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ON EQUITABLE COLORING OF COMPLETE r-PARTITE GRAPHS

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Abstract: Let $\chi_=(G)$ denote the equitable chromatic number of a graph G and let $K(m_1, m_2, \ldots, m_r)$ denote a complete r-partite graph with $m_1 \leq m_2 \leq \ldots \leq m_r$. Note that $\chi_=(K(m_1, m_2, \ldots, m_r)) \geq 1 + \sum_{i=2}^{r} \lceil m_i/(m_1+1) \rceil$. In this paper, we investigate the neccessary conditions on the number of vertices such that this bound is attained. By using those conditions, we obtain the algorithm to find the equitable chromatic number of a complete r-partite graph $K(m_1, m_2, \ldots, m_r)$ with complexity $O(rm_1)$.

Moreover for a complete bipartite graph $K(m_1, m_2)$ and a given integer m_1, we find the minimum integer C such that for every integer $m_2 \geq C$ implies $\chi_=(K(m_1, m_2)) = 1 + \lceil m_2/(m_1+1) \rceil$. For a complete r-partite graph $K(m_1, m_2, \ldots, m_r)$ with $m_1 \leq m_2 \leq \ldots \leq m_r$; $r \geq 3$ and a given integer m_1, we find the minimum integer C^* such that for every integer $m_2 \geq C^*$ implies $\chi_=(K(m_1, m_2, \ldots, m_r)) = 1 + \sum_{i=2}^{r} \lceil m_i/(m_1+1) \rceil$.

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1. Introduction

In this paper, all graphs are finite, loopless and without multiple edges. If the set of vertices of a graph G can be partitioned into k classes $V_1, V_2, \ldots, V_k$ such that each V_i is an independent set and the condition $||V_i| - |V_j|| \leq 1$ holds,

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then we say that G is *equitably k-colorable* and $V_1, V_2, \ldots, V_k$ are *equitable color classes.* The smallest integer k for which G is equitably k-colorable is known as the *equitable chromatic number of* G and denoted by $\chi_=(G)$, while the smallest positive integer k such that for any $k' \geq k$ there exists an equitable coloring of a graph G with k' colors is said to be the *equitable chromatic threshold* of G and is denoted by $\chi_=^*(G)$. The notion of equitable colorability was introduced by Meyer [8]. Let $\Delta(G)$ be the largest degree of any vertex in the graph G. Let $\lceil x \rceil$ and $\lfloor x \rfloor$ denote, respectively, the smallest integer not less than x and the largest integer not greater than x.

Given the positive integer $m_1, m_2, \ldots, m_r$, the *complete r-partite graph* $K(m_1, m_2, \ldots, m_r)$ is the graph whose vertex set is the union $P_1 \cup P_2 \cup \cdots \cup P_r$ of r partite sets, with each P_i consisting of m_i vertices, and with two vertices adjacent if and only if they belong to different partite sets. In particular, we call G a *complete bipartite graph* for $r = 2$. In this paper, we let $m_1 \leq m_2 \leq \ldots \leq m_r$.

In 1973, Meyer [8] studied the equitable chromatic number of a tree T and showed that $\chi_=(T) \leq \lceil \Delta(T)/2 \rceil + 1$. But Guy [3] reported the mistake of this proof and Eggleton showed that a tree T can be equitably k-colorable for all $k \geq \lceil \Delta(T)/2 \rceil + 1$, which extends Meyer's result.

The Equitable Coloring Conjecture (ECC) which is stronger than Brooks' Theorem [1] is posed by Meyer [8].

Theorem 1. (Brooks' Theorem, see [1]) *Let G be a connected graph. If G is neither a complete graph nor an odd cycle, then $\chi(G) \leq \Delta(G)$.*

The Equitable Coloring Conjecture (ECC). *Let G be a connected graph. If G is neither a complete graph nor an odd cycle, then*

$$\chi_=(G) \leq \Delta(G).$$

This conjecture has been verified for all graphs with 6 or fewer vertices by Meyer [8].

One well-known result of Hajnal and Szemerédi [4], when rephrased in terms of the equitable colorability, has already shown the following.

Theorem 2. (see[4]) *A graph G (not necessarily connected) is equitably k-colorable if $k \geq \Delta(G) + 1$.*

Later, Kierstead and Kostochka [5] gave a simpler proof of Hajnal and Szemerédi Theorem in the direct form of equitable coloring.

Since every connected bipartite graph G admits a unique bipartition (X, Y) of its vertex set, we can denote G by $G(X, Y)$. In 1992, Bor-Liang Chen and Ko-Wei Lih [2] studied the equitable coloring of trees, have shown the following.

Theorem 3. (see [2]) *Let $G = G(X, Y)$ be a connected bipartite graph. Then $\chi_=(G) = 2$ if and only if $||X| - |Y|| \leq 1$.*

Theorem 4. (see [2]) *Let $G = G(X, Y)$ be a bipartite graph with g vertices. If G has r components and $r \geq g/f$ for some positive integer f, then G is equitably f-colorable.*

In 1992, Ko-Wei Lih and Pou-Lin Wu [7] studied the equitable coloring of bipartite graphs, have shown the following.

Theorem 5. (see [7]) *Let $G = G(X, Y)$ be a connected bipartite graph. If G is different from any complete bipartite graph $K_{n,n}$, then G can be equitably colored with $\Delta(G)$ colors.*

Theorem 6. (see [7]) *The complete bipartite graph $K_{n,n}$ can be equitably colored with k colors if and only if $\lceil n/\lfloor k/2 \rfloor \rceil - \lfloor n/\lceil k/2 \rceil \rfloor \leq 1$.*

Theorem 7. (see [7]) *Let $G(X, Y)$ be a connected bipartite graph with ϵ edges. Suppose $|X| = m \geq n = |Y|$ and $\epsilon < \lfloor m/(n+1) \rfloor (m - n) + 2m$. Then $\chi_=(G) \leq \lceil m/(n+1) \rceil + 1$.*

In 2000, Lam et al. [6] derived an explicit formula for the equitable chromatic number of complete r-partite graphs.

Theorem 8. (see [6]) *Let $G := K(m_1, m_2, \ldots, m_r)$ be a complete r-partite graph. Suppose M is the largest integer such that*

$$m_i (mod\ M) < \lceil m_i/M \rceil \quad for \quad i = 1, 2, \ldots, r.$$

Then $\chi_=(G) = \sum_{i=1}^{r} \lceil m_i/(M+1) \rceil$.

In Section 2, we find the necessary condition on (m_1, m_2) with $m_1 \leq m_2$ such that $\chi_=(K(m_1, m_2)) = 1 + \lceil m_2/(m_1+1) \rceil$ and find the minimum integer C such that for every integer $m_2 \geq C$ implies $\chi_=(K(m_1, m_2)) = 1 + \lceil m_2/(m_1+1) \rceil$.

In Section 3, we find the necessary condition on $(m_1, m_2, \ldots, m_r)$ such that $\chi_=(K(m_1, m_2, \ldots, m_r)) = 1 + \sum_{i=2}^{r} \lceil m_i/(m_1+1) \rceil$ and find the minimum integer C^* such that for every integer $m_2 \geq C^*$ implies $\chi_=(K(m_1, m_2, \ldots, m_r)) = 1 + \sum_{i=2}^{r} \lceil m_i/(m_1+1) \rceil$.

In section 4, we give the algorithm which find the equitable chromatic number of a complete r-partite graph $K(m_1, m_2, \ldots, m_r)$.

2. Complete Bipartite Graphs

In this section, we study the complete bipartite graph $K(m_1, m_2)$ whose vertex set is the union of the two partite sets X and Y with $|X| = m_1 \leq m_2 = |Y|$.

Lemma 9. *Let $K(m_1, m_2)$ be a complete bipartite graph. Then*

$$\chi_=(K(m_1, m_2)) \geq 1 + \lceil m_2/(m_1 + 1) \rceil.$$

Proof. Let $K(m_1, m_2)$ be equitably k-colorable and $V_1, V_2, \ldots, V_k$ be equitable color classes of $K(m_1, m_2)$. WLOG, $V_1 \subseteq X$. Then $|V_1| \leq m_1$. Since $m_1 + m_2 = |V(K(m_1, m_2))| = |V_1| + |V_2| + \cdots + |V_k|$, we have $m_1 + m_2 \leq m_1 + (k-1)(m_1+1)$. This implies $k \geq 1 + \lceil m_2/(m_1+1) \rceil$. Then $\chi_=(K(m_1, m_2)) \geq 1 + \lceil m_2/(m_1 + 1) \rceil$. □

Lemma 10. *Let $K(m_1, m_2)$ be a complete bipartite graph. If $V_1, V_2, \ldots, V_{1+\lceil m_2/(m_1+1) \rceil}$ are equitable color classes of $K(m_1, m_2)$, then $X = V_i$ for some $i \in \{1, 2, \ldots, 1 + \lceil m_2/(m_1 + 1) \rceil\}$.*

Proof. Suppose $V_1 \cup V_2 \subseteq X$. So $|V_i| \leq (m_1/2) + 1 \leq m_1$ for all i. Then $|\{V_i : V_i \subseteq Y\}| \geq \lceil m_2/m_1 \rceil$. So we have $|\{V_i : V_i \subseteq V(G)\}| \geq 2 + \lceil m_2/m_1 \rceil > 1 + \lceil m_2/(m_1 + 1) \rceil$ which is a contradiction. This completes the proof. □

By division algorithm, $m_2 = q(m_1 + 1) + t$ where $0 \leq t \leq m_1$ and $q \in \mathbb{N} \cup \{0\}$. We characterize $K(m_1, m_2)$ with $\chi_=(K(m_1, m_2)) = 1 + \lceil m_2/(m_1 + 1) \rceil$ in Theorem 13.

Lemma 11. *Let $K(m_1, m_2)$ be a complete bipartite graph. $K(m_1, m_2)$ has $1 + \lceil m_2/(m_1 + 1) \rceil$ equitable color classes of size m_1 or $m_1 + 1$ if and only if $t = 0$ or $m_1 - q \leq t \leq m_1$.*

Proof. Let $V_1, V_2, \ldots, V_{1+\lceil m_2/(m_1+1) \rceil}$ be equitable color classes of $K(m_1, m_2)$ with $|V_i| = m_1$ or $m_1 + 1$. Let a and b be numbers of equitable color classes that are subsets of Y of size m_1 and $m_1 + 1$, respectively. Then we can write $m_2 = am_1 + b(m_1 + 1) = (a + b - 1)(m_1 + 1) + (m_1 + 1 - a)$.

If $a = 0$, then m_2 is divisible by $m_1 + 1$. This implies $t = 0$.

If $a > 0$, then we have $q = a + b - 1$ and $t = m_1 + 1 - a$ which implies $m_1 - q = m_1 - a - b + 1 \leq m_1 - a + 1 = t \leq m_1$.

Conversely, assume that $t = 0$ or $m_1 - q \leq t \leq m_1$.

If $t = 0$, we partition Y into q classes of size $m_1 + 1$.

If $t = m - q$, we partition Y into $q + 1$ classes of size m_1.

If $m_1 - q + 1 \leq t \leq m_1$, we partition Y into classes of size

$$\lceil m_2/(q + 1) \rceil, \lceil (m_2 - 1)/(q + 1) \rceil, \ldots, \lceil (m_2 - q)/(q + 1) \rceil.$$

It is easy to see that

$$\lceil m_2/(q + 1) \rceil + \lceil (m_2 - 1)/(q + 1) \rceil + \cdots + \lceil (m_2 - q)/(q + 1) \rceil = m_2$$

and $\lceil m_2/(q+1)\rceil = m_1+1$. This partition together with X forms a $(1+\lceil m_2/(m_1+1)\rceil)$-equitable coloring of $K(m_1,m_2)$ with color classes of size m_1 or m_1+1. $\square$

Lemma 12. *Let $K(m_1,m_2)$ be a complete bipartite graph. $K(m_1,m_2)$ has $1+\lceil m_2/(m_1+1)\rceil$ equitable color classes of size m_1-1 or m_1 if and only if $m_1-2q-1\le t\le m_1-q$.*

Proof. Let $V_1,V_2,\ldots,V_{1+\lceil m_2/(m_1+1)\rceil}$ be equitable color classes of $K(m_1,m_2)$ with $|V_i|=m_1-1$ or m_1. Let a and b be numbers of equitable color classes that are subsets of Y of size m_1-1 and m_1, respectively. Then we can write $m_2=a(m_1-1)+bm_1\,; a,b\ge 0,\ a+b\ge 1$. Note that $a+b=\lceil m_2/(m_1+1)\rceil$.

If $a+b=q$, then $m_2=(a+b)(m_1+1)-2a-b$. We obtain $t=-2a-b=0$ which is a contradiction.

If $a+b=q+1$, then $m_2=(a+b-1)(m_1+1)+(m_1+1-2a-b)$. We obtain $t=m_1+1-2a-b$ which implies $m_1-2q-1\le t\le m_1-q$.

Conversely, assume that $m_1-2q-1\le t\le m_1-q$.

If $t=m-2q-1$, we partition Y into $q+1$ classes of size m_1-1.

If $m_1-2q\le t\le m_1-q$, we partition Y into classes of size

$$\lceil m_2/(q+1)\rceil, \lceil (m_2-1)/(q+1)\rceil, \ldots, \lceil (m_2-q)/(q+1)\rceil.$$

It is easy to see that

$$\lceil m_2/(q+1)\rceil + \lceil (m_2-1)/(q+1)\rceil + \cdots + \lceil (m_2-q)/(q+1)\rceil = m_2$$

and $\lceil m_2/(q+1)\rceil = m_1$. This partition together with X forms a $(1+\lceil m_2/(m_1+1)\rceil)$-equitable coloring of $K(m_1,m_2)$ with color classes of size m_1-1 or m_1. $\square$

Theorem 13. *Let $K(m_1,m_2)$ be a complete bipartite graph.*

$\chi_=(K(m_1,m_2))=1+\lceil m_2/(m_1+1)\rceil$ if and only if $t=0$ or $m_1-2q-1\le t\le m_1$.

Proof. It is a direct consequence of Lemmas 11 and 12. $\square$

Corollary 14. *Let $K(m_1,m_2)$ be a complete bipartite graph with $m_1\le m_2$. Given an integer m_1, there exists the smallest integer C such that for every integer $m_2\ge C$ implies $\chi_=(K(m_1,m_2))=1+\lceil m_2/(m_1+1)\rceil$. Then $C=(m_1-1)(\lceil m_1/2\rceil-1)$.*

Proof. By Theorem 13, C is the smallest integer such that $\chi_{=}(K(m_1, m_2)) = 1 + \lceil m_2/(m_1+1)\rceil$ for all $|Y| = m_2 \geq C$ if (i) $|Y| = m_2 \geq C$ implies $t = 0$ or $m_1 - 2q - 1 \leq t \leq m_1$ and (ii) $|Y| = m_2 = C - 1$ implies $1 \leq t \leq m_1 - 2q - 2$.

We consider the following 2 cases:

Case 1: m_1 is odd.

Then $(m_1 - 1)(\lceil m_1/2\rceil - 1) = ((m_1 - 3)/2)(m_1 + 1) + 2$. To show (i), we let $m_2 \geq ((m_1 - 3)/2)(m_1 + 1) + 2$.

If $q = (m_1 - 3)/2$ and $2 \leq t \leq m_1$, then we have $m_1 - 2q - 1 = m_1 - 2((m_1 - 3)/2) - 1 = m_1 - (m_1 - 3) - 1 = 2$. This implies $t \geq 2 = m_1 - 2q - 1$.

If $q \geq ((m_1 - 3)/2) + 1$ and $0 \leq t \leq m_1$, then we have $m_1 - 2q - 1 \leq m_1 - 2((m_1 - 3)/2 + 1) - 1 = m_1 - (m_1 - 3 + 2) - 1 = 0$. But $t \geq 0$, so $t \geq m_1 - 2q - 1$.

Consequently, (i) holds.

To show (ii), we let $m_2 = ((m_1 - 3)/2)(m_1 + 1) + 2 - 1 = ((m_1 - 3)/2)(m_1 + 1) + 1$. We have $q = (m_1 - 3)/2$ and $t = 1$ such that $m_1 - 2q - 2 = m_1 - 2((m_1 - 3)/2) - 2 = 1$. This implies $1 \leq t = m_1 - 2q - 2$.

Case 2: m_1 is even.

Then $(m_1 - 1)(\lceil m_1/2\rceil - 1) = ((m_1 - 4)/2)(m_1 + 1) + 3$.

To show (i), we let $m_2 \geq ((m_1 - 4)/2)(m_1 + 1) + 3$.

If $q = (m_1 - 4)/2$ and $3 \leq t \leq m_1$, then we have $m_1 - 2q - 1 = m_1 - 2((m_1 - 4)/2) - 1 = m_1 - (m_1 - 4) - 1 = 3$. This implies $t \geq 3 = m_1 - 2q - 1$.

If $q \geq ((m_1 - 4)/2) + 1$ and $0 \leq t \leq m_1$, then we have $m_1 - 2q - 1 \leq m_1 - 2((m_1 - 4)/2 + 1) - 1 = m_1 - (m_1 - 4 + 2) - 1 = 1$. This implies $t = 0$ or $t \geq 1 \geq m_1 - 2q - 1$.

Consequently, (i) holds.

To show (ii), we let $m_2 = ((m_1 - 4)/2)(m_1 + 1) + 3 - 1 = ((m_1 - 4)/2)(m_1 + 1) + 2$. We have $q = (m_1 - 4)/2$ and $t = 2$ such that $m_1 - 2q - 2 = m_1 - 2((m_1 - 4)/2) - 2 = 2$. This implies $1 \leq t = m_1 - 2q - 2$.

Therefore, if C is the smallest integer such that $\chi_{=}(K(m_1, m_2)) = 1 + \lceil m_2/(m_1 + 1)\rceil$ for every integer $m_2 \geq C$, then $C = (m_1 - 1)(\lceil m_1/2\rceil - 1)$. $\square$

3. Complete r-Partite Graphs

In this section, we study the complete r-partite graph $K(m_1, m_2, \ldots, m_r)$ with $m_1 \leq m_2 \leq \ldots \leq m_r$.

Put $G = K(m_1, m_2, \ldots, m_r)$, $M = 1 + \sum_{i=2}^{r}\lceil m_i/(m_1 + 1)\rceil$ and will be use from now on. By division algorithm, for each $i \in \{1, 2, \ldots, r\}$ we write $m_i = q_i(m_1 + 1) + t_i$ with $0 \leq t_i \leq m_1$ and $q_i \in \mathbb{N} \cup \{0\}$.

Lemma 15. *Let G be a complete r-partite graph. Then $\chi_=(G) \geq M$.*

Proof. Let G be equitably k-colorable and $V_1, V_2, \ldots, V_k$ be equitable color classes of G. Since G is a complete r-partite graph, for each $j = 1, 2, \ldots, k$, $V_j \subseteq P_i$ for some $i \in \{1, 2, \ldots r\}$. WLOG, $V_1 \subseteq P_1$. Then $|V_1| \leq m_1$. By Lemma 9, the number of equitable color classes that are subsets of P_i is at least $\lceil m_i/(m_1+1)\rceil$ for all $i = 2, 3, \ldots, r$. This implies $k \geq 1 + \sum_{i=2}^{r}\lceil m_i/(m_1+1)\rceil = M$. Thus $\chi_=(G) \geq M$. □

Lemma 16. *Let G be a complete r-partite graph. If $V_1, V_2, \ldots, V_M$ are equitable color classes of G, then $P_1 = V_i$ for some $i \in \{1, 2, \ldots, M\}$.*

Proof. Let $V_1, V_2, \ldots, V_M$ be equitable color classes of G. Suppose $V_1 \cup V_2 \subseteq P_1$. Then $|V_1| \leq m_1 - 1$ and $|V_2| \leq m_1 - 1$. Therefore $M \geq 2 + \sum_{i=2}^{r}\lceil m_i/((m_1 - 1) + 1)\rceil = 2 + \sum_{i=2}^{r}\lceil m_i/m_1\rceil > 1 + \sum_{i=2}^{r}\lceil m_i/(m_1+1)\rceil = M$ which is a contradiction. This yields $P_1 = V_i$ for some $i \in \{1, 2, \ldots, M\}$. □

Lemma 17. *Let G be a complete r-partite graph. If $V_1, V_2, \ldots, V_M$ are equitable color classes of G, then there are exactly $\lceil m_i/(m_1+1)\rceil$ color classes that are subsets of P_i for each $i \in \{2, 3, \ldots, r\}$.*

Proof. Let $V_1, V_2, \ldots, V_M$ be equitable color classes of G. Suppose there are exactly k color classes that are subsets of P_i with $k < \lceil m_i/(m_1+1)\rceil$. By Lemma 16, $\chi_=(K(m_1, m_i)) = 1 + k < 1 + \lceil m_i/(m_1+1)\rceil$ which contradicts to Lemma 9. □

Lemma 18. *Let G be a complete r-partite graph. G has M equitable color classes of size m_1 or $m_1 + 1$ if and only if $t_i = 0$ or $m_1 - q_i \leq t_i \leq m_1$ for all $i = 2, 3, \ldots, r$.*

Proof. Suppose G has M equitable color classes. By Lemma 17, there are exactly $\lceil m_i/(m_1+1)\rceil$ color classes that are subsets of P_i for each $i \in \{2, 3, \ldots, r\}$. The remainder of the proof is similar to one in Lemma 11. □

Lemma 19. *Let G be a complete r-partite graph. G has M equitable color classes of size $m_1 - 1$ or m_1 if and only if $m_1 - 2q_i - 1 \leq t_i \leq m_1 - q_i$ for all $i = 2, 3, \ldots, r$.*

Proof. Suppose G has M equitable color classes. By Lemma 17, there are exactly $\lceil m_i/(m_1+1)\rceil$ color classes that are subsets of P_i for each $i \in \{2, 3, \ldots, r\}$. The remainder of the proof is similar to one in Lemma 12. □

Theorem 20. *Let G be a complete r-partite graph. Then $\chi_=(G) = M$ if and only if one of the following holds*
(*i*) $t_i = 0$ *or* $m_1 - q_i \le t_i \le m_1$ *for all* $i = 2, 3, \ldots, r$;
(*ii*) $m_1 - 2q_i - 1 \le t_i \le m_1 - q_i$ *for all* $i = 2, 3, \ldots, r$.

Proof. Assume that $\chi_=(G) = M$. Let $V_1, V_2, \ldots, V_M$ be equitable color classes of G. By Lemma 16, $P_1 = V_j$ for some $j = 1, 2, \ldots, M$. Since $|P_1| = m_1$, we consider the following 2 cases:

Case 1. $|V_i| \in \{m_1, m_1+1\}$ for all $i = 1, 2, \ldots, M$. By Lemma 18, we obtain (i).

Case 2. $|V_i| \in \{m_1-1, m_1\}$ for all $i = 1, 2, \ldots, M$. By Lemma 19, we obtain (ii).

The converse is obtained by Lemma 18 and Lemma 19. □

Theorem 21. *Let $K(m_1, m_2, \ldots, m_r)$ be a complete r-partite graph with $m_1 \le m_2, \ldots \le m_r$. Given an integer m_1, there exists the smallest integer C^* such that $m_2 \ge C^*$ implies $\chi_=(K(m_1, m_2, \ldots, m_r)) = M$. Then:*
(*i*) $C^* = 1$ *for* $m_1 = 1$,
(*ii*) $C^* = 2$ *for* $m_1 = 2$, *and*
(*iii*) $C^* = m_1(m_1 - 1)$ *for* $m_1 \ge 3$.

Proof. First, we consider the case $m_1 = 1$. Let $m_2 \ge 1$.

Then $m_i = q_i(m_1 + 1) + t_i = 2q_i + t_i \;\; ; 0 \le t_i \le 1$ for all $i = 2, 3, \ldots, r$. If $t_i \ne 0$, then $m_1 - q_i \le 1 = t_i \le m_1$. Lemma 18 implies G has M equitable color classes of size m_1 or $m_1 + 1$. Thus $C^* = 1$.

Next, we consider the case $m_1 = 2$. Let $m_2 \ge 2$.

Then $m_i = q_i(m_1 + 1) + t_i = 3q_i + t_i \;\; ; 0 \le t_i \le 2$ for all $i = 2, 3, \ldots, r$.

If $m_i = 2$, then $m_i = 0(m_1 + 1) + 2$. That is $q_i = 0$ and $t_i = 2$. Hence $m_1 - q_i = 2 - 0 = 2 = t_i$.

If $m_i \ge 3$, then $q_i \ge 1$. That is $t_i = 0$ or $m_1 - q_i \le m_1 - 1 = 1 \le t_i \le m_1$.

Lemma 18 implies G has M equitable color classes of size m_1 or $m_1 + 1$. Thus $C^* = 2$.

Finally, we consider the case $m_1 \ge 3$. Assume $m_2 \ge m_1(m_1 - 1)$. Similar to the proof of Corollary 14, G satisfies $t_i = 0$ or $m_1 - q_i \le t_i \le m_1$ for all $i = 2, 3, \ldots, r$. Lemma 18 implies G has M-equitable color classes of size m_1 or $m_1 + 1$. Thus $C^* \le m_1(m_1 - 1)$.

Suppose $C^* < m_1(m_1 - 1)$. Let $m_2 = m_1(m_1 - 1) - 1$ and $m_3 = (m_1 + 1)^2$. By assumption of C^*, G has M-equitable color classes. Lemma 17 implies there are $\lceil (m_1(m_1 - 1) - 1)/(m_1 + 1) \rceil = m_1 - 1$ and $\lceil (m_1 + 1)^2/(m_1 + 1) \rceil = m_1 + 1$

color classes that are subsets of P_2 and P_3, respectively. Since $m_2 = m_1(m_1 - 1) - 1 = (m_1 - 2)(m_1 + 1) + 1$, we have $q_i = m_1 - 2$ and $t_i = 1$ such that $m_1 - 2q_i - 1 \leq t_i \leq m_1 - q_i$. Lemma 19 implies G has M-equitable color classes of size $m_1 - 1$ or m_1. But m_2 is not divisible by m_1, so there is a color class of size $m_1 - 1$ that is a subset of P_2. Since there are exactly $m_1 + 1$ color classes that are subsets of P_3, a size of each color class is $\lceil (m_1 + 1)^2/(m_1 + 1) \rceil = m_1 + 1$, which is a contradiction. Hence $C^* = m_1(m_1 - 1)$ for $m_1 \geq 3$. □

Observation. *Let $K(m_1, m_2, \ldots, m_r)$ be a complete r-partite graphwith $m_1 \leq m_2, \ldots \leq m_r$. If $m_2 \geq C^*$ where C^* is defined in Theorem 21, then each equitable M-coloring has color classes of size m_1 or $m_1 + 1$.*

4. Algorithm to Find the Equitable Chromatic Number of $K(m_1, m_2, \ldots, m_r)$

In this section, we give the algorithm to find the equitable chromatic number of a complete r-partite graph $K(m_1, m_2, \ldots, m_r)$ with $m_1 \leq m_2 \leq \ldots \leq m_r$.

Let $M(m_1, m_2, \ldots, m_r) := \sum_{i=1}^{r} \lceil m_i/(m_1 + 1) \rceil$. Lemmas 3.4 and 3.5 leads directly to an algorithm which finds the equitable chromatic number of G.

Since the fewest possible equitable chromatic number is $M_1 = M(m_1, m_2, \ldots, m_r)$, we begin the algorithm by checking whether G satisfies conditions in Lemma 3.4 or 3.5 to have $\chi_=(G) = M_1$. If it is not the case, then the second fewest possible equitable chromatic number, says M_2, has the corresponding coloring partition P_1 into two color classes of sizes $\lfloor m_1/2 \rfloor$ and $\lceil m_1/2 \rceil$. In case that $\lfloor m_1/2 \rfloor = \lceil m_1/2 \rceil$, we have that G has equitable coloring with at least one color class of size $\lfloor m_1/2 \rfloor$ if and only if $K(\lfloor m_1/2 \rfloor, m_2, \ldots, m_r)$ satisfies conditions in Lemma 3.4 or 3.5. In this case $\chi_=(G) = 1 + M(\lfloor m_1/2 \rfloor, m_2, \ldots, m_r)$. In case that $\lfloor m_1/2 \rfloor \neq \lceil m_1/2 \rceil$, we have that G has equitable coloring with color classes of sizes $\lfloor m_1/2 \rfloor$ and $\lceil m_1/2 \rceil$ if and only if $K(\lfloor m_1/2 \rfloor, m_2, \ldots, m_r)$ satisfies conditions in Lemma 3.4. In this case $\chi_=(G) = 1 + M(\lfloor m_1/2 \rfloor, m_2, \ldots, m_r)$.

We continue this process until we find $K(\lfloor m_1/k \rfloor, m_2, \ldots, m_r)$ satisfies conditions in Lemma 3.4 or 3.5. if m_1 is divisible by k or $K(\lfloor m_1/k \rfloor, m_2, \ldots, m_r)$ satisfies conditions in Lemma 3.4 if m_1 is not divisible by k. Reaching the desired k, we have $\chi_=(K(m_1, m_2, \ldots, m_r)) = k - 1 + M(\lfloor m_1/k \rfloor, m_2, \ldots, m_r)$. From this process, we obtain the following theorem.

Theorem 22. *Let $K(m_1, m_2, \ldots, m_r)$ be a complete r-partite graph with $m_1 \leq m_2 \leq \ldots \leq m_r$. If $l :=$ minimum positive integer such that for $k := \lfloor m_1/l \rfloor$ where $h := m_1 - kl$, $q_i := \lfloor m_i/(k + 1) \rfloor$ where $t_i := m_i - q_i(k + 1)$ and* *one of the following conditions holds:*

(i) $h = 0$ *and* $t_i = 0$ *or* $k - q_i \leq t_i \leq k$

(ii) $h = 0$ *and* $k - 2q_i - 1 \leq t_i \leq k - q_i$

(iii) $h \neq 0$ *and* $t_i = 0$ *or* $k - q_i \leq t_i \leq k$ *for all* $i = 1, 2, \ldots, r$, *then* $\chi_=(K(m_1, m_2, \ldots, m_r)) = l + \sum_{i=2}^{r} \lceil m_i/(k+1) \rceil$.

Algorithm Find the equitable chromatic number of $K(m_1, m_2, \ldots, m_r)$.

begin

$N := \min\{m_1, m_2, \ldots, m_r\}$

$l := 0$

success1 := 0

success2 := 0

success3 := 0

while ($success1 + sucess2 < r$ and $success2 + success3 < r$)

$l := l + 1$

$k := \lfloor N/l \rfloor$

$h := N - kl$

if $h = 0$

for $i := 1$ to r

$q_i := \lfloor m_i/(k+1) \rfloor$

$t_i := m_i - q_i(k+1)$

if $t_i = 0$ success1 := success1+1

if $k - q_i < t_i \leq k$ success1 := success1+1

if $t_i = k - q_i$ success2 := success2+1

if $k - 2q_i - 1 \leq t_i < k - q_i$ success3 := success3+1

end (for)

else

for $i := 1$ to r

$q_i := \lfloor m_i/(k+1) \rfloor$

$t_i := m_i - q_i(k+1)$

if $t_i = 0$ success1 := success1+1

if $k - q_i \leq t_i \leq k$ success1 := success1+1

if $t_i = k - q_i$ success2 := success2+1

end (for)

end (while)

$M := 0$

for $i := 1$ to r

$a_i := \lceil m_i/(k+1) \rceil$

$M := M + a_i$

The equitable chromatic number of $K(m_1, m_2, \ldots, m_r)$ is M.

end (Algorithm).

Notice that the algorithm in the worst case for the while loop terminates when k reaches $\lfloor m_1/2 \rfloor + 1$ which makes $\lfloor m_1/k \rfloor = 1$. Since the complexity of the second for loop is $O(r)$, the overall complexity of the algorithm is $O(rm_1)$.

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References

[1] R.L. Brooks, On colouring the node of a network, *Proc. Cambridge Philos. Soc.*, **37** (1941), 194-197.

[2] B.-L. Chen, K.-W. Lih, Equitable coloring of trees, *J. Combin. Theory Ser B*, **61** (1994), 83-87.

[3] R.K. Guy, Monthly research problems, 1969-1975, *Amer. Math. Monthly*, **82** (1975), 995-1004.

[4] A. Hajnal, E. Szemerédi, Proof of a conjecture of Erdős, In: *Combinatorial Theory and Its Applications* (Ed-s: A. Rényi, V.T. Sós), **II**, Colloq. Math. Soc. János Bolyai 4, North-Holland, Amsterdam (1970), 601-623.

[5] H.A. Kierstead, A.V. Kostochka, A short proof of the Hajnal-Szemerédi Theorem on equitable colouring, *Combin. Probab. Comput.*, **17** (2008), 265-270.

[6] P.C.B. Lam et al, On the equitable chromatic number of complete n-partite graphs, *Discrete Applied Mathematics,* **113** (2001), 307-310.

[7] K.-W. Lih, P.-L. Wu, On equitable coloring of bipartite graphs, *Discrete Math.,* **151** (1996), 155-160.

[8] W. Meyer, Equitable coloring, *Amer. Math. Monthly,* **80** (1973), 920-922.