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DYNAMICAL BEHAVIOR OF A LESLIE MODEL WITH PEST CONTROL

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Abstract: In the paper we propose a Leslie model with impulsive state feedback control on the basis of the theory of economic threshold. Using the existence criteria of periodic solution of the general impulsive autonomous system, we obtain that the system with impulsive state feedback control has periodic solution of order one. Sufficient conditions for stability of periodic solution of order one are given by using stability criteria of periodic solution of the general impulsive autonomous system. Our results are confirmed by numerical simulations.

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Key Words: Leslie model, state feedback control, globally asymptotical stability, periodic solution

1. Introduction

People commonly apply the theory of economic threshold to control pest in modern. The so-called economic threshold indicates that the pest population mounts up to the density so that people take preventive measures to achieve economical benefit. Obviously, the economic threshold depends on the status of victims of crops, the type of control measures taken, the amount of pest populations and their relationship with the crop, and many other factors. The superiority of applying the economic threshold is to be able to better carry on the pest control measures and fine adjustment according to the situation of harmful insect's density and their harm. Prevention activities will be more

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effective and more close to the purpose of making the maximum gains. If it must use the pesticide in the prevention, then the amount of pesticide spraying according to the economic threshold standard is less than the amount of pesticide spraying of the general regular preventive. Therefore, it will certainly reduce pollution levels for environment. Some scholars have estimated that if people only spray pesticides in the real need time in the United States, then the consumption of pesticides can be less than 50 percent of the general regular preventive spraying. Based on the theory of economic threshold, we can come off the purpose of pest control by impulsive feedback control. As for some models with impulsive effect, Bainov [1], Laksmikantham [4] have investigated and been well studied. As for some models with impulsive state feedback control, Zeng [8], Jiang [3] and Tang [7] discussed prey-predator models and obtained the complete expression of the periodic solution. However, so far, few papers have discussed the Leslie system using the impulsive differential equation with state feedback control. In this paper, we will discuss the existence and stability of periodic solution of the Leslie model with impulsive state feedback control according to the existence criteria [8] and the stability theorem [6] of periodic solution of the general impulsive autonomous system. This paper is organized as follows. The model and some preliminary results are presented in the next section. In Section 3, the existence and stability of periodic solution of order one of differential equation with impulsive state feedback control are investigated. Numerical simulations are given in Section 4. Finally, some conclusions and biological discussions are provided in Section 5.

2. Model Formulation and Preliminaries

Leslie (1948, see [5]) introduced the famous Leslie predator-prey system

$$\begin{cases} \dfrac{dx}{dt} = ax - bx^2 - cxy, \\ \dfrac{dy}{dt} = ey - \dfrac{fy^2}{x}. \end{cases} \tag{2.1}$$

In the system (2.1), the density equation of prey is as same as the Volterra model with damping. The predator is similar to the equation of Logistic model, but the second item has been revised and takes into account the density of prey. That is, it has been assumed that the prey grows logistically with growth rate a and carrying capacity a/b in the absence of predation. The predator consumes the prey according to function $\Phi(x) = cx$ and grows logistically with growth rate e and carrying capacity x/f proportional to the population size of the prey (or prey abundance). The parameter f is a measure of the food quality that the prey provides and converts to predator birth. Leslie introduced a predator-prey model where the carrying capacity of the predator's environment was proportional to the number of prey, and still stressed the fact that there were upper limits to the rates of increasing of both prey x and predator y, which were not recognized in the Lotka-Volterra model. These upper limits can be approached under favorable conditions: for the predators, when the amount of prey per predator is large; for the prey, when the amount of predators (and perhaps the amount of prey also) is small.

In some cases, if the amount of the pest is lower than a economic threshold, it is not necessary to adopt control measures. But excessive amount of the pest will often have caused a series of adverse effects, such as the depopulation of predator population, the rampancy of the pest population, and so on. What is especially serious, the environment and ecosystem will be seriously damaged. In order to overcome the above shortcomings, we often use the method of capturing the pests and putting in the nature enemies. In the system, when the pest reaches the threshold, we take some measures to kill the pest in the system so that the system is under the good condition. To control the pests under the threshold, by using the monitoring system, we will kill the pests and put in natural enemies at the same time when the amount of pests reaches a threshold. Therefore, the feedback control can be carried out by the impulsive state control.

In this paper, we consider that the prey is impulsively captured and the predator is impulsively put in when x reaches the threshold. System (2.1) can be modified as follows by introducing the impulsive state feedback control:

$$\begin{cases} \left.\begin{aligned} \frac{dx}{dt} &= ax - bx^2 - cxy, \\ \frac{dy}{dt} &= ey - \frac{fy^2}{x}, \end{aligned}\right\} x < h, \\ \left.\begin{aligned} \triangle x &= -px, \\ \triangle y &= qy + \tau, \end{aligned}\right\} x = h, \\ x(0) = x_0 > 0, y(0) = y_0 > 0 \end{cases} \tag{2.2}$$

where $\triangle x(t) = x(t^+) - x(t)$ and $\triangle y(t) = y(t^+) - y(t)$, $\tau \geq 0$, $0 < p < 1$ is constant and is also the fraction of the density of the prey that decreases due to the feedback control when the density of prey x reaches h, and $0 < q < 1$ is constant and is also the fraction of the density of predator y that increases due to the feedback control when the density of prey x reaches h. that is, when the amount of the prey reaches the threshold h at the time $t_i(h)$, controlling measures are taken and the amount of prey and predator abruptly turn to

$(1-p)h$ and $(1+q)y(t_i(h))+\tau$, respectively. The parameters a, b, c, d, e, f and h in system (2.2) are positive constants.

In this paper, we mainly discuss the existence and stability of periodic solution of system (2.2) by the existence criteria [8] and stability criteria [6] of the general impulsive autonomous system. Due to the layout constraints, we omit the definitions of semi-dynamical system, the set of impulses, trajectory $\widetilde{\pi_x}$ and periodic solution of order k (the definitions see Lakshmikantham et al [4]). We also omit the Brouwer's fixed-point theorem (see Griffel [2]), the existence theorem of periodic solution for the general autonomous impulsive differential equations (see Theorem 1 of Zeng [8]), and the stability criteria of periodic solution (see P.E. Simeonov [6]).

3. Existence and Stability of Periodic Solution of System (2.2)

Firstly, we will study the qualitative characteristic of system (2.2) without the impulsive effect. If no impulsive effect is introduced, then system (2.2) is

$$\begin{cases} \dfrac{dx}{dt} = ax - bx^2 - cxy, \\ \dfrac{dy}{dt} = ey - \dfrac{fy^2}{x}. \end{cases} \tag{3.1}$$

Clearly, system (3.1) has a positive equilibrium $(af/(bf+ce), ae/(bf+ce))$. The Jacobian matrix at equilibrium $E = (af/(bf+ce), ae/(bf+ce))$ is given by

$$J_{(x,y)} = \begin{pmatrix} -\dfrac{abf}{bf+ce} & -\dfrac{acf}{bf+ce} \\ \dfrac{e^2}{f} & -e \end{pmatrix}.$$

Therefore, the eigenvalue equation is

$$\lambda^2 + \frac{abf + (bf+ce)e}{bf+ce}\lambda + \frac{abef + ace^2}{bf+ce} = 0.$$

Obviously,

$$\lambda_1\lambda_2 = \frac{abef + ace^2}{bf+ce} > 0,$$

$$\lambda_1 + \lambda_2 = -\frac{abf + (bf+ce)e}{bf+ce} < 0.$$

Then the equilibrium $E = (af/(bf + ce), ae/(bf + ce))$ is asymptotically stable node or focus. Here λ_1, λ_2 are two roots of

$$\lambda^2 + \frac{abf + (bf + ce)e}{bf + ce}\lambda + \frac{abef + ace^2}{bf + ce} = 0.$$

$E = (af/(bf + ce), ae/(bf + ce))$ is asymptotically stable node if $(\sqrt{a} - \sqrt{e})^2 \geq ace/(bf + ce)$ or focus if $(\sqrt{a} - \sqrt{e})^2 < ace/(bf + ce)$.

Further, we can obtain that the positive equilibrium (x^*, y^*) of system (3.1) is globally stable, where $x^* = af/(bf + ce), y^* = ae/(bf + ce)$.

3.1. Existence and Stability of Semi-Trivial Periodic Solution with $\tau = 0$

Suppose system (2.2) has a periodic solution $(\xi(t), \eta(t))$ with period T. In this paper the periodic solution $(\xi(t), \eta(t))$ is called a semi-trivial periodic solution if its second component is zero. Let $y(t) = 0$ for $t \in (0, +\infty)$, and $\tau = 0$, then from system (2.2) we have

$$\begin{cases} \dfrac{dx}{dt} = ax - bx^2, & x < h, \\ \triangle x = -px, & x = h. \end{cases} \tag{3.2}$$

Setting $x_0 = x(0) = (1 - p)h$, the solution of equation $dx/dt = ax - bx^2$ is $x(t) = ag\exp(at)/[1 + bg\exp(at)]$, where $g = (1 - p)h/[a - b(1 - p)h]$. Set $T = \frac{1}{a}\ln\frac{a-b(1-p)h}{(1-p)(a-bh)}$, then $x(T) = h$ and $x(T^+) = x_0$. This means that system (2.2) has the following semi-trivial periodic solution for $(k - 1)T < t \leq kT$:

$$\begin{cases} \xi(t) = \dfrac{ag\exp(a(t - (k - 1)T))}{1 + bg\exp(a(t - (k - 1)T))}, \\ \eta(t) = 0. \end{cases} \tag{3.3}$$

By establishing Poincare map, we can obtain the stability conditions of this semi-trivial periodic solution. Due to the constraints of the layout, we only give the theorem and omit the construction of Poincare map and the proof of the following theorem.

Theorem 3.1. *Assume that the following condition holds:*

$$0 < (1 + q)\left(\frac{a-b(1-p)h}{(1-p)(a-bh)}\right)^{\frac{e}{a}} < 1. \tag{3.4}$$

Then system (2.2) has a stable semi-trivial periodic solution

$$\begin{cases} \xi(t) = \dfrac{ag\exp(a(t-(k-1)T))}{1+bg\exp(a(t-(k-1)T))}, \\ \eta(t) = 0. \end{cases}$$

3.2. Existence for Periodic Solution of Order One

From discussions of the qualitative characteristic of system (2.2) without the impulsive effect, we can see that (x^*, y^*) is a globally stable node or focus, where $x^* = af/(bf+ce), y^* = ae/(bf+ce)$. Because excessive number of the pest will often have caused a series of adverse effects, if $h \geq x^*$, then it has not the significance to control by impulsive control. So we mainly pay attention to the case that the following assumption holds: $(H)\quad h < x^*, x_0 < x^*$. By construct a closed region such that all the solution of system (2.2) enter the closed region and retain there, we can apply the existence criteria (Theorem 1 of [8]) to prove periodic solution of order one of system (2.2). Similarly, we only give the main result and omit the process and the proof.

Theorem 3.2. *Suppose that* $0 < h < x^*, x_0 < h$, $ey_0 - fy_0^2/x_0 \leq 0$, *then system (2.2) has a periodic solution of order one.*

3.3. Stability for Periodic Solution of Order One

Next, we analyze the stability for periodic solution of order one in system (2.2). We can apply the stability theorem [6] of periodic solution of the general impulsive autonomous system to obtain the following theorem:

Theorem 3.3. *System (2.2) with the conditions of Theorem 3.2 has a periodic solution of order one. Therefore, the periodic solution of order one is orbitally asymptotically stable if*

$$\left| \frac{(1+q)(1-p)\Big(a - b(1-p)h - c\big((1+q)\eta_0 + \tau\big)\Big)}{a - bh - c\eta_0} \exp\left(\int_0^T G(t)dt\right) \right| < 1.$$

Where $x = \xi(t), y(t) = \eta(t)$ is the $T-$periodic solution of system (2.2), $\eta_0 = \eta(0), P(x,y) = ax - bx^2 - cxy, Q(x,y) = y(e - \frac{fy}{x}), G(t) = \frac{\partial P}{\partial x}(\xi(t), \eta(t)) + \frac{\partial Q}{\partial y}(\xi(t), \eta(t))$.

4. Numerical Simulation

Now we consider the following example:

$$\begin{cases} \left.\begin{aligned} \frac{dx}{dt} &= x(5-3x-3y), \\ \frac{dx}{dt} &= y\left(5-\frac{2y}{x}\right), \end{aligned}\right\} x < h, \\ \left.\begin{aligned} \triangle x &= -0.2x, \\ \triangle y &= 0.2y+\tau, \end{aligned}\right\} x = h, \end{cases} \tag{4.1}$$

In numerical simulation, let $a = 5, b = 3, c = 3, e = 5, f = 2$. If $h = 0.6 > x^* = af/(bf + ce) = \frac{10}{21}, p = 0.2, q = 0.2, x_0 = 0.1 < x^*, y_0 = 0.415, \tau = 1$, then the time series and phase portrait can be seen in Figure 1. By analysis of Section 3, we know that impulsive control is no effect for system (2.2) if $h > x^*$ holds. As shown in Figure 1, numerical simulation also suggests that system (2.2) with the coefficients above admits no impulse to occur. By Theorem 3.2, we know that system (2.2) has a periodic solution of order one under conditions of Theorem 3.2. As shown in Figure 2, if $p = 0.2, q = 0.2, x_0 = 0.1 < x^*, y_0 = 0.415, h = 0.45 < x^* = \frac{10}{21}, \tau = 1$, then system (2.2) has a periodic solution of order one which verifies theoretical results in this paper.

5. Conclusion

In this paper, we built a Leslie model with impulsive state feedback control. Firstly, we investigated qualitative characteristic of the system without impulsive effect, and verified that the system was globally asymptotically stable. We obtained that the system with impulsive state feedback control had a periodic solution of order one, and sufficient conditions for existence and stability of periodic solution of order one. The results show that the Leslie model with impulsive state feedback control tends to a stable state or periodic, and the behavior of impulsive state feedback control on the density of population plays an important role on the periodic or stable state of system (2.2). When the amount of prey reaches an appropriate threshold, a state feedback measure for controlling amount of prey is taken. According to the analysis of Section 3, this measure is effective based on the fact that the system has stable periodic solution under some conditions. According to the theoretical results, the system will tend to a stable production level or the biome will be periodic. The key to the system by applying the system with feedback control is to give the suitable feedback state (the value of h) and the control parameters (p, q and τ)

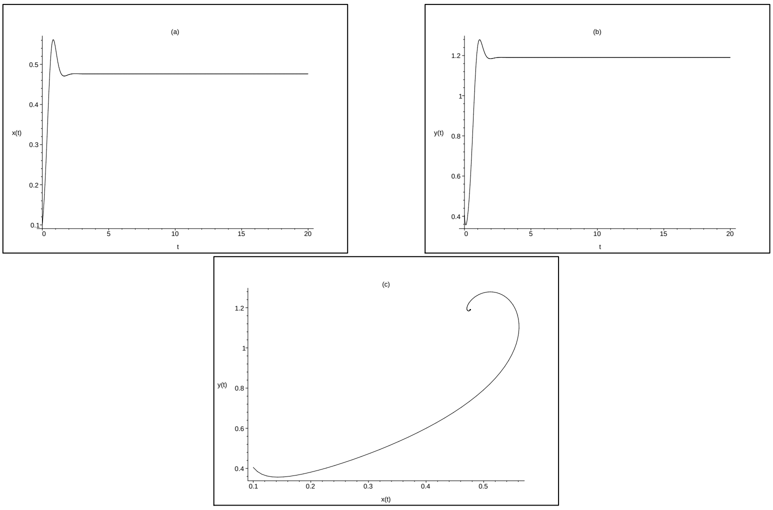

Figure 1: Time series and portrait phase of system (2.2) when $a = 5, b = 3, c = 3, e = 5, f = 2, x(0) = 0.1, y(0) = 0.415, p = 0.2, q = 0.2, \tau = 1, h = 0.6 > x^*$.

according to practice. It is seen from Figure 2 that there are positive periodic trajectories under the impulsive state feedback control. Therefore, the periodic system can be achieved if the value of h (that is economic threshold) and the suitable initial value of the predator and the prey are taken.

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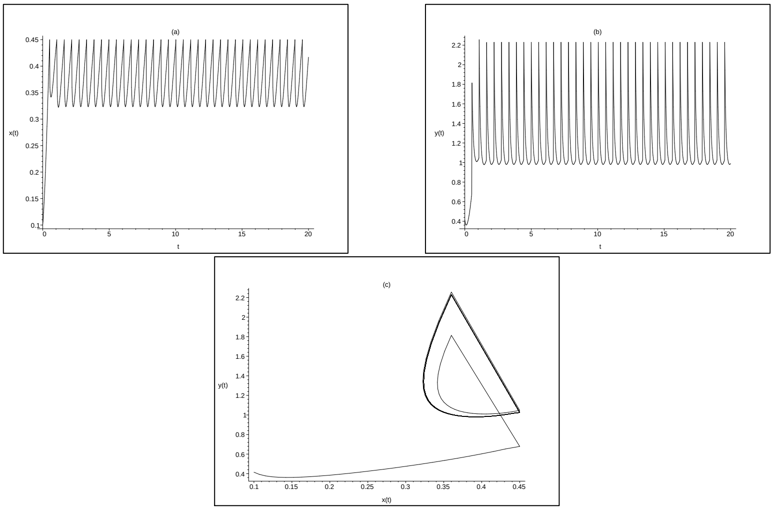

Figure 2: Time series and portrait phase of system (2.2) when $a = 5, b = 3, c = 3, e = 5, f = 2, x(0) = 0.1, y(0) = 0.415, p = 0.2, q = 0.2, \tau = 1, h = 0.45 < x^*$.

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