International Journal of Pure and Applied Mathematics
Volume 71 No. 3 2011, 329-334

TEST RANK OF THE LIE ALGEBRA F/R'

Nazar Şahin Öğüşlü
Department of Mathematics
Çukurova University
Adana, TURKEY

Abstract: Let F be a free Lie algebra of rank $n \geq 2$ and $R = F^{n_1,n_2,\ldots,n_k}$. We prove that the test rank of the Lie algebra F/R' is $n-1$.

AMS Subject Classification: 17B01, 17B40
Key Words: automorphism, free Lie algebras, test elements, test sets, test rank

1. Introduction

Test elements and test sets have an important place in Groups and Lie algebras. In [7] it was proved that test rank of a free metabelian group is less by 1 than its rank. The similar result was obtained for polynilpotent groups of finite rank [5] and for free solvable groups [8]. Analogous studies have been done for free Lie algebras. In [2] it was proved that test rank of a free metabelian Lie algebra of rank n is equal to $n-1$ and also in [3] it was proved that for an ideal R of a free Lie algebra F of finite rank n such that the universal enveloping algebra $U(F/R)$ is an integral domain satisfying the Ore condition, test rank of the free Lie algebra F/R' is equal to $n-1$ or n. In [9], it was proved that the test rank of a free solvable Lie algebra of rank $n \geq 2$ is equal to $n-1$.

Let F be a free Lie algebra of rank $n \geq 2$ over a field K. Let F^k be the $k-$th term of the lower central series of F. Then Let denote by $F^{n_1,n_2,\ldots,n_k}$ n_k-th term of the lower central series of $F^{n_1,n_2,\ldots,n_{k-1}}$. In this note we prove that the test rank of the Lie algebra F/R' is equal to $n-1$ where $R = F^{n_1,n_2,\ldots,n_k}$ and R' is the commutator subalgebra $[R, R]$.

Received: June 6, 2009

2. Preliminaries

Let L be a free Lie algebra of rank n over a field K.

Definition 1. A subset $\{g_1, ..., g_m\}$, $m \leq n$ of L, is said to be test set if for every endomorphisim φ of L, the conditions $\varphi(g_i) = g_i$, $i = 1, 2, ..., m$ imply that φ is an automorphism.

Definition 2. The least length of a test set of L is called the test rank of L.

By $U(L)$ we denote the universal enveloping algebra of L. It is well known that $U(L)$ is an integral domain. Denote by a commutator the product in a Lie algebra L over a field K. We consider L as a Lie subalgebra of $U(L)$ with respect to Lie operation $[u, v] = uv - vu$, $u, v \in U(L)$.

Let F be a free Lie algebra with free generators $x_1, x_2, ..., x_n$ over a field K. We denote by $\frac{\partial}{\partial x_i}$, $1 \leq i \leq n$ the left Fox derivatives [4]. The operators $\frac{\partial}{\partial x_i} : U(F) \to U(F)$ are linear mappings such that $\frac{\partial x_j}{\partial x_i} = \delta_{ij}$ (Kronecker delta), $\frac{\partial(u+v)}{\partial x_i} = \frac{\partial u}{\partial x_i} + \frac{\partial v}{\partial x_i}$, $\frac{\partial(uv)}{\partial x_i} = \frac{\partial u}{\partial x_i}\varepsilon(v) + u\frac{\partial v}{\partial x_i}$, where $\varepsilon : U(F) \to K$ is the homomorphism defined as $\varepsilon(x_i) = 0$ for all $i = 1, ..., n$. By Δ we denote the augmentation ideal of $U(F)$. The ideal Δ is a free left $U(F)$-module with basis $\{x_1, x_2, ..., x_n\}$. Thus any element $u \in \Delta$ can be uniquely written in the form $u = \sum\limits_{i=1}^{n} \frac{\partial u}{\partial x_i} x_i$.

Let Let R be an ideal of F. By I_R we denote the right ideal of $U(F)$ generated by R.

Let $a, b \in F$ and $m \in \mathbb{Z}^+$. In terms of shortness, we write $[a, b^m]$ instead of $[[...[[a, \underbrace{b], b], ...,], b}_{m-\text{times}}]$.

3. Test Rank of F/R'

Let F be a free Lie algebra generated by the free generating set $\{x_1, x_2, ..., x_n\}$ and R be an ideal of F. For any element u of F we denote by $\overline{u}$ and $\widehat{u}$ the images of u under the natural homomorphisms $F \to F/R$ and $F \to F/R'$ respectively,where R' is the derived subalgebra $[R, R]$. On the universal enveloping algebra $U\left(F/R'\right)$, the left Fox derivatives $\frac{\partial}{\partial x_i}$ are defined so that their values are in $U(F/R)$. For every element $\widehat{u}$ of $F/R^{'}$ the equality

$$\sum_{i=1}^{n} \frac{\partial \widehat{u}}{\partial x_i} \overline{x_i} = \overline{u}$$

is satisfied in $U(F/R)$. Therefore, if

$$\overline{u} = \sum_{i=1}^{n} \overline{c_i x_i}$$

for some $\overline{c_i} \in U(F/R)$, then there exists an element $\widehat{v} \in F/R^{'}$ such that $\overline{u} = \overline{v}$ and $\overline{c_i} = \frac{\partial \widehat{v}}{\partial x_i}$, $1 \leq i \leq n$.

The subalgebra $R/R^{'}$ of $F/R^{'}$ is endowed with a left $U(F/R)$-module structure. The module action on the elements of $R/R^{'}$ is defined as

$$v_1 ... v_t . \widehat{u} = [v_1, [..., [v_t, \widehat{u}], ...]],$$

where v_1,...,$v_t \in F/R$, $\widehat{u} \in R/R^{'}$.

Proposition 3. *Let R be an ideal of F and $\widehat{u}(\widehat{x_1}, ..., \widehat{x_n}) \in F/R^{'}$. Then*

$$\widehat{u}(\alpha_1 \widehat{x_1} + \widehat{r_1}, ..., \alpha_n \widehat{x_n} + \widehat{r_n}) = \widehat{u}(\alpha_1 \widehat{x_1}, ..., \alpha_n \widehat{x_n}) + \sum_{i=1}^{n} \frac{\partial \widehat{u}(\alpha_1 \widehat{x_1}, ..., \alpha_n \widehat{x_n})}{\partial x_i} . \widehat{r_i} \quad (1)$$

Proof. Without loss of generality we may assume that $\widehat{u}(\widehat{x_1}, ..., \widehat{x_n})$ is a monomial. Now (1) is may be obtained by induction on the length of $\widehat{u}(\widehat{x_1}, ..., \widehat{x_n})$. □

Lemma 4. *Let F be a free Lie algebra generated by the free generating set $\{x_1, x_2, ..., x_n\}$, $R = F^{n_1, n_2, ..., n_k}$ and $L = F/R^{'}$. Let for $2 \leq i \leq n$ $g_i(x_1, x_i)$ be elements of $R \backslash R^{'}$ depending on the free generators x_1 and x_i. Let φ be an endomorphism of L such that $\varphi(\widehat{x_i}) = \widehat{y_i}$. If for $2 \leq i \leq n$,*

$$\varphi([[g_i(\widehat{x_1}, \widehat{x_i}), \widehat{x_1}^m], \widehat{x_i}^m]) = [[g_i(\widehat{x_1}, \widehat{x_i}), \widehat{x_1}^m], \widehat{x_i}^m],$$

then, for all $1 \leq j \leq n$, $y_j \notin F^{'}$.

Proof. Since $g_i = \frac{\partial g_i}{\partial x_1} . x_1 + \frac{\partial g_i}{\partial x_i} . x_i$ and $g_i \in R \backslash R^{'}$ then in $U(F/R)$ $\frac{\partial g_i}{\partial x_1} \neq 0$ and $\frac{\partial g_i}{\partial x_i} \neq 0$. Let for an $k \in \mathbb{Z}^+$ $\frac{\partial g_i}{\partial x_1}, \frac{\partial g_i}{\partial x_i} \in \Delta^k$. If

$$\varphi([[g_i(\widehat{x_1}, \widehat{x_i}), \widehat{x_1}^m], \widehat{x_i}^m]) = [[g_i(\widehat{x_1}, \widehat{x_i}), \widehat{x_1}^m], \widehat{x_i}^m],$$

then,

$$[[g_i(\widehat{y_1},\widehat{y_i}),\widehat{y_1}^m],\widehat{y_i}^m]=[[g_i(\widehat{x_1},\widehat{x_i}),\widehat{x_1}^m],\widehat{x_i}^m]$$

and

$$[[g_i(y_1,y_i),y_1^m],y_i^m]=[[g_i(x_1,x_i),x_1^m],x_i^m] \bmod R'.$$

Now we compute Fox derivatives of the left and right parts of this equality, in $U(F/R)$ we obtain:

$$y_i^m y_1^m \frac{\partial g_i(y_1,y_i)}{\partial x_1}=x_i^m x_1^m \frac{\partial g_i}{\partial x_1} \bmod I_R. \tag{2}$$

Since in $U(F/R)$ $\frac{\partial g_i}{\partial x_1}\neq 0$, then $\frac{\partial g_i(y_1,y_i)}{\partial x_1}\neq 0$ in $U(F/R)$. Hence, since $\frac{\partial g_i(y_1,y_i)}{\partial x_1}=\frac{\partial g_i}{\partial y_1}\frac{\partial y_1}{\partial x_1}+\frac{\partial g_i}{\partial y_i}\frac{\partial y_i}{\partial x_1}$, then in $U(F/R)$ $\frac{\partial g_i}{\partial y_1}\neq 0$ or $\frac{\partial g_i}{\partial y_i}\neq 0$. Let for an $s\in\mathbb{Z}^+$ $\frac{\partial g_i}{\partial y_1},\frac{\partial g_i}{\partial y_i}\in\Delta^s$. Then it is clear that $k\leq s$. Let $\frac{\partial g_i(y_1,y_i)}{\partial x_1}\in\Delta^t$. Since $\frac{\partial g_i(y_1,y_i)}{\partial x_1}=\frac{\partial g_i}{\partial y_1}\frac{\partial y_1}{\partial x_1}+\frac{\partial g_i}{\partial y_i}\frac{\partial y_i}{\partial x_1}$, we obtain $s\leq t$. Hence we obtain $k\leq t$. Now we suppose that y_1 or $y_i\in F'$. Then the right part of the equality (2) belongs to Δ^{2m+k} and the left part of the equality (2) belongs to Δ^{2m+k+l}, $l\geq 1$. This is a contradiction. Hence, for all $1\leq j\leq n$, $y_j\notin F'$. □

Remark 5. The associative algebra $U(F/R)$ is an Ore domain and it can be embeddable in a skew field $Q(F/R)$ of fractions [6]. Thus the tensor product $V=R/R'\otimes_{U(F/R)}Q(F/R)$ is a vector space over $Q(F/R)$ and $\dim V=n-1$. Consider the nonzero elements $\widehat{g_j}=\widehat{g_j}(\widehat{x_1},\widehat{x_j})$, $j=2,...,n$ in R/R' such that $\widehat{g_j}$ is a monomial of $\widehat{x_1}$ and $\widehat{x_j}$. The vector space V generated as a module over $Q(F/R)$ by the set $\{g_2,...,g_n\}$. For more details see [6].

The proof of the following theorem can be found in [1].

Theorem 6. *Let R be a verbal ideal of F such that F/R is a free polynilpotent Lie algebra. If an endomorphism φ of F/R' acts identically on R/R' then φ is an inner automorphism of F/R' induced by some element of R/R'.*

Theorem 7. *Let F be a free Lie algebra generated by the free generating set $\{x_1,x_2,...,x_n\}$, $R=F^{n_1,n_2,...,n_k}$ and $L=F/R'$. Then test rank of L is $n-1$.*

Proof. Let m be a positive integer. Consider $g_2,...,g_n\in R\backslash R'$ such that for $2\leq i\leq n$, g_i belongs to the subalgebra generated by x_1 and x_i. Let $\widehat{h_i}=[[\widehat{g_i},\widehat{x_1}^m],\widehat{x_i}^m]$. Verify that $\left\{\widehat{h_2},...,\widehat{h_n}\right\}$ is a test set. Take the endomorphism

φ such that $\varphi(\widehat{x_i}) = \widehat{y_i}$, $1 \le i \le n$ and suppose that $\varphi\left(\widehat{h_i}\right) = \widehat{h_i}$, $2 \le i \le n$. We will prove that φ is an automorphisim. Since $\varphi\left(\widehat{h_i}\right) = \widehat{h_i}$, then,

$$[[g_i(\widehat{y_1}, \widehat{y_i}), \widehat{y_1}^m], \widehat{y_i}^m] = [[g_i(\widehat{x_1}, \widehat{x_i}), \widehat{x_1}^m], \widehat{x_i}^m]$$

and

$$[[g_i(y_1, y_i), y_1^m], y_i^m] = [[g_i(x_1, x_i), x_1^m], x_i^m] \operatorname{mod} R'.$$

Compute the Fox derivatives of the both side of this equality. Then we obtain that

$$y_i^m y_1^m \frac{\partial g_i(y_1, y_i)}{\partial x_1} = x_i^m x_1^m \frac{\partial g_i}{\partial x_1} \operatorname{mod} I_R.$$

By lemma 4, $y_1, ..., y_n \notin F'$. The rest of the proof is identical with the proof of main theorem of [9]. By the corollary of lemma 2 in [9] we obtain that $y_i = \alpha_i x_i + u_i$, $\alpha_i \in K$, $u_i \in R$. Define the endomorphisms θ and Ψ of L such that for $1 \le i \le n$, $\theta(\widehat{x_i}) = \alpha_i \widehat{x_i}$ and $\Psi(\widehat{x_i}) = \widehat{x_i} + \widehat{v_i}$ $\left(\widehat{v_i} = \theta^{-1}(\widehat{u_i}) \in R/R'\right)$. Then we can write $\varphi = \Psi\theta$. By lemma 4, for $1 \le i \le n$, $\alpha_i \ne 0$. Thus, θ is an automorphisim. By proposition 3,

$$\widehat{h_j} = \varphi\left(\widehat{h_j}\right) = \theta\left(\widehat{h_j}\right) + \sum_{i=1}^{n} \frac{\partial \theta\left(\widehat{h_j}\right)}{\partial x_i} . \widehat{u_i}.$$

Since $u_i \in R$, all nonzero summands $\frac{\partial \theta\left(\widehat{h_j}\right)}{\partial x_i} . \widehat{u_i}$ have greater length than $\theta\left(\widehat{h_j}\right)$. Therefore, $\sum_{i=1}^{n} \frac{\partial \theta\left(\widehat{h_j}\right)}{\partial x_i} . \widehat{u_i} = 0$. It means that $\theta\left(\widehat{h_j}\right) = \widehat{h_j}$ $(2 \le j \le n)$. Hence, it is sufficient to prove that Ψ is an automorphism. In other words, we may assume that $\theta = 1$. By Remark 5 the elements $\widehat{h_2}, ..., \widehat{h_n}$ generate the vector space $R/R' \otimes_{U(F/R)} Q(F/R)$ as a module over $Q(F/R)$. Therefore, for every element $\widehat{r}$ of R/R' we can find $c_1, c_2, ..., c_n \in U(F/R)$ such that $c_1\widehat{r} = c_2\widehat{h_2} + ... + c_n\widehat{h_n}$. Since φ induces the identity over F/R; it can be extended to an endomorphism of the $Q(F/R)$-module R/R'. This extension is the identity and consequently φ induces the identical endomorphism over R/R'. By theorem 6, φ is an inner automorphism induced by an element of R/R'. This shows that $\left\{\widehat{h_2}, ..., \widehat{h_n}\right\}$ is a test set. In [3] it was shown that the minimal cardinality of a test set of L is $n - 1$. Hence test rank of L is equal to $n - 1$. □

Acknowledgments

The author gratefully acknowledges the financial support for his PhD thesis of the Scientific and Technical Research Council of Turkey (TÜBİTAK-B İDEB).

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