WEAKLY ULTRA SEPARATION AXIOMS VIA $\alpha\psi$-OPEN SETS

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Abstract: In this paper, we introduce the concept of weakly ultra-$\alpha\psi$-separation of two sets in a topological space using $\alpha\psi$-open sets. The $\alpha\psi$-closure and the $\alpha\psi$-kernel are defined in terms of this weakly ultra-$\alpha\psi$-separation. We also investigate some of the properties of weak separation axioms like $\alpha\psi$-$T_0$ and $\alpha\psi$-$T_1$ spaces.

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1. Introduction

The notion of $\alpha\psi$-closed set was introduced and studied by R. Devi et al (see [2]). In this paper, we define that a set $A$ is weakly ultra-$\alpha\psi$-separated from $B$ if there exists a $\alpha\psi$-open set $G$ containing $A$ such that $G \cap B = \phi$. Using this concept, we define the $\alpha\psi$-closure and the $\alpha\psi$-kernel. Also we define the $\alpha\psi$-derived set and the $\alpha\psi$-shell of a set $A$ of a topological space $(X, \tau)$.

Throughout this paper, spaces means topological spaces on which no separation axioms are assumed unless otherwise mentioned. Let $A$ be a subset of a space $X$. The closure and the interior of $A$ are denoted by $cl(A)$ and $int(A)$, respectively.

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2. Preliminaries

Before entering to our work, we recall the following definitions, which are useful in the sequel.

**Definition 1.** A subset $A$ of a space $(X, \tau)$ is called

(i) a *semi-open* set (see [4]) if $A \subseteq \text{cl}(\text{int}(A))$ and a *semi-closed* set if $\text{int}(\text{cl}(A)) \subseteq A$

(ii) an *$\alpha$-open* set (see [5]) if $A \subseteq \text{int}(\text{cl}(\text{int}(A)))$ and an *$\alpha$-closed* set if $\text{cl}(\text{int}(\text{cl}(A))) \subseteq A$.

The *semi-closure* (resp. *$\alpha$-closure*) of a subset $A$ of a space $(X, \tau)$ is the intersection of all *semi-closed* (resp. *$\alpha$-closed*) sets that contain $A$ and is denoted by $\text{scl}(A)$ (resp. $\text{\alpha\text{-}cl}(A)$).

**Definition 2.** A subset $A$ of a topological space $(X, \tau)$ is called

(i) a *semi-generalized closed* (briefly *sg-closed*) set (see [1]) if $\text{scl}(A) \subseteq U$ whenever $A \subseteq U$ and $U$ is *semi-open* in $(X, \tau)$. The complement of *sg-closed* set is called *sg-open* set,

(ii) a *$\psi$-closed* set (see [6]) if $\text{scl}(A) \subseteq U$ whenever $A \subseteq U$ and $U$ is *sg-open* in $(X, \tau)$. The complement of *$\psi$-closed* set is called *$\psi$-open* set and

(iii) a *$\alpha\psi$-closed* set (see [2]) if $\text{\psi\text{-}cl}(A) \subseteq U$ whenever $A \subseteq U$ and $U$ is *$\alpha$-open* in $(X, \tau)$. The complement of *$\alpha\psi$-closed* set is called *$\alpha\psi$-open* set.

The *$\alpha\psi$-closure* of a subset $A$ of a space $(X, \tau)$ is the intersection of all *$\alpha\psi$-closed* sets that contain $A$ and is denoted by $\alpha\psi\text{cl}(A)$ or $\alpha\psi\text{-cl}(A)$. The *$\alpha\psi$-interior* of a subset $A$ of a space $(X, \tau)$ is the union of all *$\alpha\psi$-open* sets that are contained in $A$ and is denoted by $\alpha\psi\text{int}(A)$. By $\alpha\psi\text{O}(X, \tau)$ or $\alpha\psi\text{O}(X)$, we denote the family of all *$\alpha\psi$-open* sets of $(X, \tau)$.

**Definition 3.** The intersection of all *$\alpha\psi$-open* subsets of $(X, \tau)$ containing $A$ is called the *$\alpha\psi$-kernel* of $A$ (briefly, $\alpha\psi\text{-ker}(A)$), i.e.

$$\alpha\psi\text{-ker}(A) = \cap\{G \in \alpha\psi\text{O}(X) : A \subseteq G\}.$$ 

**Definition 4.** Let $x \in X$. Then *$\alpha\psi$-kernel* of $x$ is denoted by $\alpha\psi\text{-ker}\{x\} = \cap\{G \in \alpha\psi\text{O}(X) : x \in G\}$.

**Definition 5.** Let $X$ be a topological space and $x \in X$, then a subset $N_x$ of $X$ is called an *$\alpha\psi$-neighborhood* (briefly, *$\alpha\psi$-nbd*) of $X$ if there exists an *$\alpha\psi$-open* set $G$ such that $x \in G \subseteq N_x$. 
Definition 6. In a space $X$, a set $A$ is said to be *weakly ultra-$\alpha\psi$-separated* from a set $B$ if there exists an $\alpha\psi$-open set $G$ such that $A \subseteq G$ and $G \cap B = \phi$ or $A \cap \alpha\psi\text{cl}(B) = \phi$.

By the definition 6, we have the following for $x, y \in X$ of a topological space,

(i) $\alpha\psi\text{-cl}({x}) = \{ y : \{ y \} \text{ is not weakly ultra-} \alpha\psi\text{-separated from} \{ x \} \}$
(ii) $\alpha\psi\text{-ker}({x}) = \{ y : \{ y \} \text{ is not weakly ultra-} \alpha\psi\text{-separated from} \{ x \} \}$.

Definition 7. For any point $x$ of a space $X$:

(i) The $\alpha\psi$-derived (briefly, $\alpha\psi\text{-d}(\{x\})$) set of $x$ is defined to be the set $\alpha\psi\text{-d}(\{x\}) = \alpha\psi\text{-cl}(\{x\}) \setminus \{x\} = \{ y : y \neq x \text{ and } \{ y \} \text{ is not weakly ultra-} \alpha\psi\text{-separated from} \{ x \} \}$,

(ii) the $\alpha\psi$-shell (briefly, $\alpha\psi\text{-shl}(\{x\})$) of a singleton set $\{x\}$ is defined to be the set $\alpha\psi\text{-shl}(\{x\}) = \alpha\psi\text{-ker}(\{x\}) \setminus \{x\} = \{ y : y \neq x \text{ and } \{ y \} \text{ is not weakly ultra-} \alpha\psi\text{-separated from} \{ y \} \}$.

Definition 8. Let $X$ be a topological space. Then we define

(i) $\alpha\psi\text{-N-D} = \{ x : x \in X \text{ and } \alpha\psi\text{-d}(\{x\}) = \phi \}$.
(ii) $\alpha\psi\text{-N-shl} = \{ x : x \in X \text{ and } \alpha\psi\text{-shl}(\{x\}) = \phi \}$.
(iii) $\alpha\psi\langle x \rangle = \alpha\psi\text{-cl}(\{x\}) \cap \alpha\psi\text{-ker}(\{x\})$.

3. $\alpha\psi$-$T_i$ Spaces, $i=0, 1$

Definition 9. A topological space $X$ is said to be $\alpha\psi$-$T_0$ if for $x, y \in X$, $x \neq y$, there exists $U \in \alpha\psi\text{O}(X)$ such that $U$ contains only one of $x$ or $y$ but not the other.

Definition 10. (see [3]) A topological space $X$ is said to be $\alpha\psi$-$T_1$ if for $x, y \in X$, $x \neq y$, there exist $U, V \in \alpha\psi\text{O}(X)$ such that $x \in U$ and $y \in V$ but $y \notin U$ and $x \notin V$.

Remark 11. Every $\alpha\psi$-$T_1$ space is $\alpha\psi$-$T_0$.

Theorem 12. A topological space $X$ is an $\alpha\psi$-$T_0$ space if and only if the $\alpha\psi$-closure of distinct points are distinct.
Proof. Let \( x \neq y \) implies \( \alpha\psi\text{-cl}\{x\} \neq \alpha\psi\text{-cl}\{y\} \). Then there exists at least one \( z \in \alpha\psi\text{-cl}\{x\} \) but \( z \notin \alpha\psi\text{-cl}\{y\} \). Let \( x \in \alpha\psi\text{-cl}\{y\} \). Then \( \alpha\psi\text{-cl}\{x\} \subseteq \alpha\psi\text{-cl}\{y\} \), which is a contradiction that \( z \notin \alpha\psi\text{-cl}\{y\} \). Hence \( x \in X - \alpha\psi\text{-cl}\{y\} \). Conversely, let \( X \) be an \( \alpha\psi\text{-T}_0 \) space. Take \( x, y \in X \) and \( x \neq y \). Then there exists an \( \alpha\psi\text{-open set} \) \( G \) such that \( x \in G \) and \( y \notin G \), which implies \( y \in X - G = F(\text{say}) \). Now \( \alpha\psi\text{-cl}\{y\} = \cap \{ F : \text{cl}\{y\} \subseteq F \text{ and } F \text{ is an } \alpha\psi \text{-closed set} \} \). This implies that \( y \notin \alpha\psi\text{-cl}\{y\} \) and \( x \notin \alpha\psi\text{-cl}\{y\} \).

Theorem 13. A space \( X \) is \( \alpha\psi\text{-T}_0 \) if and only if any of the following conditions holds.

(i) For arbitrary \( x, y \in X \), \( x \neq y \), either \( \{x\} \) is weakly ultra-\( \alpha\psi \)-separated from \( \{y\} \) or \( \{y\} \) is weakly ultra-\( \alpha\psi \)-separated from \( \{x\} \).

(ii) \( y \in \alpha\psi\text{-cl}\{x\} \) implies \( x \notin \alpha\psi\text{-cl}\{y\} \).

(iii) For all \( x, y \in X \) if \( x \neq y \), then \( \alpha\psi\text{-cl}\{x\} \neq \alpha\psi\text{-cl}\{y\} \).

Proof. (i) Obvious from the definitions of \( \alpha\psi\text{-T}_0 \) and weakly ultra-\( \alpha\psi \)-separation.

(ii) By assumption, \( y \in \alpha\psi\text{-cl}\{x\} \) and so \( \{y\} \) is not weakly ultra-\( \alpha\psi \)-separated from \( \{x\} \). As \( X \) is \( \alpha\psi\text{-T}_0 \), \( \{x\} \) should be weakly ultra-\( \alpha\psi \)-separated from \( \{y\} \), that is \( x \notin \alpha\psi\text{-cl}\{y\} \).

(iii) If \( X \) is \( \alpha\psi\text{-T}_0 \), then for all \( x, y \in X \) and \( x \neq y \), \( \alpha\psi\text{-cl}\{x\} \neq \alpha\psi\text{-cl}\{y\} \) as evidenced by (ii). Now let us prove the converse. Let \( \alpha\psi\text{-cl}\{x\} \neq \alpha\psi\text{-cl}\{y\} \). Then there exists \( z \in X \), such that \( z \in \alpha\psi\text{-cl}\{x\} \) and \( z \notin \alpha\psi\text{-cl}\{y\} \). If \( \{x\} \) is not weakly ultra-\( \alpha\psi \)-separated from \( \{y\} \), then \( x \in \alpha\psi\text{-cl}\{y\} \). So \( \alpha\psi\text{-cl}\{x\} \subseteq \alpha\psi\text{-cl}\{y\} \). Then \( z \in \alpha\psi\text{-cl}\{y\} \), which is a contradiction. □

Theorem 14. A space \( X \) is \( \alpha\psi\text{-T}_0 \) if and only if \( \alpha\psi\text{-d}(\{x\}) \cap \alpha\psi\text{-shl}(\{x\}) = \phi \).

Proof. Let \( X \) be \( \alpha\psi\text{-T}_0 \). Suppose we have \( \alpha\psi\text{-d}(\{x\}) \cap \alpha\psi\text{-shl}(\{x\}) = \phi \). Let \( z \in \alpha\psi\text{-d}(\{x\}) \) and \( z \in \alpha\psi\text{-shl}(\{x\}) \). Then \( z \neq x \) and \( z \in \alpha\psi\text{-cl}(\{x\}) \) and \( z \in \alpha\psi\text{-ker}(\{x\}) \). Then \( \{z\} \) is not weakly ultra-\( \alpha\psi \)-separated from \( \{x\} \) and also \( \{x\} \) is not weakly ultra-\( \alpha\psi \)-separated from \( \{z\} \), which is a contradiction.

Conversely, let \( \alpha\psi\text{-d}(\{x\}) \cap \alpha\psi\text{-shl}(\{x\}) = \phi \). Then there exists \( z \neq x \), \( z \in \alpha\psi\text{-cl}(\{x\}) \) and \( z \notin \alpha\psi\text{-ker}(\{x\}) \). Hence if we have \( \{z\} \), which is not weakly ultra-\( \alpha\psi \)-separated from \( \{x\} \), then \( \{x\} \) is weakly ultra-\( \alpha\psi \)-separated from \( \{z\} \). Hence, \( X \) is \( \alpha\psi\text{-T}_0 \). □
Theorem 15. A topological space $X$ is $\alpha\psi$-$T_1$ if and only if $\{x\}$ is $\alpha\psi$-closed in $X$ for every $x \in X$.

Proof. If $\{x\}$ is $\alpha\psi$-closed in $X$, for $x \neq y$, $X - \{x\}$, $X - \{y\}$ are $\alpha\psi$-open sets such that $y \in X - \{x\}$ and $x \in X - \{y\}$. Therefore, $X$ is $\alpha\psi$-$T_1$.

Conversely, if $X$ is $\alpha\psi$-$T_1$ and if $y \in X - \{x\}$ then $x \neq y$. Therefore, there exists an $\alpha\psi$-open sets $U_x$, $V_y$ in $X$ such that $x \in U_x$ but $y \notin U_x$ and $y \in V_y$ but $x \notin V_y$. Let $G$ be the union of all such $V_y$. Then $G$ is an $\alpha\psi$-open set and $G \subseteq X - \{x\} \subseteq X$. Therefore, $X - \{x\}$ is a $\alpha\psi$-open set in $X$. \qed

Theorem 16. A topological space is an $\alpha\psi$-$T_1$ space if and only if every subset of $X$ is an $\alpha\psi$-closed set.

Proof. Assume that every singleton subset $\{x\}$ of $X$ is a $\alpha\psi$-closed set. A finite subset of $X$ is the union of finite number of singleton sets. Hence $\alpha\psi$-closed. Conversely, every singleton set $\{x\}$ is a finite subset of $X$. \qed

Theorem 17. A space $X$ is $\alpha\psi$-$T_1$ if and only if one of the following conditions holds.

(i) For arbitrary $x, y \in X$, $x \neq y$, $\{x\}$ is weakly ultra-$\alpha\psi$-separated from $\{y\}$.

(ii) For every $x \in X$, $\alpha\psi$-cl($\{x\}$) = $\{x\}$.

(iii) For every $x \in X$, $\alpha\psi$-d($\{x\}$) = $\phi$ or $\alpha\psi$-$N-D = X$.

(iv) For every $x \in X$, $\alpha\psi$-ker($\{x\}$) = $\{x\}$.

(v) For every $x \in X$, $\alpha\psi$-shl($\{x\}$) = $\phi$ or $\alpha\psi$-$N-shl = X$.

(vi) For every $x, y \in X$, $x \neq y$, $\alpha\psi$-cl($\{x\}$) $\cap$ $\alpha\psi$-cl($\{y\}$) = $\phi$.

(vii) For every arbitrary $x, y \in X$, $x \neq y$, we have $\alpha\psi$-ker($\{x\}$)$\cap$ $\alpha\psi$-ker($\{y\}$) = $\phi$.

Proof. (i) This is just a reformulation of the definition of $\alpha\psi$-$T_1$.

(ii) If $\{x\}$ is weakly ultra-$\alpha\psi$-separated from $\{y\}$, then for $y \neq x$, we have $y \notin \alpha\psi$-cl($\{x\}$), and hence $x \notin \alpha\psi$-ker($\{y\}$). Therefore we get that $\alpha\psi$-ker($\{y\}$) = $\{y\}$. Its converse is just a reformulation of the above proof.

(iii), (iv) and (v) are obvious.

(vi) As $X$ is $\alpha\psi$-$T_1$, $\alpha\psi$-cl($\{x\}$) = $\{x\}$ and $\alpha\psi$-cl($\{y\}$) = $\{y\}$. So, when $x \neq y$, $\alpha\psi$-cl($\{x\}$) $\cap$ $\alpha\psi$-cl($\{y\}$) = $\phi$. 

(vii) Obvious from (vi).

**Theorem 18.** If $X$ is an $\alpha\psi$-$T_1$ space, then the intersection of $\alpha\psi$-nbd of an arbitrary point of $X$ is a singleton set.

**Proof.** Let $X$ be an $\alpha\psi$-$T_1$ space. Also let $x \in X$ and $N_x$ be the $\alpha\psi$-nbd of $x$. If $y$ is a point of $X$ and $y \neq x$, then there exists an $\alpha\psi$-open set containing $x$ but not $y$. Since $y$ is arbitrary, $N_x$ has no point other than $x$. Conversely, the intersection of $\alpha\psi$-nbd of $x$ is the singleton set $\{x\}$, which does not contain any other point $y$. Hence $X$ is an $\alpha\psi$-$T_1$ space.

**References**


