International Journal of Pure and Applied Mathematics
Volume 71 No. 2 2011, 207-215

UPPER BOUNDS ON THE RADIO NUMBER OF SOME TREES

P. Venkata Subba Reddy[1], K. Viswanathan Iyer[2 §]
[1,2]Department of Computer Science and Engineering
National Institute of Technology
Tiruchirapalli, 620015, Tamil Nadu, INDIA

Abstract: Let G be a simple,connected and undirected graph with diameter d. For a positive integer k ($\leq d$), a radio k-labeling f of G is an assignment of non-negative integers, called labels to the vertices of G such that if $u, v \in V(G)$ are distinct then $d(u, v) + |f(u) - f(v)| \geq k + 1$ where $d(u, v)$ is the distance between u and v. The maximum label (positive integer) assigned by f to some vertex of G is called the span of f. The radio number of G denoted by $rn(G)$ is the minimum span over all radio d-labelings of G. In this paper, we prove an upper bound for the radio number of binomial tree, Fibonacci trees and uniform caterpillar.

AMS Subject Classification: 05C12, 05C15, 05C78
Key Words: radio labeling, radio number, binomial tree, Fibonacci tree and uniform caterpillar

1. Introduction

Motivated by the channel assignment problem introduced by Hale [4] in wireless networks, radio labeling in graphs has been studied from various perspectives. Let $G = (V(G), E(G))$ be a graph, simple,connected and undirected. Let $diam(G) = d$ and $k \leq d$ be a positive integer. We use standard terms or notations as used in common texts such as [1, 2]. A radio k-labeling f of G is an assignment of non-negative integers, called labels to the vertices of G such that if $u, v \in V(G)$ are distinct then

$$d(u, v) + |f(u) - f(v)| \geq k + 1 \quad (1)$$

where $d(u, v)$ is the distance between u and v. The radio k-labeling number $rc_k(f)$ of a radio k-labeling f of G is the maximum label assigned to a vertex

Received: April 15, 2011
§Correspondence author

of G. The *radio k-chromatic number* $rc_k(f)$ is $min\{rc_k(f)\}$ over all radio k-labelings f of G. A radio k-labeling f of G is a minimum radio k-labeling if $rc_k(f) = rc_k(G)$. Then $rc_1(G) = \chi(G)$. A radio d-labeling of G is a radio labeling of G, and the radio d-chromatic number $rc_d(G)$ is the *radio number* $rn(G)$ and has been studied extensively in the past decade (see [3, 5, 9, 6, 8, 7]).

Kchikech et al. [5] have given an upper bound for the radio number of a tree of order $n \geq 5$ that is neither a star nor a path. As a corollary we get upper bounds for radio number of a binomial tree B_n, a binary Fibonacci tree T_n of order n $(n \geq 0)$ and a uniform caterpillar UC_n as given below:

(i) $rn(B_n) \leq 2(2^n - 1)(n - 1) - 1$.

(ii) $rn(T_n) \leq 2(n - 1)F_{n+3} - (4n - 3)$.

(iii) $rn(UC_n) \leq (N - 1)(n - 2) - 1$, where $N = n + (n - 2)(\Delta - 2)$.

In this paper, we give improved upper bounds for the radio numbers of B_n, T_n and UC_n; specifically, we show that $rn(B_n) \leq 2(2^n - 1)(n - 1) - 1 - 3.2^{n-3}$, $rn(T_n) \leq 2(n - 1)F_{n+3} - 2F_{n+4} + n + 10$, $rn(UC_n) \leq (N - 1)(n - 2) - (n^2 - 4n - 2)/12$, if n is even and $\Delta \leq (n^2 - 4n - 2)/12$ and $rn(UC_n) \leq (N - 1)(n - 2) - (n^2 - 4n - 7)/2$, if $n \geq 7, 9, 11, \cdots$.

2. Upper Bounds for Radio Number

2.1. Binomial Tree

Definition 1. A binomial tree B_n of order n $(n \geq 0)$ is an ordered tree defined recursively as (a) B_0 is a one-vertex graph (b) B_n consists of two copies of B_{n-1} such that the root of one is the leftmost child of the root of the other.

Theorem 2. *Let B_n be a binomial tree of order n with $n \geq 3$. Then* $rn(B_n) \leq 2(2^n - 1)(n - 1) - 1 - 3.2^{n-3}$.

Proof. Let $k = diam(B_n) = 2n - 1$. We provide a radio labeling f of B_n in two steps: first we number the vertices of B_n with labels from $\{1, 2, \dots, 2^n\}$ and then we specify the labels $f(i)$ for all $1 \leq i \leq 2^n$.

We first assign indices to vertices of B_n recursively as follows:

(i) The vertices of B_3 are indexed 1 to 8 from left to right starting with the root.

(ii) Since B_k consists of two copies of B_{k-1}(retaining the indices of B_{k-1}), if a vertex in the subtree rooted at the leftmost child of the root of B_k has index x then it is changed to $2^{k-1} + x$.

Now we define a radio labeling $f\colon V(B_n) \to \{0, 1, 2, \dots, \}$ recursively by

$$f(x) = \begin{cases} 0, & \text{if } x = 1. \\ f(x-1) + 2n - 1, & \text{if } x \equiv 1 \bmod 8. \\ f(x-1) + 2n + 1 - d(x, x-1), & \text{if } x \equiv 0 \bmod 8. \\ f(x-1) + 2n - d(x, x-1), & \text{otherwise.} \end{cases} \tag{2}$$

Clearly for all $1 \le p \le 2^n - 7$ if $p \equiv 1 \bmod 8$ then $G[\{p, \dots, p+7\}] \cong G[\{1, \dots, 8\}]$ and $l = \sum_{i=p}^{p+7} d(i, i+1) = 19$. If $i < j$ then $f(i) < f(j)$ hence $f(1) < f(2) < \dots < f(2^n)$ and so the maximum label used is $f(2^n)$. Since $n \ge 3$ we have

$$\begin{aligned} f(2^n) &= f(2^n - 1) + 2n + 1 - d(2^n, 2^n - 1) \\ &= f(2^n - 2) + 4n + 1 - \sum_{i=2^n-2}^{2^n-1} d(i, i+1) \\ &= f(2^n - 8) + 8.2n - l \\ &= f(8) + (2^{n-3} - 1)(8.2n - l) \\ f(2^n) &= 2(2^n - 1)(n-1) - 1 - 3.2^{n-3}. \end{aligned}$$

Then, for any two integers u and $v, 1 \le u < v \le 2^n$, let us show that the condition 1 is satisfied. For all $1 \le i \le 2^{n-3}$, let $B(i, 3)$ denote $G[\{8i - 7, \dots, 8i\}]$. Clearly $V(B_n) = \bigcap_{i=1}^{2^{n-3}} V(B(i, 3))$. For any two vertices $u, v \in V(B_n)$ such that $u < v$ one of the following is true:

(i)There exists an i such that $u, v \in B(i, 3)$. It is easy to verify that (1) is satisfied.

(ii) There exist two distinct integers i and j such that $u \in B(i, 3)$ and $v \in B(j, 3)$. From (2) we have $|f(u) - f(v)| \ge k \ge k + 1 - d(u, v)$ as $d(u, v) \ge 1$. Hence (1) is satisfied.

Therefore f is a radio labeling of B_n. Hence $rn(B_n) \le 2(2^n - 1)(n-1) - 1 - 3.2^{n-3}$. □

2.2. Binary Fibonacci Tree

Definition 3. A binary Fibonacci tree BFT_n of order n ($n \ge 0$) is a variant of a binary tree. An order 0 binary Fibonacci tree has a single node, and an order 1 tree is P_2. For $n > 1$, BFT_n has a root whose left subtree is a binary Fibonacci tree of order $n - 1$ and whose right subtree is a binary Fibonacci tree of order $n - 2$.

We assume that the vertices of BFT_n are labeled level by level from 1 to $F_{n+3}-1$ sequentially from left to right starting with the root.

Lemma 4. *Let BFT_n be a binary Fibonacci tree of order n with $n \geq 2$. Let $D(n) = \sum_{i=1}^{n'-1} d(i,i+1)$, where $n' = F_{n+3}-1$. Then $D(n) = 2F_{n+5} - 5(n+2)$.*

Proof. From the structure of BFT_n, we can write the recurrence for $D(n)$ as

$$D(n) = D(n-1) + D(n-2) + 5(n-1),$$

with the initial conditions $D(2), D(3)$ being $6, 17$ respectively.

The generating function $G(z)$ for the sequence $D(n)$ is given by

$$G(z) = \frac{1}{1-z-z^2}\Big(6 + 11z + \frac{15z^2}{1-z} + \frac{5z^3}{(1-z)^2}\Big).$$

It then follows that $D(n) = 2F_{n+5} - 5(n+2)$. □

Let T be a tree rooted at a vertex r. A vertex x is called a *descendant* of another vertex y (or y is an *ancestor* of x) if y is on the unique path of T from r to x. Define the *level* of $x \in V(T)$ (w.r.t r) by $L(x) = d(r,x)$. Let $P(x)$ and $a(x,y)$ denote the parent of vertex x and closest common ancestor of both x and y respectively.

Theorem 5. *Let BFT_n be a binary Fibonacci tree of order n with $n \geq 3$ and let n' be the total number of node in it. Then $rn(BFT_n) \leq 2(n-1)F_{n+3} - 2F_{n+4} + n + 10$.*

Proof. Let $k = diam(T_n) = 2n-1$. Define a labeling $f\colon V(T_n) \to \{0,1,2,\ldots,\}$ recursively by

$$f(x) = \begin{cases} 0, & \text{if } x = 1. \\ f(x-1) + 2n - d(x,x-1), & \text{otherwise.} \end{cases} \qquad (3)$$

If $i < j$ then $f(i) < f(j)$ hence $f(1) < f(2) < \ldots < f(n')$ and so the maximum label used is $f(n')$. For all $n \geq 4$, we have

$$\begin{aligned} f(n') &= (n'-1)2n - D(T_n) \\ &= (F_{n+3}-2)2n - 2F_{n+5} + 5(n+2) \\ f(n') &= 2(n-1)F_{n+3} - (2F_{n+4} - n - 10). \end{aligned}$$

Next we show that f is a radio labeling of T_n. Let x and y are two vertices such that $x < y$. If $y - x = 1$ then we see that condition (1) is satisfied. Hence we take $y - x \neq 1$. We have the following cases:

Case 1. $L(x) = L(y)$. There exist two vertices z and $z+1$ such that $x \leq z < z+1 \leq y$ and $L(x) = L(z) = L(z+1)$. Hence $L(a(z, z+1)) \geq L(a(x, y))$, so $d(z, z+1) \leq d(x, y)$. From the labeling f we have

$$\begin{aligned} |f(x) - f(y)| &= \sum_{i=x}^{y-1} 2n - d(i, i+1) \\ &= \sum_{i=x, i\neq z}^{y-1} 2n - d(i, i+1) + 2n - d(z, z+1) \\ &\geq 2n - d(z, z+1) \\ &\geq 2n - d(x, y). \end{aligned}$$

Case 2. $L(y) - L(x) = 1$.

Case 2.1. $d(x, y) = 1$ or $1 \leq L(y) \leq \lfloor n/2 \rfloor$. Since T_n is a complete binary tree upto level $\lfloor n/2 \rfloor$, there exist two vertices x' and $x'+1$ such that $x \leq x' < y$ and $d(x', x'+1) = 2$. Hence $|f(x') - f(x'+1)| \geq 2(n-1)$. From the labeling f it is clear that for all $x', y' \in V(T_n), |f(x') - f(y')| \geq 1$. Since $L(y) - L(x) = 1$ we have $y - x \geq 2$. Therefore we must have $|f(x) - f(y)| \geq 2n - 1$.

Case 2.2. $d(x, y) \neq 1$ and $L(y) > \lfloor n/2 \rfloor + 1$. From the structure of T_n it is clear that similar to case 1 there exist two vertices z and $z+1$ such that $d(z, z+1) \leq d(x, y)$.

Case 2.3. $L(y) - L(x) > 1$. Follows from the case 2.2.

Therefore f is a radio labeling of T_n and $rn(T_n) \leq 2(n-1)F_{n+3} - 2F_{n+4} + n + 10$. □

2.3. Fibonacci Tree

Definition 6. A Fibonacci tree FT_n is defined recursively as follows:

(a) Fibonacci tree of order 0 and 1 is a single node.

(b) FT_n $(n \geq 2)$ is constructed by attaching a Fibonacci tree of order $n-2$ as the leftmost child of the Fibonacci tree of order $n-1$.

We assume that the vertices of FT_n are labeled level by level from 1 to F_{n+2} sequentially from left to right starting with the root.

Theorem 7. *Let FT_n be a Fibonacci tree of order n with $n \geq 3$ and $S = \sum_{i=1}^{F_{n+2}-1} d(i, i+1)$. Then $rn(FT_n) \leq (F_{n+2} - 1)n - S$.*

Proof. Let $k = diam(FT_n) = n - 1$. We assume that the vertices of FT_n are labeled from 1 to F_{n+2} sequentially from left to right starting with the root. Define a labeling $f\colon V(T_n') \to \{0, 1, 2, \ldots, \}$ recursively by

$$f(x) = \begin{cases} 0, & \text{if } x = 1. \\ f(x-1) + n - d(x, x-1), & \text{otherwise.} \end{cases} \tag{4}$$

If $i < j$ then $f(i) < f(j)$ hence $f(1) < f(2) < \ldots < f(F_{n+2})$ and so the maximum label used is $f(F_{n+2})$. For all $n \geq 4$, we have

$$\begin{aligned} f(F_{n+2}) &= (F_{n+2} - 1)n - S \\ &\leq (F_{n+2} - 1)(n-1) - 1. \end{aligned}$$

For any two vertices x and $y, 1 \leq x < y \leq F_{n+2}$, let us show that the condition (1) is satisfied. As the case $y - x = 1$ is obvious, we assume $y - x > 1$. We have the following cases:

Case 1: $y - x > 1$ and $P(x) = P(y)$. From the structure of FT_n it is clear that similar to case 2.2 above there exist two vertices z and $z + 1$ such that $d(z, z+1) \leq d(x, y)$.

Case 2: $y - x > 1$ and $P(x) \neq P(y)$. We have the following subcases:

Case 2.1: $L(x) = L(y)$. Proof is similar to the case 1 of previous theorem.

Case 2.2: $L(y) - L(x) > 1$. There exist two vertices z and $z + 1$ such that $d(z, z+1) = 2$ hence $d(z, z+1) \leq d(x, y)$. Proof is similar to the case 1 of this theorem.

Case 3.3: $L(y) - L(x) = 1$. Either there exist two vertices z and $z+1$ such that $d(z, z+1) \leq d(x, y)$ or $y - x \geq n - 1$ or both. The result is obvious if the former is true. Otherwise there exist at least $n - 1$ number of $i's$ such that for all $x \leq i < y, n - d(i, i+1) \geq 1$. Hence $|f(x) - f(y)| \geq n - 1$ which implies $|f(x) - f(y)| \geq n - d(x, y)$.

Therefore f is a radio labeling of FT_n. Hence $rn(FT_n) \leq (F_{n+2} - 1)n - S$. □

2.4. Uniform Caterpillar

Definition 8. A graph G is called a *caterpillar* if G is a tree such that the removal of the pendent vertices produces a path, the *spine* of the caterpillar. A *uniform caterpillar*, UC_n is a caterpillar with only degree one and degree Δ vertices with $n-2$ vertices on the spine.

Theorem 9. *Let UC_n be a uniform caterpillar so that $n = max\{i : P_i \text{ is a subgraph of } UC_n\}$. Let $N = n + (n-2)(\Delta - 2)$, $l = (n^2 - 4n - 2)/12$ and $p = (n^2 - 4n - 7)/2$. Then*

$$rn(UC_n) \leq \begin{cases} (N-1)(n-2) - l, & \text{if } n \text{ is even and } \Delta \leq l\ . \\ (N-1)(n-2) - p, & \text{if } n \geq 7 \text{ and odd.} \end{cases}$$

Proof. Label the vertices of UC_n as $v_1, \ldots, v_N$ such that $v_1 \ldots v_n \cong P_n$ and for all $i > n, v_i \in N(v_j)$ when $j = \left\lceil \frac{i-n}{\Delta-2} \right\rceil + 1$.

Radio Labeling of UC_n: We provide a radio labeling f of UC_n in two steps. First we define a position function g that renames the vertices of UC_n using the set $\{1, \ldots, N\}$, then we specify the labels $f(x)$ such that $f(x) < f(y)$ iff $x < y$.

Assuming $j = \left\lceil \frac{i-n}{\Delta-2} \right\rceil + 1$ for a given i, we define the position function $g\colon \{v_1, \ldots, v_N\} \to \{1, \ldots, N\}$ as follows:

For $n = 2k$ we define

$$g(v_i) = \begin{cases} N - \Delta + 1, & \text{if } i = 1. \\ 2(\Delta-1)x_i - 3\Delta + 5, & \text{if } 2 \leq i \leq k-1. \\ 1, & \text{if } i = k\ . \\ N - \Delta + 2, & \text{if } i = k+1. \\ 2((\Delta-1)(x_i - k - 2) + \Delta), & \text{if } k+2 \leq i \leq n-1. \\ \Delta, & \text{if } i = n. \\ g(v_j) + i - n - (j-2)(\Delta-2), & \text{if } i > n. \end{cases}$$

For $n = 2k+1$ we define

$$g(v_i) = \begin{cases} 3, & \text{if } i = 1. \\ n-1, & \text{if } i = 2. \\ 2i, & \text{if } 3 \le i \le k-1\ . \\ 1, & \text{if } i = k. \\ 4, & \text{if } i = k+1. \\ n, & \text{if } i = k+2. \\ 2(i-k)+1, & \text{if } k+3 \le i \le n-1. \\ 2, & \text{if } i = n-1. \\ 5, & \text{if } i = n. \\ g(v_j) + i - n - (j-2)(\Delta - 2), & \text{if } i > n. \end{cases}$$

We now define a radio labeling $f\colon \{1, \ldots, N\} \to \{0, 1, \ldots\}$ as

$$f(x) = \begin{cases} 0, & \text{if } x = 1. \\ f(x-1) + n - d(x, x-1), & \text{otherwise.} \end{cases}$$

It is easy to verify that f is a radio labeling of UC_n. Clearly if n is even the maximum label used is

$$f(N) = (N-1)n - D_s^0 \tag{5}$$

where $D_s^0 = \sum_{i=1}^{N-1} d(i, i+1)$, which simplifies to $2\Delta(n-5) + n^2/2 - 4n + 6$. By substituting the value of D_s^0 in Eq.(5), we get $f(N) = (N-1)(n-2) - (n^2 - 4n - 12\Delta)/2$.

If n is odd the maximum label used is

$$f(N) = (N-1)n - D_s^1. \tag{6}$$

where $D_s^1 = \sum_{i=1}^{N-1} d(i, i+1)$, which simplifies to $2n\Delta + n^2/2 - 4n - 4\Delta + 9/2$. By substituting the value of D_s^1 in Eq.(6), we get $f(N) = (N-1)(n-2) - (n^2 - 4n - 7)/2$. □

References

[1] R. Balakrishnan, K. Renganathan, *A Textbook of Graph Theory*, Springer (2000).

[2] G. Chartrand, P. Zhang, *Introduction to Graph Theory*, McGraw-Hill (2006).

[3] J. Georges, D. Mauro, M. Stein, Labeling products of complete graphs with a condition at distance two, *SIAM J. Discrete Math.*, **14** (2001), 28-35.

[4] W.K. Hale, Frequency assignment: Theory and applications, *IEEE Proc.*, **68**, No. 12 (1980).

[5] M. Kchikech, R. Khennoufa, O. Togni, Linear and cyclic radio k-labelings of trees, *Discussiones Mathematicae Graph Theory*, **27**, No. 1 (2007), 105-123.

[6] D.D.F. Liu, M. Xie, Radio number for square cycles, *Cong. Numer.*, **169** (2004), 101-125.

[7] D.D.F. Liu, M. Xie, Radio number for square paths, *Ars Combin.*, **90**, 307-319.

[8] D.D.F. Liu, M. Xie, Radio number for trees, *Discrete Math.*, **308** (2008), 1153-1164.

[9] D.D. Liu, X. Zhu, Multilevel distance labelings for paths and cycles, *SIAM J. Discrete Math.*, **19** (2005), 610-621.