb{R}$ satisfy (2.5), where $g_A(0) = g_A(1)$ and $g_A(x) = g_A(n)$ for $x \geq n$.

For $A = \{x_0\}$, $x_0 \in \{0, ..., n\}$, the solution $g_{x_0} = g_{\{x_0\}}$ of (2.5) can be written as

$$g_{x_0}(x) = \begin{cases} \frac{\binom{n}{x_0} p^{x_0-x} \mathcal{B}_{n,p}(1-h_{C_{x-1}})}{x\binom{n}{x} q^{x_0-(x-1)}} & \text{if } x_0 < x, \\ \\ -\frac{\binom{n}{x_0} p^{x_0-x} \mathcal{B}_{n,p}(h_{C_{x-1}})}{x\binom{n}{x} q^{x_0-(x-1)}} & \text{if } x_0 \geq x \geq 1, \end{cases} \tag{2.7}$$

where $C_x = \{0, ..., x\}$.

To prove the result, the following lemma is also need.

Lemma 2.1. *For $x_0 \in \{0, 1, ..., n\}$ and $x \in \mathbb{N}$, let $\Delta g_{x_0}(x) = g_{x_0}(x+1) - g_{x_0}(x)$, then we have the following.*

$$|g_{x_0}(x)| \leq \begin{cases} \frac{1-q^n}{np} & \textit{if } x_0 = 0, \\ \min\left\{\frac{1-p^n}{x_0 q}, \frac{1-p^{n+1}q^{n+1}}{(n+1)pq}\right\} & \textit{if } x_0 > 0 \end{cases} \tag{2.8}$$

and

$$|\Delta g_{x_0}(x)| \leq \begin{cases} \frac{1-q^n}{np} & \textit{if } x_0 = 0, \\ \min\left\{\frac{1-p^n}{x_0 q}, \frac{1-p^{n+1}q^{n+1}}{(n+1)pq}\right\} & \textit{if } x_0 > 0 \end{cases} \tag{2.9}$$

Proof. For $x_0 = 0$, it follows from [1] that g_0 is positive and decreasing in $x \in \{1, ..., n\}$ and $\Delta g_0(x) = 0$ for $x = 0$ and $x \geq n$. Therefore, we have $g_0(1) \geq g_0(x) \geq |\Delta g_0(x)|$ for every $x \in \{1, ..., n\}$ and, by (2.7), $g_0(1) = \frac{1-q^n}{np}$, which implies $|g_0(x)| \leq \frac{1-q^n}{np}$ and $|\Delta g_0(x)| \leq \frac{1-q^n}{np}$.

For $x_0 > 0$, it follows from [1] that g_{x_0} is positive and decreasing in $x \in \{x_0+1, ..., n\}$ and is negative and decreasing in $x \in \{1, ..., x_0\}$ and $\Delta g_{x_0}(x) = 0$ for $x = 0$ and $x \geq n$. Therefore, we have that $|g_{x_0}(x)| \leq \Delta g_{x_0}(x_0)$ and $|\Delta g_{x_0}(x)| \leq \Delta g_{x_0}(x_0)$ and

$$\begin{aligned} \Delta g_{x_0}(x_0) &= \frac{1}{(n-x_0)p} \sum_{k=x_0+1}^{n} \binom{n}{k} p^k q^{n-k} + \frac{1}{qx_0} \sum_{k=0}^{x_0-1} \binom{n}{k} p^k q^{n-k} \\ &= \frac{1}{qx_0} \left\{ \sum_{k=x_0+1}^{n} \binom{n}{k-1} p^{k-1} q^{n+1-k} \frac{x_0(n+1-k)}{k(n-x_0)} + \sum_{k=0}^{x_0-1} \binom{n}{k} p^k q^{n-k} \right\} \\ &\leq \frac{1}{qx_0} \left\{ \sum_{k=x_0}^{n-1} \binom{n}{k} p^k q^{n-k} + \sum_{k=0}^{x_0-1} \binom{n}{k} p^k q^{n-k} \right\} \end{aligned}$$

$$= \frac{1-p^n}{qx_0}$$

and

$$\begin{aligned}
\Delta g_{x_0}(x_0) &= \frac{\sum_{k=x_0+1}^{n} \frac{n!}{k!(n-k)!(n-x_0)} p^k q^{n+1-k}}{pq} + \frac{\sum_{k=0}^{x_0-1} \frac{n!}{x_0 k!(n-k)!} p^{k+1} q^{n-k}}{pq} \\
&= \frac{\sum_{k=x_0+1}^{n} \frac{(n+1)!(n+1-k)}{k!(n+1-k)!(n-x_0)} p^k q^{n+1-k}}{(n+1)pq} + \frac{\sum_{k=0}^{x_0-1} \frac{(n+1)!(k+1)}{(k+1)!(n-k)! x_0} p^{k+1} q^{n-k}}{(n+1)pq} \\
&\le \frac{1}{(n+1)pq} \left\{ \sum_{k=x_0+1}^{n} \binom{n+1}{k} p^k q^{n+1-k} + \frac{1}{qx_0} \sum_{k=1}^{x_0} \binom{n+1}{k} p^k q^{n+1-k} \right\} \\
&= \frac{1-p^{n+1}-q^{n+1}}{(n+1)pq},
\end{aligned}$$

gives

$$|g_{x_0}(x)| \le \min\{\frac{1-p^n}{x_0 q}, \frac{1-p^{n+1}-q^{n+1}}{(n+1)pq}\}$$

and

$$|\Delta g_{x_0}(x)| \le \min\{\frac{1-p^n}{x_0 q}, \frac{1-p^{n+1}-q^{n+1}}{(n+1)pq}\}.$$

So, from both cases, (2.8) and (2.9) are obtained. □

3. Result

The following theorem shows a result in the binomial approximation to the distribution of a non-negative integer-valued random variable X, in terms of the point metric and its non-uniform bound, which is obtained by Stein's method and w-functions.

Theorem 3.1. *Let a non-negative integer-valued random variable X with $p(x) > 0$ for every $x \in \mathcal{S}(x)$ and together with corresponding w-function $w(X)$ be defined as above. Then the following inequalities hold:*

1. For $x_0 = 0$,

$$|p(0) - b(0; n, p)| \le \frac{1-q^n}{np} \left\{ E\left|(n-X)p - \sigma^2 w(X)\right| + |np - \mu| \right\} \tag{3.1}$$

and, if $np=\mu$*, then*

$$|p(0)-b(0;n,p)| \leq \frac{1-q^n}{np} E\left|(n-X)p-\sigma^2 w(X)\right|. \tag{3.2}$$

2. For $x_0 \in \{1,...,n\}$,

$$|p(x_0)-b(x_0;n,p)| \leq \min\left\{\frac{1-p^n}{x_0 q}, \frac{1-p^{n+1}-q^{n+1}}{(n+1)pq}\right\} \times \left\{E\left|(n-X)p-\sigma^2 w(X)\right| + |np-\mu|\right\} \tag{3.3}$$

and, if $np=\mu$*, then*

$$|p(x_0)-b(x_0;n,p)| \leq \min\left\{\frac{1-p^n}{x_0 q}, \frac{1-p^{n+1}-q^{n+1}}{(n+1)pq}\right\} E\left|(n-X)p-\sigma^2 w(X)\right|. \tag{3.4}$$

Proof. Substituting h by $h_{\{x_0\}}$, x by X and taking expectation in (2.5), we obtain

$$p(x_0)-b(x_0;n,p) = E[(n-X)pg(X+1)-qXg(X)], \tag{3.5}$$

where $g=g_{x_0}$ is defined in (2.7) and

$$\begin{aligned} E[(n-X)pg(X+1)-qXg(X)] &= E[npg(X+1)-pX\Delta g(X)-Xg(X)] \\ &= E[npg(X+1)]-pE[X\Delta g(X)]-E[Xg(X)] \\ &= npE[g(X+1)]-pE[X\Delta g(X)] \\ &\quad -E[(X-\mu)g(X)]-\mu E[g(X)] \\ &= npE[\Delta g(X)]-pE[X\Delta g(X)] \\ &\quad -E[(X-\mu)g(X)]+(np-\mu)E[g(X)] \end{aligned}$$

Since $E[w(X)]=1$ and $|\Delta g(x)|$ is bounded, then $E|w(X)\Delta g(X)|<\infty$. Thus, by (2.4), it follows that

$$\begin{aligned} E[(n-X)pg(X+1)-qXg(X)] &= npE[\Delta g(X)]-pE[X\Delta g(X)] \\ &\quad -\sigma^2 E[w(X)\Delta g(X)]+(np-\mu)E[g(X)] \\ &= E\{[(n-X)p-\sigma^2 w(X)]\Delta g(X)\} \\ &\quad +(np-\mu)E[g(X)], \end{aligned}$$

which, by (3.5), yields

$$|p(x_0)-b(x_0;n,p)| = \left|E\left\{[(n-X)p-\sigma^2 w(X)]\Delta g(X)\right\}+(np-\mu)E[g(X)]\right|$$

$$\leq E\{|(n-X)p-\sigma^2 w(X)||\Delta g(X)|\} + |np-\mu|E|g(X)|.$$

Hence, by using Lemma 2.1, the theorem is proved. □

The following corollary is a consequence of Theorem 3.1.

Corollary 3.1. *If $(n-x)p-\sigma^2 w(x) \geq / < 0$ for every $x \in \mathcal{S}(x)$, then*

1. For $x_0 = 0$,

$$|p(0) - b(0;n,p)| \leq \frac{1-q^n}{np}\left\{\left|(n-\mu)p-\sigma^2\right| + |np-\mu|\right\} \tag{3.6}$$

and, if $np = \mu$, then

$$|p(0) - b(0;n,p)| \leq \frac{1-q^n}{np}\left|\mu q-\sigma^2\right|. \tag{3.7}$$

2. For $x_0 \in \{1, ..., n\}$,

$$\begin{aligned}|p(x_0) - b(x_0;n,p)| \leq \min&\left\{\frac{1-p^n}{x_0 q}, \frac{1-p^{n+1}-q^{n+1}}{(n+1)pq}\right\} \\ &\times \left\{\left|(n-\mu)p-\sigma^2\right| + |np-\mu|\right\}\end{aligned} \tag{3.8}$$

and, if $np = \mu$, then

$$|p(x_0) - b(x_0;n,p)| \leq \min\left\{\frac{1-p^n}{x_0 q}, \frac{1-p^{n+1}-q^{n+1}}{(n+1)pq}\right\}\left|\mu q-\sigma^2\right|. \tag{3.9}$$

4. Applications

This section, we apply the result in Theorem 3.1 to approximate some distributions such as hypergeometric, negative hypergeometric and Pólya distributions.

4.1. Application to Hypergeometric Distribution

Suppose a random sample of size n is drawn without replacement from a finite population containing N elements of two types of which m are of type $\mathcal{I}$ and $N-m$ are of type $\mathcal{II}$. Let X be the number of type $\mathcal{I}$ elements in the sample.

Then X has a hypergeometric distribution with parameters N, n and m and has probability function as follows:

$$p(x) = \frac{\binom{m}{x}\binom{N-m}{n-x}}{\binom{N}{n}}, \ x = 0, 1, ..., \min\{n, m\}.$$

Here, its mean and variance are $\mu = \frac{nm}{N}$ and $\sigma^2 = \frac{nm(N-n)(N-m)}{N^2(N-1)}$, respectively. It is well-known that the hypergeometric distribution can be approximated by the binomial distribution. For this application, we give a result of the binomial approximation to the hypergeometric distribution in terms of the point metric $|p(x_0) - b(x_0; n, p)|$, where $x_0 \in \{0, 1, ..., \min\{n, m\}\}$.

Following the relation (2.2), we have $w(x) = \frac{(n-x)(m-x)}{N\sigma^2}$. If $\min\{n, m\} = n$, we put $p = \frac{m}{N}$ in Theorem 3.1, then $(n-x)p - \sigma^2 w(x) = \frac{(n-x)m}{N} - \frac{(n-x)(m-x)}{N} \geq 0$ for all $0 \leq x \leq n$. By applying Corollary 3.1, a result of this approximation can be expressed as the following.

Corollary 4.1. *For* $p = \frac{m}{N}$,

$$|p(x_0) - b(x_0; n, p)| \leq \begin{cases} \frac{(1-q^n)q(n-1)}{N-1} & \text{if } x_0 = 0, \\ \min\left\{\frac{1-p^n}{x_0}, \frac{1-p^{n+1}-q^{n+1}}{(n+1)p}\right\}\frac{(n-1)np}{N-1} & \text{if } 1 \leq x_0 \leq n. \end{cases}$$

Similarly, if $\min\{n, m\} = m$, we replace n and p in Theorem 3.1 by m and $\frac{n}{N}$, respectively, and using Corollary 3.1, we can obtain an another result of this approximation as the following corollary.

Corollary 4.2. *If* $p = \frac{n}{N}$, *then we have*

$$|p(x_0) - b(x_0; m, p)| \leq \begin{cases} \frac{(1-q^m)q(m-1)}{N-1} & \text{if } x_0 = 0, \\ \min\left\{\frac{1-p^m}{x_0}, \frac{1-p^{m+1}-q^{m+1}}{(m+1)p}\right\}\frac{(m-1)mp}{N-1} & \text{if } 1 \leq x_0 \leq m. \end{cases}$$

Remark 4.1. It should be noted that each result in Corollaries 4.1 and 4.2 gives a good binomial approximation if N is large and n and m are small, or $\frac{n}{N}$ and $\frac{m}{N}$ are small.

4.2. Application to Negative Hypergeometric Distribution

Let us consider the process of sampling without replacement as mentioned in previous subsection. If elements in a random sample are drawn without replacement from this population until the number of types $\mathcal{II}$ elements reaches a fixed positive integer r and let X be the number of types $\mathcal{I}$ elements in the sample. Then X has a negative hypergeometric distribution with parameters N, m and r, and its probability function can be expressed as

$$p(x) = \frac{\binom{r+x-1}{x}\binom{N-r-x}{m-x}}{\binom{N}{m}}, \ x = 0, 1, ..., m,$$

where $r \in \{1, ..., N-m\}$ and $\mu = \frac{rm}{N-m+1}$ and $\sigma^2 = \frac{rm(N-m-r+1)(N+1)}{(N-m+1)^2(N-m+2)}$ are the mean and variance of X, respectively. It is observed that, if $N, r \to \infty$ such that $\frac{r}{N-m+1}$ tends to a constant θ, then a negative hypergeometric distribution with parameters N, m and r converges to a binomial distribution with parameters m and θ. Therefore, we can also use the binomial probability function to approximate the negative hypergeometric probability function by using certain conditions of this convergence.

Using the relation (2.2), then $w(x) = \frac{(r+x)(m-x)}{(N-m+1)\sigma^2}$. Thus, replacing n by m and p by $\frac{r}{N-m+1}$ in Theorem 3.1, we have $(m-x)p - \sigma^2 w(x) = \frac{r(m-x)}{N-m+1} - \frac{(r+x)(m-x)}{N-m+1} \leq 0$ for all $0 \leq x \leq m$. By Corollary 3.1, the following corollary is obtained.

Corollary 4.3. *For $r \in \{1, ..., N-m\}$, if $p = \frac{r}{N-m+1}$, then*

$$|p(x_0) - b(x_0; m, p)| \leq \begin{cases} \frac{(1-q^m)q(m-1)}{N-m+2} & \text{if } x_0 = 0, \\ \min\left\{\frac{1-p^m}{x_0}, \frac{1-p^{m+1}-q^{m+1}}{(m+1)p}\right\} \frac{(m-1)mp}{N-m+2} & \text{if } 1 \leq x_0 \leq m. \end{cases}$$

4.3. Application to Pólya Distribution

Suppose that a single urn contain r red and $N-r$ black balls. Draw a ball at random, note the color, and return it into the urn together with an additional ball of the same color. Repeat this way for m draws. Let X be the number of red balls taken out in the m drawings, then the distribution of X is a Pólya distribution with parameters N, m and r. The probability function of X is given by

$$p_(x) = \frac{\binom{r+x-1}{x}\binom{N-r+m-x-1}{m-x}}{\binom{N+m-1}{m}}, \ x = 0, 1, ..., m$$

and the mean and variance of X are $\mu = \frac{rm}{N}$ and $\sigma^2 = \frac{rm(N+m)(N-r)}{N^2(N+1)}$, respectively.

Using the relation (2.2), we then obtain $w(x) = \frac{(r+x)(m-x)}{N\sigma^2}$. Setting $n = m$ and $p = \frac{r}{N}$ in Theorem 3.1, it follows that $(m-x)p - \sigma^2 w(x) = -\frac{(m-x)x}{N} \leq 0$

for all $0 \leq x \leq m$. The following corollary is directly obtained from Corollary 3.1.

Corollary 4.4. *If $p = \frac{r}{N}$, then we have the following.*

$$|p(x_0) - b(x_0; m, p)| \leq \begin{cases} \frac{(1-q^m)q(m-1)}{N+1} & \text{if } x_0 = 0, \\ \min\left\{\frac{1-p^m}{x_0}, \frac{1-p^{m+1}-q^{m+1}}{(m+1)p}\right\}\frac{(m-1)mp}{N+1} & \text{if } 1 \leq x_0 \leq m. \end{cases}$$

Remark 4.2. Each result in Corollaries 4.3 and 4.4 yields a good approximation as N is large and m is small, or $\frac{m}{N}$ is small.

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