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A COMMON FIXED POINTS THEOREM FOR OCCASIONALLY WEAKLY COMPATIBLE PAIRS ON D*-METRIC SPACES

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Abstract: In this paper we have proved common fixed point theorem for occasionally weakly compatible pairs on D* Matric space.

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1. Introduction

Fuzzy set was defined by Zadeh [15]in 1965. Kramosil and Michalek [13] introduced fuzzy metric space and the concept of fuzzy topological spaces introduced by fuzzy metric which have very important applications in quantum Physics. On the other hand, there have been a number of generalizations of metric spaces. One of such generalization is generalized metric space (or D-Metric space) initiated by B.C. Dhage [1] in 1992. He proved existence of unique fixed point of self-map satisfying contractive condition in complete and bounded D-metric spaces. Dealing with D-metric spaces, Ahmad et al [2], B.C. Dhage (see [1], [3], [4]), Rhoades [5], Singh and Sharma [6], and others made significant contribution in fixed point theory of D-metric space. Unfortunately, almost all theorems in D-metric spaces are not valid (see [7], [8], [9]).

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In this paper, we introduce D*-metric which is a probable modification of definition of D-metric introduced by Dhage (see [1], [3]) and prove some properties in D*-metric spaces.

In what follows (X, D) will denote a D*-metric spaces, N the set of all natural numbers and $\mathbb{R}^+$ the set all the positive real numbers.

2. Preliminary Notes

Definition 2.1. Let X be a non empty set. A generalized D^*-metric on X is a function, $D^* : X^3 \to [0, \infty)$, that satisfies the following conditions for all $x, y, z, a \in X$:

(1) $D * (x; y; z) \geq 0$;

(2) $D^*(x; y; z) = 0$ if and only if $x = y = z$,

(3) $D^*(x; y; z) = D^*(\rho\{x; y; z\})$, (symmetry) where ρ is a permutation function,

(4) $D^*(x; y; z) \leq D^*(x; y; a) + D^*(a; z; z)$.

Then the function D^* is called a generalized D^*-metric and the pair $(X; D^*)$ is called a generalized D^*-metric space

Example 2.2. Let $\{P = (x; y) \in \mathbb{R}^2 : x; y \geq 0\}$, $X = R$ and $D^* : X \times X \times X \to \mathbb{R}^2$ defined by $D^*(x; y; z) = (|x + y| + |y + z| + |x + z|; \alpha(|x + y| + |y + z| + |x + z|))$, where $\alpha \geq 0$ is a constant. Then $(X; D^*)$ is a generalized D^*- metric space.

Proposition 2.3. *If $(X; D^*)$ be generalized D^* metric space, Then for all $x; y; z \in X$, then $D^*(x; x; y) = D^*(x; y; y)$*

Proof. *Let $D^*(x; x; y) \leq D^*(x; x; x) + D^*(x; y; y) = D^*(x; y; y)$ and similarly $D^*(y; y; x) \leq D^*(y; y; y) + D^*(y; x; x) = D^*(y; x; x)$. Hence we have $D^*(x; x; y) = D^*(x; y; y)$.*

Let (X, D^*) be a D^*-metric space. For $r > 0$, define

$$B_{D^*}(x, r) = \{y \in X : D^*(x, y, y) < r\}.$$

Example 2.4. Let $X = \mathbb{R}$. Denote $D^*(x, y, z) = |x - y| + |y - z| + |z - x|$ for all $x, y, z \in \mathbb{R}$. Thus

$$\begin{aligned} B_{D^*}(1, 2) &= \{y \in \mathbb{R} : D^*(1, y, y) < 2\} \\ &= \{y \in \mathbb{R} : |y - 1| + |y - 1| < 2\} \end{aligned}$$

$$= \{y \in \mathbb{R} : |y-1| < 1\}$$
$$= (0, 2).$$

Definition 2.5. Let (X, D^*) be a D^* -metric space and $A \subset X$.

(1)If for every $x \in A$, there exists $r > 0$ such that $B^*_D(x, r) \subset A$, then subset A is called open subset of X.

(2) Subset A of X is said to be D^*-bounded if there exists $r > 0$ such that $D^*(x, y, y) < r$ for all $x, y \in A$.

(3)A sequence $\{x_n\}$ in X converges to x if and only if $D^*(x_n, x_n, x) = D^*(x, x, x_n) \to 0$ as $n \to \infty$. That is, for each $\epsilon > 0$ there exists $n_0 \in N$ such that

$$D^*(x, x, x_n) < \epsilon, \quad \forall n \geq n_0. \tag{$*$}$$

This is equivalent; for each $\epsilon > 0$ there exists $n_0 \in N$ such that

$$D^*(x, x_n, x_m) < \epsilon, \quad \forall n, m \geq n_0. \tag{$**$}$$

then

$$D^*(x, x_n, x_m) = D^*(x_n, x, x_m) < D^*(x_n, x, x_m) + D^*(x, x_n, x_m) < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon,$$

$\forall n, m \geq n_0$. Conversely, set $m = n$ in (**), then we have $D^*(x_n, x_n, x) < \epsilon$.

(4) A sequence $\{x_n\}$ in X is called a Cauchy sequence if for each $\epsilon > 0$, there exists $n_0 \in N$ such that $D^*(x_n, x_n, x_m) < \epsilon$ for each $n, m \geq n_0$. The D^* -metric space (X, D^*) is said to be complete if every Cauchy sequence is convergent.

Let τ be the set of all $A \subset X$ with $x \in A$ if and only if there exists $r > 0$ such that $B_{D^*}(x, r) \subset A$. Then τ is a topology on X (induced by the D^* -metric D^*).

Lemma 2.6. (see [14]) *Let (X, D^*) be a D^* -metric space. If $r > 0$, then ball $B_{D^*}(x, r)$ with center $x \in X$ and radius r is open ball.*

Definition 2.7. Let (X, D^*) be a D^* -metric space. D^* is said to be a continuous function on X^3 if

$$\lim D^*(x_n, y_n, z_n) = D^*(x, y, z)$$

whenever a sequence $\{(x_n, y_n, z_n)\}$ in X^3 converges to a point (x, y, z) in X^3, that is,

$$\lim x_n = x, \lim y_n = y, \lim z_n = z.$$

Lemma 2.8. (see [14]) *Let (X, D^*) be a D^* -metric space. Then D^* is a continuous function on X^3.*

Lemma 2.9. (see [14]) *Let (X, D^*) be a D^* -metric space. If sequence $\{x_n\}$ in X converges to x, then x is unique.*

Lemma 2.10. (see [14]) *Let (X, D^*) be a D^* -metric space. If sequence $\{x_n\}$ in X is convergent to x, then sequence $\{x_n\}$ is a Cauchy sequence.*

Lemma 2.11. (see [14]) *Let (X, D^*) be a D^* -metric space. If sequence $\{x_n\}$ in X is convergent to x, then sequence $\{x_n\}$ is a Cauchy sequence.*

Definition 2.12. Let A and S be two mappings from a D^*-metric space (X, D^*) into itself. Then $\{A, S\}$ is said to be weakly commuting pair if

$$D^*(ASx, SAx, SAx) = D^*(Ax, Sx, Sx),$$

for all $x \in X$.

Definition 2.13. Let X be a set, f, g self maps of X. A point x in X is called a coincidence point of f and g iff $fx = gx$. We shall call $w = fx = gx$ a point of coincidence of f and g.

Definition 2.14. (see [10]) A pair of maps S and T is called weakly compatible pair if they commute at coincidence points.

The concept occasionally weakly compatible is introduced by M. Al-Thagafi and Naseer Shahzad [11]. It is stated as follows.

Definition 2.15. Two self maps f and g of a set X are occasionally weakly compatible (owc) iff there is a point x in X which is a coincidence point of f and g at which f and g commute.

A. Al-Thagafi and Naseer Shahzad [11] shown that occasionally weakly is weakly compatible but converse is not true.

Example 2.16. (see [11]) Let R be the usual metric space. Define $S, T : R \to R$ by $Sx = 2x$ and $Tx = x^2$ for all $x \in R$. Then $Sx = Tx$ for $x = 0, 2$ but $ST0 = TS0$, and $ST2 \neq TS2$. S and T are occasionally weakly compatible self maps but not weakly compatible.

Lemma 2.17. (see [11]) *Let X be a set, f, g owc self maps of X. If f and g have a unique point of coincidence, w = fx = gx, then w is the unique common fixed point of f and g.*

Clearly, a commuting pair is weakly commuting, but not conversely

3. Main Results

A class of implicit relation. Throughout this section (X, D^*) denotes a D^*-metric space and Φ denotes a family of mappings such that each $\phi \in \Phi$, $\phi : (R^+)^5 \to R^+$, and ϕ is continuous and increasing in each coordinate variable. Also $\gamma(t) = \phi(t, t, a_1 t, a_2 t, t) < t$ for every $t \in R^+$ where $a_1 + a_2 = 3$.

Example 3.1. Let $\phi : (R^+)^5 \to R^+$ be defined by

$$\phi(t_1, t_2, t_3, t_4, t_5) < \frac{1}{7}(t_1, t_2, t_3, t_4, t_5).$$

The following lemma is the key in proving our result.

Lemma 3.2. *For every $t > 0$, $\gamma(t) < t$ if and only if $\lim_{n\to\infty} \gamma^n(t) = 0$, where γ^n denotes the composition of γ with itself n times.*

Theorem 3.3. *Let A be a self-mapping of complete D^* -metric space (X, D^*), and let S, T be continuous self-mappings on X satisfying the following conditions:*

(i) $\{A, S\}$ and $\{A, T\}$ are occasionally weakly compatible pairs such that $A(X) \subset S(X) \bigcap T(X)$.

(ii) there exists a $\phi \in \Phi$ such that for all $x, y \in X$,,

$$D^*(Ax, Ay, Az) \leq \phi\{D^*(Sx, Ty, Tz), D^*(Sx, Ax, Ax), D^*(Sx, Ay, Ay), D^*(Ty, Ax, Ax), D^*(Ty, Ay, Ay)\}. \quad (1)$$

Then A, S, and T have a unique common fixed point in X.

Proof. Let $x_0 \in X$ be an arbitrary point in X. Then $Ax_0 \in X$. Since $A(X)$ is contained in $S(X)$, there exists a point $x_1 \in X$ such that $Ax_0 = Sx_1$. Since $A(X)$ is also contained in $T(X)$, we can choose a point $x_2 \in X$ such that $Ax_1 = Tx_2$. Continuing this way, we define by induction a sequence $\{x_n\}$ in X such that

$$Sx_{2n+1} = Ax_{2n} = y_{2n}, \quad n = 0, 1, 2, ..., \quad (2)$$

and

$$Tx_{2n+1} = Ax_{2n+1} = y_{2n+1}, \quad n = 0, 1, 2, ..., \quad (3)$$

For simplicity, we set

$$d_n = D^*(y_n, y_{n+1}, y_{n+1}, \quad n = 0, 1, 2.... \quad (4)$$

We prove that $d_{2n} = d_{2n-1}$. Now, if $d_{2n} > d_{2n-1}$ for some $n \in N$, since ϕ is an increasing function, then

$$\begin{aligned} d_{2n} &= D^*(y_{2n}, y_{2n+1}, y_{2n+1}) \\ &= D(Ax_{2n}, Ax_{2n+1}, Ax_{2n+1}) = D^*(Ax_{2n+1}, Ax_{2n}, Ax_{2n}) \\ &\le \phi\Bigg(\begin{matrix} D^*(Sx_{2n+1}, Tx_{2n}, Tx_{2n}), \\ D^*(Tx_{2n}, Ax_{2n+1}, Ax_{2n+1}) \end{matrix} \\ &\qquad\qquad \begin{matrix} D^*(Sx_{2n+1}, Ax_{2n+1}, Ax_{2n+1}), & D^*(Sx_{2n+1}, Ax_{2n}, Ax_{2n}) \\ D^*(Tx_{2n}, Ax_{2n+1}, Ax_{2n+1}), & D^*(Tx_{2n}, Ax_{2n}, Ax_{2n}) \end{matrix} \Bigg) \\ &= \phi\Bigg(\begin{matrix} D^*(y_{2n}, y_{2n-1}, y_{2n-1}) \\ D^*(y_{2n-1}, y_{2n+1}, y_{2n+1}) \end{matrix} \\ &\qquad\qquad = \begin{matrix} D^*(y_{2n}, y_{2n+1}, y_{2n+1}), & D^*(y_{2n}, y_{2n}, y_{2n}) \\ D^*(y_{2n-1}, y_{2n}, y_{2n}) \end{matrix} \Bigg). \end{aligned} \tag{5}$$

Since

$$\begin{aligned} D^*(y_{2n-1}, y_{2n+1}, y_{2n+1} &= D^*(y_{2n-1}, y_{2n-1}, y_{2n}) + D^*(y_{2n}, y_{2n+1}, y_{2n+1}) \\ &= d_{2n-1} + d_{2n}, \end{aligned} \tag{6}$$

hence by the above inequality we have

$$\begin{aligned} d_{2n} \le \phi(d_{2n-1}, d_{2n}, 0, d_{2n-1} + d_{2n}, d_{2n-1}) &\le \phi(d_{2n}, d_{2n}, d_{2n}, 2d_{2n}, d_{2n}) \\ &< d_{2n}, \end{aligned} \tag{7}$$

a contradiction. Hence $d_{2n} = d_{2n-1}$. Similarly, one can prove that $d_{2n+1} = d_{2n}$ for $n = 0, 1, 2, ...$ Consequently, $\{d_n\}$ is a nonincreasing sequence of nonnegative reals. Now,

$$\begin{aligned} d_1 &= D^*(y_1, y_2, y_2) = D^*(Ax_1, Ax_2, Ax_2), \\ &\le \phi\left(\begin{matrix} D^*(Sx_1, Tx_2, Tx_3), & D^*(Sx_1, Ax_1, Ax_1), & D^*S(x_1, Ax_2, Ax_2), \\ D^*(Tx_2, Ax_1, Ax_1), & D^*(Tx_2, Ax_2, Ax_2) \end{matrix} \right) \\ &= \phi\left(\begin{matrix} D^*(y_0, y_1, y_1), & D^*(y_0, y_1, y_1), & D^*(y_0, y_2, y_2), \\ D^*(y_1, y_1, y_1), & D^*(y_1, y_2, y_2) \end{matrix} \right) \\ &= \phi(d_0, d_0, d_0 + d_1, 0, d_0) \\ &\le \phi(d_0, d_0, 2d_0, d_0, d_0) = \gamma(d_0). \end{aligned} \tag{8}$$

In general, we have $d_n = \gamma^n(d_0)$. So if $d_0 > 0$, then Lemma 3.2 gives $lim_{n\to\infty} d_n = 0$. For $d_0 = 0$, we clearly have $lim_{n\to\infty} d_n = 0$, since then $d_n = 0$ for each n. Now we prove that sequence $\{Ax_n = y_n\}$ is a Cauchy sequence. Since $lim_{n\to\infty} d_n = 0$, it is sufficient to show that the sequence $\{Ax_{2n} = y_{2n}\}$ is a Cauchy sequence. Suppose that $\{Ax_{2n} = y_{2n}\}$ is not a Cauchy sequence. Then there is an $\epsilon > 0$ such that for each even integer $2k$, for $k = 0, 1, 2, ...,$ there exist even integers $2n(k)$ and $2m(k)$ with $2k \leq 2n(k) \leq 2m(k)$ such that

$$D^*(Ax_{2n(k)}, Ax_{2n(k)}, Ax_{2m(k)}) > \epsilon. \tag{9}$$

Let, for each even integer $2k, 2m(k)$ be the least integer exceeding $2n(k)$ satisfying (8). Therefore

$$D^*(Ax_{2n(k)}, Ax_{2n(k)}, Ax_{2m(k)-2}) \leq \epsilon, D^*(Ax_{2n(k)}, Ax_{2n(k)}, Ax_{2m(k)}) > \epsilon. \tag{10}$$

Then, for each even integer $2k$ we have

$$\begin{aligned} &\epsilon < D^*(Ax_{2n(k)}, Ax_{2n(k)}, Ax_{2m(k)}) \\ &\quad = D^*(Ax_{2n(k)}, Ax_{2n(k)}, Ax_{2m(k)-2}) + D^*(Ax_{2m(k)-2}, Ax_{2n(k)-2}, Ax_{2m(k)-1}) \\ &\quad + D^*(Ax_{2m(k)}, Ax_{2m(k)-1}, Ax_{2m(k)}) \\ &\quad = D^*(Ax_{2n(k)}, Ax_{2n(k)}, Ax_{2m(k)-2}) + d_{2m(k)-2} + d_{2m(k)-1}. \end{aligned} \tag{11}$$

So, by (9) and $d_n \to 0$, we obtain

$$\lim_{k\to\infty} D^*(Ax_{2n(k)}, Ax_{2n(k)}, Ax_{2m(k)}) = \epsilon. \tag{12}$$

It follows immediately from the triangular inequality that

$$|D^*(Ax_{2n(k)}, Ax_{2n(k)}, Ax_{2m(k)-1}) - D^*(Ax_{2n(k)}, Ax_{2n(k)}, Ax_{2m(k)})| \leq d_{2m(k)-1},$$

$$\begin{aligned} &|D^*(Ax_{2n(k)+1}, Ax_{2n(k)+1}, Ax_{2m(k)-1}) - D^*(Ax_{2n(k)}, Ax_{2n(k)}, Ax_{2m(k)})| \\ &\qquad < d_{2m(k)-1} + d_{2m(k)}. \end{aligned} \tag{13}$$

Hence by (9),as $k \to \infty$, we obtain

$$\begin{aligned} D^*(Ax_{2n(k)}, Ax_{2n(k)}, Ax_{2m(k)-1}) &\to \epsilon, \\ D^*(Ax_{2n(k)+1}, Ax_{2n(k)+1}, Ax_{2m(k)-1}) &\to \epsilon. \end{aligned} \tag{14}$$

Now

$$\begin{aligned} &D^*(Ax_{2n(k)}, Ax_{2n(k)}, Ax_{2m(k)}) \leq D^*(Ax_{2n(k)}, Ax_{2n(k)}, Ax_{2n(k)+1}) \\ &\quad + D^*(Ax_{2n(k)+1}, Ax_{2n(k)}, Ax_{2m(k)}) \end{aligned}$$

$$\leq d_{2n(k)} + \phi \left(\begin{array}{c} D^*(Ax_{2n(k)}, Ax_{2m(k)-1}, Ax_{2n(k)-1}), \\ D^*(Ax_{2m(k)-1}, Ax_{2n(k)+1}, Ax_{2n(k)+1}), \\ d_{2n(k)}, \quad D^*(Ax_{2n(k)}, Ax_{2m(k)}, Ax_{2m(k)}), \\ d_{2m(k)-1} \end{array} \right). \tag{15}$$

Using (13), $lim_{k\to\infty} d_n = 0$, and continuity and nondecreasing property of ϕ in each coordinate variable, we have

$$\epsilon \leq \phi(\epsilon, 0, \epsilon, \epsilon, 0) + \leq \phi(\epsilon, \epsilon, 2\epsilon, \epsilon, \epsilon) = \gamma(\epsilon) < \epsilon \tag{16}$$

as $k \to \infty$, which is a contradiction. Thus $\{Ax_n = y_n\}$ is a Cauchy sequence and hence by completeness of X, it converges to $z \in X$. That is,

$$lim_{n\to\infty} Ax_n = lim_{n\to\infty} y_n = z \tag{17}$$

Since the sequences $\{Sx_{2n+1} = y_{2n+1}\}$ and $\{Tx_{2n} = y_{2n}\}$ are subsequences of $\{Ax_{2n} = y_{2n}\}$; they have the same limit z. As S and T are continuous, we have $STx_{2n} \to Sz$ and $TSx_{2n} \to Tz$.

Now consider

$$\begin{aligned} D^*(STx_{2n}, TSx_{2n+1}, TSx_{2n+1}) &= D^*(SAx_{2n-1}, TAx_{2n}, TAx_{2n}) \\ &\leq D^*(SAx_{2n-1}, ASx_{2n-1}, ASx_{2n-1}) \\ &+ D^*(ASx_{2n-1}, ASx_{2n-1}, ATx_{2n}) \\ &+ D^*(ATx_{2n}, ATx_{2n}, TAx_{2n}). \end{aligned} \tag{18}$$

Using (ii) and the occasionally weakly compatibility of $\{A, S\}$ and $\{A, T\}$,we get

$$\begin{aligned} &D^*(STx_{2n}, TSx_{2n+1}, TSx_{2n+1}) \\ &= D^*(Sx_{2n-1}, Ax_{2n-1}, Ax_{2n-1}) \\ &\quad + D^*(ASx_{2n-1}, ATx_{2n}, ATx_{2n}) + D^*(Ax_{2n}, Ax_{2n}, Tx_{2n}) \\ &\leq D^*(Sx_{2n-1}, Ax_{2n-1}, Ax_{2n-1}) \\ &+ \phi \left(\begin{array}{cc} D^*(S^2x_{2n-1}, T^2x_{2n}, T^2x_{2n}), & D^*(S^2x_{2n-1}, ASx_{2n-1}, ASx_{2n}), \\ & D^*(S^2x_{2n-1}, ATx_{2n}, ATx_{2n}), \\ D^*(T^2x_{2n}, ASx_{2n-1}, ASx_{2n-1}), & D^*(T^2x_{2n}, ATx_{2n}, ATx_{2n}) \end{array} \right) \\ &+ D^*(Ax_{2n}, Ax_{2n}, Tx_{2n}) \\ &\leq D^*(Sx_{2n-1}, Ax_{2n-1}, Ax_{2n-1}) \end{aligned}$$

$$+\phi\left(\begin{array}{r} D^*(S^2x_{2n-1},T^2x_{2n},T^2x_{2n}), \\ D^*(S^2x_{2n-1},S^2x_{2n-1},SAx_{2n-1})+D^*(Sx_{2n-1},Sx_{2n-1},Ax_{2n-1}), \\ D^*(S^2x_{2n-1},TAx_{2n},TAx_{2n})+D^*(Tx_{2n},Tx_{2n},Ax_{2n}), \\ D^*(T^2x_{2n},SAx_{2n-1},SAx_{2n-1})+D^*(Sx_{2n-1},Sx_{2n-1},Ax_{2n-1}), \\ D^*(T^2x_{2n},TAx_{2n},TAx_{2n})+D^*(Tx_{2n},Ax_{2n},Ax_{2n}) \end{array}\right)$$
$$+D^*(Ax_{2n},Ax_{2n},Tx_{2n}) \tag{19}$$

If $D^*(Sz,Tz,Tz)>0$, then as $n\to\infty$ we have

$$\begin{aligned} &D^*(Sz,Tz,Tz)\le D^*(z,z,z) \\ &+\phi\left(\begin{array}{ccc} D^*(Sz,Tz,Tz), & D^*(Sz,Sz,Sz)+0, & D^*(Sz,Tz,Tz)+0, \\ D^*(Tz,Sz,Sz)+0, & D^*(Tz,Tz,Tz)+0 & \end{array}\right)+0 \\ &\le\gamma(D^*(Sz,Tz,Tz))<D^*(Sz,Tz,Tz), \end{aligned} \tag{20}$$

a contradiction.Therefore, $Sz=Tz$. Now we will prove that $Az=Sz$. To end this, consider the inequality

$$\begin{aligned} D^*(SAx_{2n+1},Az,Az)\le{}& D^*(SAx_{2n+1},ASx_{2n+1},ASx_{2n+1}) \\ &+D^*(Az,Az,ASx_{2n+1}). \end{aligned} \tag{21}$$

Using (ii) and the occasionally weakly compatibility of $\{A,S\}$, we have

$$\begin{aligned} &D^*(SAx_{2n+1},Az,Az)\le D^*(SAx_{2n+1},ASx_{2n+1},ASx_{2n+1}) \\ &\quad+\phi\left(\begin{array}{ccc} D^*(Sz,Tz,TSx_{2n+1}), & D^*(Sz,Az,Az), & D^*(Tz,Az,Az), \\ D^*(Tz,Az,Az), & D^*(Tz,Az,Az) & \end{array}\right), \end{aligned}$$

then as $n\to\infty$, we have

$$\begin{aligned} &D^*(Sz,Az,Az)\le D^*(z,z,z)+ \\ &\quad\phi\left(\begin{array}{ccc} D^*(Sz,Tz,Tz), & D^*(Sz,Az,Az), & D^*(Sz,Az,Az), \\ D^*(Tz,Az,Az), & D^*(Tz,Az,Az) & \end{array}\right) \\ &\quad=\phi(0,D^*(Sz,Az,Az),D^*(Sz,Az,Az),D^*(Sz,Az,Az),D^*(Sz,Az,Az)) \\ &\quad\le\delta(D^*(Sz,Az,Az))<D^*(Sz,Az,Az), \end{aligned} \tag{22}$$

given there by $Sz=Az$. Thus $Az=Sz=Tz$. It now follows that

$$\begin{aligned} &D^*(Az,Ax_{2n},Ax_{2n})\le \\ &\phi\left(\begin{array}{ccc} D^*(Sz,Tx_{2n},Tx_{2n}), & D^*(Sz,Az,Az), & D^*(Sz,Ax_{2n},Ax_{2n}), \\ D^*(Tx_{2n},Az,Az), & D^*(Tx_{2n},Ax_{2n},Ax_{2n}) & \end{array}\right). \end{aligned} \tag{23}$$

Then as $n \to \infty$, we get

$$\begin{aligned} D^*(Az, z, z) &\le \phi(D^*(Sz, z, z), 0, D^*(Sz, z, z), D^*(z, Az, Az), 0) \\ &= \gamma(D^*(Az, z, z) < D^*(Az, z, z) \end{aligned} \tag{24}$$

a contradiction, and therefore $Az = z = Sz = Tz$. Thus z is a common fixed point of A, S, and T. The unicity of the common fixed point is not hard to verify. This completes the proof of the theorem.

Example 3.4. Let (X, D^*) be metric space, where $X = [0, 1]$ and

$$D^*(x, y, z) = |x - y| + |y - z| + |x - z|. \tag{25}$$

Define self mapping A, S and T on X as follows:

$$Sx = x, \;\; Ax = 0, \;\; Tx = \frac{x^2}{2}, \tag{26}$$

for all $x \in X$.

Let

$$\phi(t_1, t_2, t_3, t_4, t_5) = \frac{1}{7}(t_1 + t_2 + t_3 + t_4 + t_5). \tag{27}$$

Then

$$A(X) = 0 \in [0, 1] \cap [0, \frac{1}{2}] = S(X) \cap T(X). \tag{28}$$

for every $x \in X$,we have

$$\begin{aligned} D^*(ATx, TAx, TAx) = D^*(0, 0, 0) = 0 \le D^*(Ax, Tx, Tx), \\ D^*(ASx, SAx, SAx) = D^*(0, 0, 0) = 0 \le D^*(Ax, Sx, Sx). \end{aligned} \tag{29}$$

That is, the pair $(A, S)(A, T)$ are owc.

Also for all

$$\begin{aligned} D^*(Ax, Ay, Az) \le \phi(&D^*(Sx, Ty, Tz), D^*(Sx, Ax, Ax), \\ &D^*(Tx, Ay, Az)), D^*(Tx, Ax, Ax), D^*(Ty, Ay, AY)). \end{aligned} \tag{30}$$

That is, all conditions of Theorem 3.3 hold and 0 is the unique common fixed point of A,S, and T.

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