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THE EQUIVALENCE OF MANN AND ISHIKAWA ITERATION METHODS WITH ERRORS FOR UNIFORMLY CONTINUOUS AND ϕ-STRONGLY ACCRETIVE OPERATORS

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Abstract: The equivalence between the convergences of Mann and Ishikawa iteration methods with errors for uniformly continuous and ϕ-strongly accretive or ϕ-strongly pseudocontractive operators in Banach spaces are proved. Two examples which dwell upon the importance of our results are also included.

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1. Introduction and Preliminaries

Rhoades-Şoltuz [17], [18] first discussed the equivalence of Mann and Ishikawa iteration methods for Lipschitzian strongly pseudocontractive operators, and continuous and strongly pseudocontractive operators, respectively, Rhoades-Şoltuz [19] proved that the convergence of Mann iteration is equivalent to the convergence of Ishikawa iteration for asymptotically pseudocontractive operators, and Chang-Cho-Kim [2] obtained the equivalence between the convergences of modified Picard, modified Mann and modified Ishikawa iterations for contraction operators, nonexpansive operators and asymptotically nonexpansive operators, respectively, in Banach spaces. Rhoades-Şoltuz [20] studied the equivalence between the convergences of Ishikawa and Mann iterations for asymptotically nonexpansive operators in the intermediate sense and strongly successively pseudocontractive operators in Banach spaces. Recently several authors have applied Mann iteration methods, Mann iteration methods with errors, Ishikawa iteration methods and Ishikawa iteration methods with errors to approximate both fixed points of pseudocontractive, strongly pseudocontractive, ϕ-strongly pseudocontractive and ϕ-hemicontractive operators, and solutions of nonlinear equations $Tx = f$ and $x + Tx = f$ in the case when T is accretive, strongly accretive and ϕ-strongly accretive. For details, we refer to [1]-[5], [7]-[16], [21] and the references therein. In particular, Liu-Kang [10] established the convergence of Ishikawa iteration with errors for uniformly continuous and ϕ-strongly accretive operators in Banach spaces.

Motivated and inspired by the works in [2], [17]-[20], in this paper, we establish the equivalence between the convergences of Mann and Ishikawa iteration methods with errors for uniformly continuous and ϕ-strongly accretive or ϕ-strongly pseudocontractive operators in Banach spaces. Our results properly extend a host of results due to Chang-Cho-Jung-Kang [1], Chidume [3], Chidume-Osilike [4], [5], Liu [8], Liu-Kang [10], [13], Liu-Xu-Cho [15], Osilike [16] and Rhoades-Şoltuz [17]-[19] and others. Two examples which dwell upon the importance of our results are also included.

For a real Banach space X, we denote by J the normalized duality mapping from X into 2^{X^*} given by

$$Jx = \{f^* \in X^* : \|f^*\|^2 = \|x\|^2 = \langle x, f^*\rangle\}, \quad \forall x \in X,$$

where X^* denotes the dual space of X and $\langle \cdot, \cdot \rangle$ denotes the generalized duality pairing. In the sequel, I denotes the identity operator on X and $\delta(K)$ denotes the diameter of K for any $K \subseteq X$. An operator T with domain $D(T)$ and range $R(T)$ in X is called *strongly accretive* if there exists a constant $k > 0$ such that for any $x, y \in D(T)$, there exists $j(x-y) \in J(x-y)$ satisfying

$$\langle Tx - Ty, j(x-y)\rangle \geq k\|x-y\|^2. \tag{1.1}$$

Without loss of generality we may assume $k \in (0,1)$. T is called *accretive* if for any $x, y \in D(T)$, there exists $j(x-y) \in J(x-y)$ satisfying

$$\langle Tx - Ty, j(x-y)\rangle \geq 0. \tag{1.2}$$

T is called *ϕ-strongly accretive* if there exists a strictly increasing function $\phi : [0,+\infty) \to [0,+\infty)$ with $\phi(0) = 0$ such that for any $x, y \in D(T)$, there exists $j(x-y) \in J(x-y)$ satisfying

$$\langle Tx - Ty, j(x-y)\rangle \geq \phi(\|x-y\|)\|x-y\|. \tag{1.3}$$

Closely related to the class of strongly accretive operators is the class of strongly pseudocontractive operators where an operator T is called *strongly pseudocontractive* if there exists $t > 1$ such that for any $x, y \in D(T)$, there exists $j(x-y) \in J(x-y)$ satisfying

$$\langle Tx - Ty, j(x-y)\rangle \leq \frac{1}{t}\|x-y\|^2. \tag{1.4}$$

T is called *ϕ-strongly pseudocontractive* if there exists a strictly increasing function $\phi : [0,+\infty) \to [0,+\infty)$ with $\phi(0) = 0$ such that for any $x, y \in D(T)$, there exists $j(x-y) \in J(x-y)$ satisfying

$$\langle Tx - Ty, j(x-y)\rangle \leq \|x-y\|^2 - \phi(\|x-y\|)\|x-y\|. \tag{1.5}$$

T is called *ϕ-hemicontractive* if $F(T) = \{x \in D(T) : Tx = x\} \neq \emptyset$ and there exists a strictly increasing function $\phi : [0,+\infty) \to [0,+\infty)$ with $\phi(0) = 0$ such that for any $x \in D(T)$ and $q \in F(T)$, there exists $j(x-q) \in J(x-q)$ satisfying

$$\langle Tx - q, j(x-q)\rangle \leq \|x-q\|^2 - \phi(\|x-q\|)\|x-q\|. \tag{1.6}$$

It is well known that T is strongly pseudocontractive (respectively, pseudocontractive and ϕ-strongly pseudocontractive) if and only if $(I-T)$ is strongly accretive (respectively, accretive and ϕ-strongly accretive).

Lemma 1.1. ([7]) *Let $\{\alpha_n\}_{n=0}^{\infty}, \{\beta_n\}_{n=0}^{\infty}$ and $\{\gamma_n\}_{n=0}^{\infty}$ be three nonnegative real sequences satisfying the inequality*

$$\alpha_{n+1} \leq (1-\omega_n)\alpha_n + \omega_n\beta_n + \gamma_n, \quad n \geq 0,$$

where $\{\omega_n\}_{n=0}^{\infty} \subset [0,1]$, $\sum_{n=0}^{\infty}\omega_n = \infty$, $\lim_{n\to\infty}\beta_n = 0$ and $\sum_{n=0}^{\infty}\gamma_n < \infty$. Then $\lim_{n\to\infty}\alpha_n = 0$.

Lemma 1.2. ([10]) *Suppose that* X *is a real Banach space and* $T : X \to X$ *is a continuous and* ϕ*-strongly accretive operator. Then the equation* $Tx = f$ *has a unique solution for any* $f \in X$.

Lemma 1.3. ([10]) *Suppose that* X *is a real Banach space and* $T : X \to X$ *is a continuous and* ϕ*-strongly pseudocontractive operator. Then* T *has a unique fixed point in* X.

2. The Equivalence between Mann and Ishikawa Iteration Methods with Errors

First we prove the following result for uniformly continuous and ϕ-strongly accretive operators.

Theorem 2.1. *Suppose that* X *is a real Banach space and* $T : X \to X$ *is a uniformly continuous and* ϕ*-strongly accretive operator. For a given* $f \in X$, *define* $S : X \to X$ *by* $Sx = f + x - Tx$ *for* $x \in X$. *Define the Ishikawa iteration sequence with errors* $\{x_n\}_{n=0}^{\infty}$ *iteratively by*

$$\begin{aligned} &x_0, \sigma_0, \delta_0 \in X, \\ &y_n = a'_n x_n + b'_n S x_n + c'_n \delta_n, \\ &x_{n+1} = a_n x_n + b_n S y_n + c_n \sigma_n, \quad n \geq 0 \end{aligned} \tag{2.1}$$

and the Mann iteration sequence with errors $\{u_n\}_{n=0}^{\infty}$ *iteratively by*

$$\begin{aligned} &u_0, \sigma_0 \in X, \\ &u_{n+1} = a_n u_n + b_n S u_n + c_n \sigma_n, \quad n \geq 0, \end{aligned} \tag{2.2}$$

where $\{\sigma_n\}_{n=0}^{\infty}$ *and* $\{\delta_n\}_{n=0}^{\infty}$ *are arbitrary bounded sequences in* X *and* $\{a_n\}_{n=0}^{\infty}$, $\{b_n\}_{n=0}^{\infty}$, $\{c_n\}_{n=0}^{\infty}$, $\{a'_n\}_{n=0}^{\infty}$, $\{b'_n\}_{n=0}^{\infty}$ *and* $\{c'_n\}_{n=0}^{\infty}$ *are sequences in* $[0, 1]$ *satisfying the following conditions:*

$$\begin{aligned} &a_n + b_n + c_n = a'_n + b'_n + c'_n = 1, \\ &c_n = t_n b_n, \quad n \geq 0, \quad \{t_n\}_{n=0}^{\infty} \subset [0, +\infty); \end{aligned} \tag{2.3}$$

$$\lim_{n\to\infty} b_n = \lim_{n\to\infty} b'_n = \lim_{n\to\infty} c'_n = \lim_{n\to\infty} t_n = 0; \tag{2.4}$$

$$\sum_{n=0}^{\infty} b_n = +\infty. \tag{2.5}$$

Assume that either

$$\begin{aligned}&\text{the sequences } \{u_n - Tu_n\}_{n=0}^{\infty},\ \{x_n - Tx_n\}_{n=0}^{\infty} \text{ and } \{y_n - Ty_n\}_{n=0}^{\infty}\\ &\text{or the sequences } \{Tu_n\}_{n=0}^{\infty},\ \{Tx_n\}_{n=0}^{\infty} \text{ and } \{Ty_n\}_{n=0}^{\infty} \text{ are bounded.}\end{aligned} \tag{2.6}$$

Then, for $u_0 = x_0 \in X$, *the following assertions hold and are equivalent:*

(a) *the Mann iteration sequence with errors* $\{u_n\}_{n=0}^{\infty}$ *converges strongly to the unique solution of* $Tx = f$;

(b) *the Ishikawa iteration sequence with errors* $\{x_n\}_{n=0}^{\infty}$ *converges strongly to the unique solution of* $Tx = f$.

Proof. It follows from Lemma 1.2 that the equation $Tx = f$ has a unique solution $q \in X$. By a similar argument used in the proof of Theorem 3.1 in [10], we conclude that the Ishikawa iteration sequence with errors $\{x_n\}_{n=0}^{\infty}$ converges strongly to the unique solution q of $Tx = f$. It is clear that (a) follows from (b) by setting $a'_n = 1$ and $b'_n = c'_n = 0$ for $n \geq 0$.

Next we prove that (a) implies (b). By the ϕ-strong accretivity of T, we deduce that for $x, y \in X$

$$\begin{aligned}\langle Tx - Ty, j(x-y)\rangle &= \langle (I-S)x - (I-S)y, j(x-y)\rangle\\ &\geq \phi(\|x-y\|)\|x-y\|\\ &\geq A(x,y)\|x-y\|^2,\end{aligned} \tag{2.7}$$

where $A(x,y) = \frac{\phi(\|x-y\|)}{1+\|x-y\|+\phi(\|x-y\|)} \in [0,1)$ for $x, y \in X$. (2.7) implies that

$$\langle (I - S - A(x,y))x - (I - S - A(x,y))y, j(x-y)\rangle \geq 0 \tag{2.8}$$

for $x, y \in X$. It follows from Lemma 1.1 in [6] and (2.8) that

$$\|x-y\| \leq \|x - y + r[(I - S - A(x,y))x - (I - S - A(x,y))y]\| \tag{2.9}$$

for $x, y \in X$ and $r > 0$. Put $d_n = b_n + c_n$, $d'_n = b'_n + c'_n$ and

$$\begin{aligned}D = \max\{\sup\{&\|Sx_n - q\|, \|Sy_n - q\|, \|Su_n - q\|,\\ &\|\sigma_n - q\|, \|\delta_n - q\| : n \geq 0\}, \|x_0 - q\|\}.\end{aligned} \tag{2.10}$$

Observe that

$$\begin{aligned}\|Sx - Sy\| &\leq \|x-y\| + \|Tx - Ty\|\\ &\leq \phi^{-1}(\|Tx - Ty\|) + \|Tx - Ty\|\end{aligned}$$

and

$$\|Sx - Sy\| \leq \|x - Tx\| + \|y - Ty\|$$

for $x, y \in X$. Therefore (2.6) ensures that D is bounded. By induction and (2.10), we easily conclude that

$$\max\{\|x_n - q\|, \|y_n - q\|\} \le D, \quad \forall n \ge 0. \tag{2.11}$$

Using (2.1) and (2.2), we infer that

$$\begin{aligned}(1-d_n)x_n &= x_{n+1} - d_n S y_n - c_n(\sigma_n - S y_n)\\ &= [1-(1-A(x_{n+1},u_{n+1}))d_n]x_{n+1}\\ &\quad + d_n(I - S - A(x_{n+1},u_{n+1}))x_{n+1}\\ &\quad + d_n(Sx_{n+1} - Sy_n) - c_n(\sigma_n - Sy_n)\end{aligned} \tag{2.12}$$

and

$$\begin{aligned}(1-d_n)u_n &= u_{n+1} - d_n S u_n - c_n(\sigma_n - S u_n)\\ &= [1-(1-A(x_{n+1},u_{n+1}))d_n]u_{n+1}\\ &\quad + d_n(I - S - A(x_{n+1},u_{n+1}))u_{n+1}\\ &\quad + d_n(Su_{n+1} - Su_n) - c_n(\sigma_n - Su_n).\end{aligned} \tag{2.13}$$

It follows from (2.9), (2.12) and (2.13) that

$$\begin{aligned}&(1-d_n)\|x_n - u_n\|\\ &\ge [1-(1-A(x_{n+1},u_{n+1}))d_n]\|x_{n+1} - u_{n+1}\\ &\quad + \frac{d_n}{1-(1-A(x_{n+1},u_{n+1}))d_n}[(I - S - A(x_{n+1},u_{n+1}))x_{n+1}\\ &\qquad - (I - S - A(x_{n+1},u_{n+1}))u_{n+1}]\|\\ &\quad - d_n\|Sx_{n+1} - Sy_n - Su_{n+1} + Su_n\| - c_n\|Sy_n - Su_n\|\\ &\ge [1-(1-A(x_{n+1},u_{n+1}))d_n]\|x_{n+1} - u_{n+1}\|\\ &\quad - d_n(\|Sx_{n+1} - Sy_n\| + \|Su_{n+1} - Su_n\|) - c_n\|Sy_n - Su_n\|,\end{aligned}$$

which implies that

$$\begin{aligned}&\|x_{n+1} - u_{n+1}\|\\ &\le \frac{1-d_n}{1-(1-A(x_{n+1},u_{n+1}))d_n}\|x_n - u_n\|\\ &\quad + \frac{d_n}{1-(1-A(x_{n+1},u_{n+1}))d_n}(\|Sx_{n+1} - Sy_n\| + \|Su_{n+1} - Su_n\|)\\ &\quad + \frac{c_n}{1-(1-A(x_{n+1},u_{n+1}))d_n}\|Sy_n - Su_n\|\\ &\le (1 - A(x_{n+1},u_{n+1})d_n)\|x_n - u_n\|\\ &\quad + Md_n(\|Sx_{n+1} - Sy_n\| + \|Su_{n+1} - Su_n\|) + Mc_n\\ &\le (1 - A(x_{n+1},u_{n+1})b_n)\|x_n - u_n\| + Mb_n s_n + Mc_n(1 + s_n)\end{aligned} \tag{2.14}$$

for $n \geq 0$, where $s_n = \|Sx_{n+1} - Sy_n\| + \|Su_{n+1} - Su_n\|$ and M is a constant. In view of (2.1)-(2.4), (2.10) and (2.11), we know that

$$\begin{aligned}
&\|x_{n+1} - y_n\| \\
&\leq \|x_{n+1} - x_n\| + \|x_n - y_n\| \\
&\leq b_n\|Sy_n - x_n\| + c_n\|\sigma_n - x_n\| + b'_n\|Sx_n - x_n\| + c'_n\|\delta_n - x_n\| \\
&\leq 2D(d_n + d'_n) \\
&\to 0
\end{aligned}$$

as $n \to \infty$. Note that $\{u_n\}_{n=0}^{\infty}$ converges strongly to q. Thus the uniform continuity of T ensures that

$$\|Sx_{n+1} - Sy_n\| \to 0 \quad \text{and} \quad \|Su_{n+1} - Su_n\| \to 0 \quad \text{as } n \to \infty. \tag{2.15}$$

It follows from (2.3), (2.14) and (2.15) that

$$\|x_{n+1} - u_{n+1}\| \leq (1 - A(x_{n+1}, u_{n+1})b_n)\|x_n - u_n\| + Gb_n(s_n + t_n) \tag{2.16}$$

for $n \geq 0$, where G is a constant.

Set $\inf\{A(x_{n+1}, u_{n+1}) : n \geq 0\} = r$. We assert that $r = 0$. If not, then $r > 0$. (2.16) yields that

$$\|x_{n+1} - u_{n+1}\| \leq (1 - rb_n)\|x_n - u_n\| + Gb_n(s_n + t_n) \tag{2.17}$$

for $n \geq 0$. Put $\alpha_n = \|x_n - u_n\|$, $\omega_n = rb_n$, $\beta_n = Gb_n(s_n + t_n)$ and $\gamma_n = 0$ in (2.17). It follows from (2.4), (2.5), (2.17) and Lemma 1.1 that $\|x_n - u_n\| \to 0$ as $n \to \infty$, which means that $r = 0$. This is a contradiction. Therefore $r = 0$ and there exists a subsequence $\{\|x_{n_i+1} - u_{n_i+1}\|\}_{i=1}^{\infty}$ of $\{\|x_{n+1} - u_{n+1}\|\}_{n=0}^{\infty}$ satisfying

$$\|x_{n_i+1} - u_{n_i+1}\| \to 0 \quad \text{as } i \to \infty. \tag{2.18}$$

Using (2.18) we conclude that, given $\epsilon > 0$, there exists a positive integer m such that for $n \geq m$,

$$\|x_{n_m+1} - u_{n_m+1}\| < \epsilon, \tag{2.19}$$

$$G(s_n + t_n) < \min\left\{\frac{1}{2}\epsilon, \frac{\phi(\epsilon)\epsilon}{1 + \phi(\frac{3}{2}\epsilon) + \frac{3}{2}\epsilon}\right\}. \tag{2.20}$$

Now we claim that

$$\|x_{n_m+j} - u_{n_m+j}\| \leq \epsilon \tag{2.21}$$

for $j \geq 1$. In fact (2.19) means that (2.21) holds for $j = 1$. Assume that (2.21) holds for $j = k$. If $\|x_{n_m+k+1} - u_{n_m+k+1}\| > \epsilon$, we obtain by (2.16) and (2.20) that

$$\begin{aligned} &\|x_{n_m+k+1} - u_{n_m+k+1}\| \\ &\leq \|x_{n_m+k} - u_{n_m+k}\| + Gb_{n_m+k}(s_{n_m+k} + t_{n_m+k}) \\ &\leq \epsilon + \min\left\{\frac{1}{2}\epsilon, \frac{\phi(\epsilon)\epsilon}{1+\phi(\frac{3}{2}\epsilon)+\frac{3}{2}\epsilon}\right\} b_{n_m+k} \\ &\leq \frac{3}{2}\epsilon. \end{aligned} \tag{2.22}$$

Note that $\phi(\|x_{n_m+k+1} - u_{n_m+k+1}\|) > \phi(\epsilon)$. It follows from (2.22) that

$$A(x_{n_m+k+1}, u_{n_m+k+1}) \geq \frac{\phi(\epsilon)}{1+\phi(\frac{3}{2}\epsilon)+\frac{3}{2}\epsilon}. \tag{2.23}$$

By virtue of (2.16), (2.20) and (2.23), we obtain that

$$\begin{aligned} &\|x_{n_m+k+1} - u_{n_m+k+1}\| \\ &\leq \left(1 - \frac{\phi(\epsilon)}{1+\phi(\frac{3}{2}\epsilon)+\frac{3}{2}\epsilon} b_{n_m+k}\right) \|x_{n_m+k} - u_{n_m+k}\| \\ &\quad + Gb_{n_m+k}(s_{n_m+k} + t_{n_m+k}) \\ &\leq \left(1 - \frac{\phi(\epsilon)}{1+\phi(\frac{3}{2}\epsilon)+\frac{3}{2}\epsilon} b_{n_m+k}\right)\epsilon + \min\left\{\frac{1}{2}\epsilon, \frac{\phi(\epsilon)}{1+\phi(\frac{3}{2}\epsilon)+\frac{3}{2}\epsilon}\right\} b_{n_m+k} \\ &\leq \epsilon. \end{aligned}$$

That is

$$\epsilon < \|x_{n_m+k+1} - u_{n_m+k+1}\| \leq \epsilon,$$

which is impossible. Hence $\|x_{n_m+k+1} - u_{n_m+k+1}\| \leq \epsilon$. By induction, (2.19) holds for $j \geq 1$. Thus (2.19) yields that $\|x_n - u_n\| \to 0$ as $n \to \infty$. Consequently,

$$\|x_n - q\| \leq \|u_n - q\| + \|u_n - x_n\| \to 0 \quad \text{as } n \to \infty.$$

Hence the Ishikawa iteration sequence with errors $\{x_n\}_{n=0}^{\infty}$ converges strongly to the unique solution of $Tx = f$. This completes the proof. □

Next we establish the following result for uniformly continuous and accretive operators.

Theorem 2.2. *Let X be a real Banach space and $T : X \to X$ be a uniformly continuous and accretive operator. For a given $f \in X$, define $S : X \to X$*

by $Sx = f - Tx$ *for* $x \in X$. *Let* $\{x_n\}_{n=0}^{\infty}$, $\{y_n\}_{n=0}^{\infty}$, $\{u_n\}_{n=0}^{\infty}$, $\{\sigma_n\}_{n=0}^{\infty}$, $\{\delta_n\}_{n=0}^{\infty}$, $\{a_n\}_{n=0}^{\infty}$, $\{b_n\}_{n=0}^{\infty}$, $\{c_n\}_{n=0}^{\infty}\{a'_n\}_{n=0}^{\infty}$, $\{b'_n\}_{n=0}^{\infty}$ *and* $\{c'_n\}_{n=0}^{\infty}$ *be as in Theorem 2.1 satisfying* (2.1)-(2.5). *Assume that either the sequences* $\{x_n + Tx_n\}_{n=0}^{\infty}$, $\{y_n + Ty_n\}_{n=0}^{\infty}$ *and* $\{u_n + Tu_n\}_{n=0}^{\infty}$ *or the sequences*$\{Tx_n\}_{n=0}^{\infty}$, $\{Ty_n\}_{n=0}^{\infty}$ *and* $\{Tu_n\}_{n=0}^{\infty}$ *are bounded. Then, for* $u_0 = x_0 \in X$, *the following assertions both hold and are equivalent:*

(c) *the Mann iteration sequence with errors* $\{u_n\}_{n=0}^{\infty}$ *converges strongly to the unique solution of the equation* $x + Tx = f$;

(d) *the Ishikawa iteration sequence with errors* $\{x_n\}_{n=0}^{\infty}$ *converges strongly to the unique solution of the equation* $x + Tx = f$.

Proof. Put $A = I + T$. Clearly, $Sx = f + x - Ax$ for $x \in X$, $A : X \to X$ is uniformly continuous and ϕ-strongly accretive with $\phi(t) = \frac{1}{2}t$ for all $t \geq 0$, and $\{Ax_n\}_{n=0}^{\infty}$, $\{Ay_n\}_{n=0}^{\infty}$ and $\{Au_n\}_{n=0}^{\infty}$ or $\{x_n - Ax_n\}_{n=0}^{\infty}$, $\{y_n - Ay_n\}_{n=0}^{\infty}$ and $\{u_n - Au_n\}_{n=0}^{\infty}$ are bounded. Hence Theorem 2.2 follows from Theorem 2.1. This completes the proof. □

Finally, we have the following results for uniformly continuous and ϕ-strongly pseudocontractive operators.

Theorem 2.3. *Suppose that* X *is a real Banach space and* $T : X \to X$ *is a uniformly continuous and* ϕ*-strongly pseudocontractive operator. Define the Mann and Ishikawa iteration sequences with errors* $\{u_n\}_{n=0}^{\infty}$ *and* $\{x_n\}_{n=0}^{\infty}$ *by* (2.2) *and* (2.1) *with* $S = T$, *respectively. Suppose that* $\{\sigma_n\}_{n=0}^{\infty}$, $\{\delta_n\}_{n=0}^{\infty}$ *are arbitrary bounded sequences in* X *and* $\{a_n\}_{n=0}^{\infty}$, $\{b_n\}_{n=0}^{\infty}$, $\{c_n\}_{n=0}^{\infty}$, $\{a'_n\}_{n=0}^{\infty}$, $\{b'_n\}_{n=0}^{\infty}$ *and* $\{c'_n\}_{n=0}^{\infty}$ *are sequences in* $[0, 1]$ *satisfying* (2.3)-(2.6). *Then, for* $u_0 = x_0 \in X$, *the following assertions hold and are equivalent:*

(e) *the Mann iteration sequence with errors* $\{u_n\}_{n=0}^{\infty}$ *converges strongly to the unique fixed point of* T;

(f) *the Ishikawa iteration sequence with errors* $\{x_n\}_{n=0}^{\infty}$ *converges strongly to the unique fixed point of* T.

Proof. It follows from Lemma 1.3 that T has a unique fixed point $q \in X$. Put $f = 0$ and $Ax = x - Tx$ for $x \in X$. Then $A : X \to X$ is uniformly continuous and ϕ-strongly accretive. Hence Theorem 2.3 follows from Theorem 2.1. This completes the proof. □

Theorem 2.4. *Let* K *be a nonempty closed convex subset of a real Banach space* X *and* $T : K \to K$ *be a uniformly continuous and* ϕ*-strongly pseudocontractive operator. Define the Mann and Ishikawa iteration sequences with errors* $\{u_n\}_{n=0}^{\infty}$ *and* $\{x_n\}_{n=0}^{\infty}$ *by* (2.2) *and* (2.1) *with* $S = T$, *respectively.*

Suppose that q is the fixed point of T, $\{\sigma_n\}_{n=0}^{\infty}$ and $\{\delta_n\}_{n=0}^{\infty}$ are arbitrary bounded sequences in K and $\{a_n\}_{n=0}^{\infty}$, $\{b_n\}_{n=0}^{\infty}$, $\{c_n\}_{n=0}^{\infty}$, $\{a'_n\}_{n=0}^{\infty}$, $\{b'_n\}_{n=0}^{\infty}$ and $\{c'_n\}_{n=0}^{\infty}$ are sequences in $[0,1]$ satisfying (2.3)-(2.5) and

$$\text{the sequences } \{Tu_n\}_{n=0}^{\infty},\ \{Tx_n\}_{n=0}^{\infty} \text{ and } \{Ty_n\}_{n=0}^{\infty} \text{ are bounded.} \tag{2.24}$$

Then, for $u_0 = x_0 \in K$, the following assertions hold and are equivalent:

(g) *the Mann iteration sequence with errors $\{u_n\}_{n=0}^{\infty}$ converges strongly to q;*

(h) *the Ishikawa iteration sequence with errors $\{x_n\}_{n=0}^{\infty}$ converges strongly to q.*

Proof. Since T is ϕ-strongly pseudocontractive, it follows that q is the unique fixed point of T and

$$\begin{aligned}\langle (I-T)x-(I-T)y, j(x-y)\rangle &\geq \phi(\|x-y\|)\|x-y\| \\ &\geq A(x,y)\|x-y\|^2,\end{aligned} \tag{2.25}$$

where $A(x,y) = \frac{\phi(\|x-y\|)}{1+\|x-y\|+\phi(\|x-y\|)}$ for $x, y \in K$. Using (2.25) and Lemma 1.1 in [6], we deduce that

$$\|x-y\| \leq \|x-y+r[(I-T-A(x,y))x-(I-T-A(x,y))y]\|$$

for any $x, y \in K$ and $r > 0$. The remaining part of the proof is exactly same as the argument of the corresponding part of Theorem 2.1, hence it is omitted. This completes the proof. □

3. Remarks and Examples

Now we compare the results in Section 2 with the results in [1], [3]-[5], [8], [10], [13], [15]-[19].

Remark 3.1. The convergence results in Theorems 2.1 and 2.2 extend, improve and unify Theorems 2 and 3 in [3], Corollaries 6 and 7 in [4], Theorems 2 and 3 and Corollaries 2 and 3 in [5], Theorems 1-4 in [8], Theorem 3.1 and Corollary 3.1 in [10], Theorem 2.1 in [13], Theorem 3.1 in [15] and Theorem 1 in [16] in the following ways:

(1) the Mann iteration method in [4], [5], [8] and the Ishikawa iteration method in [3]-[5], [8], [16] are replaced by the more general Ishikawa iteration method with errors in the sense of Xu [21];

(2) the strong accretivity in [3]-[5], [8] and the Lipschitz continuity in [3], [8], [15], [16] are taken the place of the more general ϕ-strong accretivity and uniform continuity, respectively;

(3) the assumption that the range of $(I - T)$ or T is bounded in [3], [5], [10], [13] is weakened by the condition (2.6);

(4) the conditions $\{b_n + c_n\}_{n=0}^{\infty} \subset (0, 1)$ and $\lim_{n\to\infty} \frac{c_n}{b_n+c_n} = 0$ in [10] are replaced by the slightly more general conditions $c_n = t_n b_n$ for all $n \geq 0$ and $\lim_{n\to\infty} t_n = 0$.

Remark 3.2. The equivalence results in Theorems 2.1 and 2.2 reveal that Corollaries 6 and 7 in [4], Theorem 2 and Corollary 2 in [5], Theorems 3 and 4 in [8], Theorem 3.1 and Corollary 3.1, and Theorem 3.2 and Corollary 3.2 in [10], respectively, are pairwise equivalent.

Remark 3.3. Theorem 2.4 extends Theorems 3.4 and 4.2 in [1], Theorem 1 in [3], Theorem 1 and Corollary 1 in [4], [5], Theorem 4 in [17], Theorem 2.3 in [18] and Theorem 10 in [19] from strongly pseudocontractive operators to ϕ-strongly pseudocontractive operators and Lipschitzian operators to uniformly continuous operators.

The Examples 3.1 and 3.2 below demonstrate that Theorems 2.1, 2.2 and 2.4 generalize indeed Theorems 3.4 and 4.2 in [1], Theorems 1-3 in [3], Theorem 1, Corollaries 1, 6 and 7 in [4], Theorems 1-3 and Corollaries 1-3 in [5], Theorems 1-4 in [8], Theorems 3.1 and 3.2, Corollaries 3.1 and 3.2 in [10], Theorem 2.1 in [13], Theorem 3.1 in [15],Theorem 1 in [16], Theorem 4 in [17], Theorem 2.3 in [18] and Theorem 10 in [19].

Example 3.1. Let $X = (-\infty, +\infty)$ with the usual norm $|\cdot|$. Define $T : X \to X$ and $\phi : [0, +\infty) \to [0, +\infty)$ by

$$Tx = \begin{cases} \frac{x^2}{2(1+x)} & \text{for } x \in [0, +\infty) \\ \frac{1}{2}x - \sqrt{-x} & \text{for } x \in [-1, 0) \\ \frac{1}{2}x - 1 & \text{for } x \in (-\infty, -1) \end{cases}$$

and

$$\phi(t) = \frac{t^2}{2(1+t)} \quad \text{for } t \in [0, +\infty),$$

respectively. Set

$$\begin{aligned}
&a_n = 1 - b_n - c_n, \quad b_{3n} = \frac{1}{n+2}, \quad b_{3n+2} = \frac{1}{3n+2},\\
&c_{3n+2} = \frac{1}{(3n+2)^2}, \quad c_{3n} = c_{3n+1} = b_{3n+1} = 0,\\
&a'_n = \frac{n+1}{n+3}, \quad b'_n = c'_n = \frac{1}{n+3},\\
&t_{3n} = 0, \quad t_{3n+1} = 0, \quad t_{3n+2} = \frac{1}{3n+2}, \quad n \geq 0,
\end{aligned}$$

and $\{\sigma_n\}_{n=0}^{\infty}$ and $\{\delta_n\}_{n=0}^{\infty}$ are arbitrary bounded sequences in X. It is easy to see that $\{Tx_n\}_{n=0}^{\infty}$ and $\{Ty_n\}_{n=0}^{\infty}$ are bounded, where $\{x_n\}_{n=0}^{\infty}$ and $\{y_n\}_{n=0}^{\infty}$ are as in (2.1), and T is uniformly continuous. In order to prove that T is ϕ-strongly accretive, for any $x, y \in X$ with $x \geq y$, we consider the following cases:

Case 1. Suppose that $x, y \in [0, +\infty)$. It follows that

$$\begin{aligned}
&\langle Tx - Ty, x - y\rangle - \phi(|x-y|)|x-y|\\
&= \left(\frac{x^2}{2(1+x)} - \frac{y^2}{2(1+y)}\right)(x-y) - \frac{|x-y|^3}{2(1+|x-y|)}\\
&= \frac{1}{2}(x-y)^2\frac{x+y+xy-|x-y|}{(1+x)(1+y)(1+|x-y|)}\\
&\geq 0;
\end{aligned}$$

Case 2. Suppose that $x, y \in [-1, 0)$. It is easy to verify that

$$\begin{aligned}
&\langle Tx - Ty, x - y\rangle - \phi(|x-y|)|x-y|\\
&= \frac{1}{2}(x-y)^2\left(1 - \frac{|x-y|}{1+|x-y|}\right) - (\sqrt{-x} - \sqrt{-y})(x-y)\\
&\geq 0;
\end{aligned}$$

Case 3. Suppose that $x, y \in (-\infty, -1)$. Then we have

$$\begin{aligned}
&\langle Tx - Ty, x - y\rangle - \phi(|x-y|)|x-y|\\
&= \frac{1}{2}(x-y)^2\left(1 - \frac{|x-y|}{1+|x-y|}\right)\\
&\geq 0;
\end{aligned}$$

Case 4. Suppose that $x \in [0, +\infty)$ and $y \in [-1, 0)$. It follows that

$$\begin{aligned}
&\langle Tx - Ty, x - y\rangle - \phi(|x - y|)|x - y| \\
&= \frac{1}{2}(x - y)\frac{-y}{(1 + x)(1 + x - y)} + \sqrt{-y}(x - y) \\
&\geq 0;
\end{aligned}$$

Case 5. Suppose that $x \in [0, +\infty)$ and $y \in (-\infty, -1)$ It is easy to see that

$$\begin{aligned}
&\langle Tx - Ty, x - y\rangle - \phi(|x - y|)|x - y| \\
&= \left(\frac{x^2}{2(1 + x)} - \frac{1}{2}y + 1\right)(x - y) - \frac{|x - y|^3}{2(1 + |x - y|)} \\
&= \frac{1}{2}(x - y)\frac{-y}{(1 + x)(1 + x - y)} + (x - y) \\
&\geq 0;
\end{aligned}$$

Case 6. Suppose that $x \in [-1, 0)$ and $y \in (-\infty, -1)$. It is easy to verify that

$$\begin{aligned}
&\langle Tx - Ty, x - y\rangle - \phi(|x - y|)|x - y| \\
&= \frac{1}{2}(x - y)^2\left(1 - \frac{|x - y|}{1 + |x - y|}\right) + (1 - \sqrt{-x})(x - y) \\
&\geq 0.
\end{aligned}$$

Therefore, T is ϕ-strongly accretive. Consequently, all the assumptions of Theorems 2.1 and 2.2 are fulfilled. However, Theorems 2 and 3 in [3], Corollaries 6 and 7 in [4], Theorems 2 and 3 and Corollaries 2 and 3 in [5], Theorems 1-4 in [8], Theorems 3.1 and 3.2, Corollaries 3.1 and 3.2 in [10], Theorem 2.1 in [13], Theorem 3.1 in [15] and Theorem 1 in [16] are not applicable since $b_{3n+1} + c_{3n+1} = 0$ for all $n \geq 0$, the ranges of both T and $(I - T)$ are unbounded, and T is neither Lipschitz nor strongly accretive. In fact, we see that

$$\lim_{x \to 0^-} \frac{Tx - T0}{x - 0} = \lim_{x \to 0^-} \left(\frac{1}{2} - \frac{\sqrt{-x}}{x}\right) = +\infty,$$

which implies that T is not Lipschitz, and for any given $\epsilon \in (0, 1)$, there exist $(x_\epsilon, y_\epsilon) = (\frac{\epsilon}{1-\epsilon}, 0) \in X \times X$ such that

$$\begin{aligned}
\langle Tx_\epsilon - Ty_\epsilon, x_\epsilon - y_\epsilon\rangle - \epsilon|x_\epsilon - y_\epsilon|^2 &= \frac{x_\epsilon^3}{2(1 + x_\epsilon)} - \epsilon x_\epsilon^2 \\
&= -\frac{\epsilon}{2}x_\epsilon^2 \\
&< 0,
\end{aligned}$$

which yields that T is not strongly accretive.

Example 3.2. Let $X = (-\infty, +\infty)$ with the usual norm $|\cdot|$ and $K = [-1, +\infty)$. Define $T : K \to K$ and $\phi : [0, +\infty) \to [0, +\infty)$ by

$$Tx = \begin{cases} \frac{x}{1+x} & \text{for } x \in [0, +\infty) \\ \sqrt{-x} & \text{for } x \in [-1, 0) \end{cases}$$

and

$$\phi(t) = \frac{t^2}{1+t} \quad \text{for } t \in [0, +\infty),$$

respectively. Set

$$a_n = 1 - \frac{1}{4(n+1)} - \frac{1}{8(n+1)^2}, \quad b_n = \frac{1}{4(n+1)}, \quad c_n = \frac{1}{8(n+1)^2},$$
$$a'_n = 1 - \frac{2}{3\sqrt{n+1}}, \quad b'_n = c'_n = \frac{1}{3\sqrt{n+1}}, \quad t_n = \frac{1}{2(n+1)}, \quad n \geq 0,$$

and $\{\sigma_n\}_{n=0}^{\infty}$ and $\{\delta_n\}_{n=0}^{\infty}$ are arbitrary bounded sequences in K. It is easy to verify that T has both a fixed point $0 \in K$ and is uniformly continuous and the range of T is bounded. In order to show that T is ϕ-strongly pseudocontractive, for any $x, y \in K$with $x \geq y$, we consider the following three cases:

Case 1. Suppose that $x, y \in [0, +\infty)$. It follows that

$$\begin{aligned} &\langle Tx - Ty, x - y\rangle - |x-y|^2 + \phi(|x-y|)|x-y| \\ &= \Big(\frac{x}{1+x} - \frac{y}{1+y}\Big)(x-y) - |x-y|^2\Big(1 - \frac{|x-y|}{1+|x-y|}\Big) \\ &= (x-y)^2 \frac{|x-y| - x - y - xy}{(1+x)(1+y)(1+|x-y|)} \\ &\leq 0; \end{aligned}$$

Case 2. Suppose that $x, y \in [-1, 0)$. Then

$$\begin{aligned} &\langle Tx - Ty, x - y\rangle - |x-y|^2 + \phi(|x-y|)|x-y| \\ &= (\sqrt{-x} - \sqrt{-y})(x-y) - |x-y|^2\Big(1 - \frac{|x-y|}{1+|x-y|}\Big) \\ &\leq 0; \end{aligned}$$

Case 3. Suppose that $x \in [0, +\infty)$ and $y \in [-1, 0)$. It is easy to see that

$$\begin{aligned}
&\langle Tx - Ty, x - y\rangle - |x-y|^2 + \phi(|x-y|)|x-y| \\
&= \left(\frac{x}{1+x} - \sqrt{-y}\right)(x-y) - |x-y|^2 + \frac{|x-y|^3}{1+|x-y|} \\
&= (x-y)\frac{y}{(1+x)(1+x-y)} - \sqrt{-y}(x-y) \\
&\le 0.
\end{aligned}$$

Thus T is ϕ-strongly pseudocontractive. Consequently, Theorem 2.4 ensures the convergence and equivalence of the Mann and Ishikawa iteration methods with errors for the uniformly continuous and ϕ-strongly pseudocontractive operator T in K. But we can not invoke Theorems 3.4 and 4.2 in [1], Theorem 1 in [3], Theorem 1 and Corollary 1 in [4], [5], Theorem 4 in [17], Theorem 2.3 in [18] and Theorem 10 in [19] to show the convergence and equivalence of the Mann and Ishikawa iteration methods with errors for the uniformly continuous and ϕ-strongly pseudocontractive operator T in K since T is neither strongly pseudocontractive nor Lipschitz. In fact, for any $t > 1$ there exist $(x_t, y_t) = (\frac{t-1}{2}, 0) \in K \times K$ such that

$$\begin{aligned}
\langle Tx_t - Ty_t, x_t - y_t\rangle - \frac{1}{t}|x_t - y_t|^2 &= \frac{x_t^2}{1+x_t} - \frac{1}{t}x_t^2 \\
&= x_t^2 \frac{t-1}{(t+1)t} \\
&> 0,
\end{aligned}$$

that is, T is not strongly pseudocontractive in K. Note that

$$\lim_{x\to 0^-} \frac{Tx - T0}{x - 0} = \lim_{x \to 0^-} \frac{\sqrt{-x}}{x} = -\infty.$$

This means that T is not Lipschitz.

In the end, we pose the following problems.

Problem 3.1. If the uniform continuity and ϕ-strong pseudocontractivity of the operator T in Theorem 2.4 are replaced by the Lipschitz continuity and ϕ-hemicontractivity, respectively, dose the equivalence conclusion in Theorem 2.4 hold ?

Problem 3.2. Is it possible to extend Theorem 2.4 to the case where $T : K \to K$ is a uniformly continuous and ϕ-hemicontractive operator ?

Remark 3.4. As in the proof of Theorem 3.3 and Corollary 3.3 in [10], we can easily conclude that the convergence of the Mann and Ishikawa iteration methods with errors for uniformly continuous and ϕ-hemicontractive operators, respectively.

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