International Journal of Pure and Applied Mathematics
Volume 71 No. 3 2011, 351-360

A VISCOSITY THREE STEP ITERATION SCHEMES AND COMMON FIXED POINTS OF THREE GENERALIZED QUASI NON EXPANSIVE MAPPINGS

H.K. Pathak[1], Vinod Kumar Sahu[2 §]

[1]S.O.S. in Mathematics
Pt. Ravishankar Shukla University,
Raipur (C.G.), 492010, INDIA
[2]Department of Mathematics
Govt. V.Y.T. Autonomous P.G. College
Durg (C.G.), 491001, INDIA

Abstract: In this paper we introduce a new three step iteration schemes for approximating the common fixed point theorem of three generalized quasi-nonexpansive mappings, and prove strong convergence results in uniformly convex Banach space. These results generalize and extend some known results.

AMS Subject Classification: 47H05, 47H10, 47H17
Key Words: uniformly convex Banach space, asymptotically and generalized quasi-nonexpansive mapping, common fixed point, strong convergence

1. Introduction and Preliminaries

It is well known that the concept of asymptotically nonexpansive mappings was introdcuced by Goebel and Kirk [3] who proved that every asymptotically nonexpansive self mapping of nonempty closed bounded and convex subset of a uniformly convex Banach space has fixed point. Since 1972,the weak and strong convergence problems of iterative sequences with errors for asymptotically nonexpansive types mapping in the Hilbert space and Banach spaces setting have been studied by many authors.

Received: November 11, 2010

§Correspondence author

Let C be a nonempty subset of a real Banach space E. A self mapping $T : C \to C$ is called uniformly L-Lipschitzian if three exists some positive constant L such that

$$\|T^n x - T^n y\| \leq L\|x - y\|$$

for all $x, y \in C$ and for all $n \geq 1$.A self mapping $T : C \to C$ is said to be nonexpansive if

$$\|Tx - Ty\| \leq \|x - y\|$$

for all $x, y \in C$.

A self mapping $T : C \to C$ is said to be asymptotically nonexpansive if there exists a sequence $\{\kappa_n\} \subset [0, \infty)$,$\kappa_n \to 0$ as $n \to \infty$ such that

$$\|T^n x - T^n y\| \leq (1 + \kappa_n)\|x - y\|$$

for all $x, y \in C$ and for all $n \geq 1$. Let $F(T)$ denote the set of all fixed points 0f a mapping T.If $F(T) \neq \Phi$. Then T is called asymptotically quasi nonexpansive if there exists a sequence $\kappa_n \subset [0, \infty)$ with $\lim_{n\to\infty} \kappa_n = 0$ such that

$$\|T^n x - p\| \leq (1 + \kappa_n)\|x - p\|$$

for all $x \in C, p \in F(T)$ and $n \geq 1$.

In 1995 Lui[5] introduced the concept of Ishikawa iteration process with errors by the sequence $\{x_n\}_{n=1}^{\infty}$ defined as follows

$$x_1 \in X,$$

$$x_{n+1} = (1 - \alpha_n)x_n + \alpha_n T y_n + u_n, \quad n = 1, 2, \ldots \tag{1}$$

$$y_n = (1 - \beta_n)x_n + \beta_n T x_n + v_n,$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are two sequence in [0,1] and $\{u_n\}$ and $\{v_n\}$ are bounded sequences in E satisfying the following conditions

$$\Sigma_{n=o}^{\infty}\|u_n\| < \infty, \tag{2}$$

$$\Sigma_{n=o}^{\infty}\|v_n\| < \infty, \tag{3}$$

If $\beta_n = o, n \geq 0$ and $v_n = o, n \geq 0$ then the Ishikawa iteration process with errors (1) reduces to the Mann iteration procedure with errors in the sense of Liu which is defined recursively as follows

$$x_1 \in X,$$

$$x_{n+1} = (1 - \alpha_n)x_n + \alpha_n T x_n + u_n, \quad n = 1, 2, \ldots, \tag{4}$$

with $\{\alpha_n\} \subset [0,1]$ satisfying appropriate conditions and $\{u_n\}$ satisfying condition (2).The Ishikawa iteration process with errors (1) with null sequences $\{u_n\}$ and $\{v_n\}$ clearly reduces to the usual Ishikawa iteration procedure and similarly the Mann iteration procedure with errors (4) with null sequence $\{u_n\}$ reduces to the usual Mann iteration procedure.

A more satisfactory concept of Ishikawa and Mann iterative processes with errors was given by Y.G.Xu[6] as follows:

Let C be a nonempty convex subset of a Banach space E and a mapping $T : C \to C$.The sequence $\{x_n\}_{n=1}^{\infty}$ defined iteratively by,

$$x_1 \in C,$$

$$x_{n+1} = \alpha_n x_n + \beta_n T y_n + \gamma_n u_n, \quad n \geq 1 \tag{5}$$

$$y_n = \lambda_n x_n + \mu_n T x_n + \theta_n v_n$$

where $\{u_n\}$ and $\{v_n\}$ are bounded sequence in C and $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}, \{\lambda_n\}, \{\mu_n\}$, and $\{\theta_n\}$ are sequence in $[0,1]$ such that $\alpha_n + \beta_n + \gamma_n = \lambda_n + \mu_n + \theta_n = 1, n \geq 1$ is called the Ishikawa iteration sequence with errors.If $\lambda_n = 1, n \geq 1$ then the Ishikawa iteration with errors (5) reduces to the mann iteration with errors defined as follows scheme,

$$x_1 \in C,$$

$$x_{n+1} = \alpha_n x_n + \beta_n T y_n + \gamma_n u_n, \quad n \geq 1 \tag{6}$$

with $0 \leq \alpha_n, \beta_n, \gamma_n \leq 1, \alpha_n + \beta_n + \gamma_n = 1, n \geq 1$ and $\{u_n\}$ is bounded sequences in C.

In 2004 H.Fukhar-ud-din and S.H.Khan [2] studied an iterative process with errors in the sense of Liu for two asymptotically nonexpansive mappings in uniformly convex Banach space.

In 2006 J.U Jeong and S.H.Kim [4] studied the Ishikawa scheme with error members for a pair of asymptotically nonexpansive mapping S, T defined as follows:

$$x_1 \in X,$$

$$x_{n+1} = \alpha_n S^n y_n + \beta_n x_n + \gamma_n u_n, \quad n \geq 1 \tag{7}$$

$$y_n = \lambda_n T^n x_n + \mu_n x_n + \theta_n v_n$$

where $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}, \{\lambda_n\}, \{\mu_n\}$, and $\{\theta_n\}$ are sequences in [0,1] with $0 < \delta \leq \alpha_n, \lambda_n \leq (1-\delta) < 1$, $\alpha_n + \beta_n + \gamma_n = \lambda_n + \mu_n + \theta_n = 1$ and $\{u_n\}$ and $\{v_n\}$ are bounded sequences in C.

Let C be a nonempty closed convex subset of E a self map $f : C \to C$ is said to be contraction on C if there exists a constant $\alpha \in$ (0,1)such that

$$\|fx - fy\| \leq \alpha\|x - y\|$$

for all $x, y \in C$.

We introduce a new three step iterative scheme $\{x_n\}_{n=1}^{\infty}$ associated with three asymptotically quasi-nonexpansive mappings $S, T, R : C \to C$ as follows :

$$x_{n+1} = \alpha_n S^n y_n + \beta_n f(x_n) + \gamma_n u_n$$

$$y_n = \lambda_n T^n z_n + \mu_n g(x_n) + \theta_n v_n$$

$$z_n = \xi_n R^n x_n + \eta_n h(x_n) + \zeta_n w_n, \quad n = 1, 2, \ldots, \tag{8}$$

where $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}, \{\lambda_n\}, \{\mu_n\}, \{\theta_n\}, \{\xi_n\}, \{\eta_n\}$ and $\{\zeta_n\}$ are sequence in [0,1] such that $\alpha_n + \beta_n + \gamma_n = 1$, $\lambda_n + \mu_n + \theta_n = 1$, $\xi_n + \eta_n + \zeta_n = 1$, $n \geq 1$ and $\{u_n\}, \{v_n\}, \{w_n\}$,are bounded sequence in C.

If S, T and R are asymptotically nonexpansive mappings such that $S = T = R$,then the iterative procedure (8) reduces to the one introduced by Cho, Zhou and Guo[1] If S, T and R are asymptotically nonexpansive mappings, $S = T = R$ and $u_n = v_n = w_n = 0, n \geq 1$, then scheme (8) reduces to the three step iteration defined by Xu and Noor [7].

In this paper,we prove weak and strong convergence of the (8) to a common fixed point of $S, T$and R under certain restrictions.

In the sequal,we need of the following definations and lemmas.

Definition 1.1. (see [11]) Let E be a real Banach space, C be a nonempty subset of E and $F(T)$ denotes the set of fixed point of T. A self mapping $T : C \to C$,is said to be, generalized quasi-nonexpansive with respect to $\{\kappa_n\}$,if there exists a sequence $\{\kappa_n\} \subset [0, 1)$ with $\kappa_n \to 0$ as $n \to \infty$ such that

$$\|T^n x - p\| \leq \|x - p\| + \kappa_n\|x - T^n x\|$$

for all $x \in C$ and $p \in F(T), n \geq 1$.

Remark 1.2. If $\kappa_n \equiv 0$ for all $n \geq 1$,then the generalized quasi-nonexpansive mapping reduces to the usual asymptotically quasi-nonexpansive mapping.

Lemma 1.3. (see [10]) *Let X be a uniformly convex Banach space,*$0 < \alpha \leq t_n \leq \beta < 1$,*and $\{x_n\}$ and $\{y_n\}$ be two sequences in X such that* $\limsup_{n\to\infty} \|x_n\| \leq l, \limsup_{n\to\infty} \|y_n\| \leq l$ *and* $\lim_{n\to\infty} \|t_n x_n + (1 - t_n)y_n\| = l$ *for some* $l \geq 0$.*Then* $\lim_{n\to\infty} \|x_n - y_n\| = 0$.

Lemma 1.4. (see [12]) *Let $\{a_n\}, \{b_n\}$ and $\{\delta_n\}$ be sequences of nonnegative real numbers satisfying the inequality*

$$a_{n+1} \leq (1 + \delta_n)a_n + b_n, n \geq 1$$

If $\Sigma_{n=1}^{\infty}\delta_n < \infty$ and $\Sigma_{n=1}^{\infty}b_n < \infty$, then $lim_{n\to\infty}a_n$ exists.

We can easily prove he following lemma.

Lemma1.5. *Let $\{a_n\}, \{b_n\}$ and $\{c_n\}$ be sequences of nonnegative real numbers. If $\Sigma_{n=1}^{\infty}a_n < \infty$,$\Sigma_{n=1}^{\infty}b_n < \infty$ and $\Sigma_{n=1}^{\infty}c_n < \infty$ then $\Sigma_{n=1}^{\infty}a_n b_n < \infty$ and $\Sigma_{n=1}^{\infty}a_n b_n c_n < \infty$.*

2. Main Results

Throughout this paper, F will denote the set of common fixed points of S, T and R,i.e, $F = F(S) \cap F(T) \cap F(R)$.We begin with giving the following proposition.

Proposition 2.1. *If C is a nonempty convex subset of a real uniformly convex Banach space E and $S, T, R : C \to C$ are generalized quasi-nonexpansive mappings with sequences $\{k_n\}, \{k'_n\}, \{k''_n\}$ in $[0, 1)$ such that $\lim_{n\to\infty} k_n = \lim_{n\to\infty} k'_n = \lim_{n\to\infty} k''_n = 0$. Then S, T and R are generalized quasi-nonexpansive each with a sequence $\{\kappa_n\}_{n=1}^{\infty}$ such that $\kappa_n = \max\{k_n\, k'_n\, k''_n\}$ and a self map$f, g, h : C \to C$ is said to be contraction on C.*

Proof. Clearly: $\{\kappa_n\}_{n=1}^{\infty}$ is in $[0, 1)$, furthermore $\lim_{n\to\infty} \kappa_n = 0$.

Lemma 2.2. *Let C be a nonempty convex subset of a normed space E. Let $S, T, R : C \to C$ be generalized quasi-nonexpansive mappings and a self map $f, g, h : C \to C$ are contraction on C,then such that for all $x \in C, p \in F$ and for all $n \geq 1$*

$$\|S^n x - p\| \leq \|x - p\| + k_n\|x - S^n x\|$$

$$\|T^n x - p\| \leq \|x - p\| + k'_n\|x - T^n x\|$$

$$\|R^n x - p\| \leq \|x - p\| + k''_n\|x - R^n x\|$$

where $\{k_n\}, \{k'_n\}, \{k''_n\}$ are sequence in $[0, 1)$ with $\Sigma_{n=1}^{\infty}k_n < \infty, \Sigma_{n=1}^{\infty}k'_n < \infty$ and $\Sigma_{n=1}^{\infty}k''_n < \infty$.Let $\{x_n\}$ be the sequence defined in (8) with $\Sigma_{n=1}^{\infty}\gamma_n < \infty, \Sigma_{n=1}^{\infty}\theta_n < \infty$ and $\Sigma_{n=1}^{\infty}\zeta_n < \infty$.If $F \neq \emptyset$ then $\lim_{n\to\infty} \|x - p\|$ exists for all $p \in F$.

Proof. By Proposition (2.1), there exists a sequence $\{\kappa_n\}_{n=1}^{\infty}$ in [0.1) with $\Sigma_{n=1}^{\infty}\kappa_n < \infty$ such that for all $x \in C, p \in F$ and all $n \geq 1$.

$$\|S^n x - p\| \leq \|x - p\| + k_n\|x - S^n x\|$$

$$\|T^n x - p\| \leq \|x - p\| + k'_n\|x - T^n x\|$$

$$\|R^n x - p\| \leq \|x - p\| + k''_n\|x - R^n x\|$$

Since $\{u_n\}_{n=1}^{\infty}, \{v_n\}_{n=1}^{\infty}$ and $\{w_n\}_{n=1}^{\infty}$ are bounded sequences in C then there exists,$0 < M < \infty$,such that

$$M = \max\{\sup_{n\geq 1}\|u_n - p\|, \sup_{n\geq 1}\|v_n - p\|, \sup_{n\geq 1}\|w_n - p\|\}.$$

Now,for any $p \in F$ we have

$$\|x_{n+1} - p\| = \|\alpha_n S^n y_n + \beta_n f(x_n) + \gamma_n u_n - p\|$$

$$\leq \alpha_n\|S^n y_n - p\| + \beta_n\|f(x_n) - p\| + \gamma_n\|u_n - p\|$$

$$\leq \alpha_n[\|y_n - p\| + k_n\|y_n - S^n y_n\|] + \beta_n\|f(x_n) - f(p)\| + \gamma_n\|u_n - p\|$$

$$\leq \alpha_n\Big(\frac{1 + k_n}{1 - k_n}\Big)\|y_n - p\| + k\beta_n\|x_n - p\| + \gamma_n M$$

$$\leq \alpha_n\Big(1 + \frac{2k_n}{1 - k_n}\Big)\|y_n - p\| + k\beta_n\|x_n - p\| + \gamma_n M$$

$$\leq \alpha_n(1 + l_n)\|\lambda_n T^n z_n + \mu_n g(x_n) + \theta_n v_n - p\|$$

$$+ k\beta_n\|x_n - p\| + \gamma_n M$$

$$\leq \alpha_n(1 + l_n)[\lambda_n\|T^n z_n - p\| + \mu_n\|g(x_n) - p\| + \theta_n\|v_n - p\|]$$

$$+ k\beta_n\|x_n - p\| + \gamma_n M$$

$$\leq \alpha_n(1 + l_n)[\lambda_n\Big(\frac{1 + k'_n}{1 - k'_n}\Big)\|z_n - p\| + \mu_n\|g(x_n) - g(p)\| + \theta_n M]$$

$$+ k\beta_n\|x_n - p\| + \gamma_n M$$

$$\leq \alpha_n(1 + l_n)[\lambda_n\Big(1 + \frac{2k'_n}{1 - k'_n}\Big)\|z_n - p\| + k'\mu_n\|x_n - p\| + \theta_n M]$$

$$+ k\beta_n\|x_n - p\| + \gamma_n M$$

$$\leq \alpha_n(1 + l_n)[\lambda_n(1 + l'n)\|z_n - p\| + k'\mu_n\|x_n - p\| + \theta_n M]$$

$$+ k\beta_n\|x_n - p\| + \gamma_n M$$

$$\leq \alpha_n\lambda_n(1+l_n)(1+l'_n)\|z_n-p\| + k'\alpha_n\mu_n(1+l_n)\|x_n-p\|$$
$$+\alpha_n\theta_n(1+l_n)M + k\beta_n\|x_n-p\| + \gamma_n M$$
$$\leq \alpha_n\lambda_n(1+l_n)(1+l'_n)[\|\xi_n R^n x_n + \eta_n h(x_n) + \zeta_n w_n - p\|]$$
$$+\|x_n-p\|[k'\alpha_n\mu_n(1+l_n) + k\beta_n] + M[\alpha_n\theta_n(1+l_n)+\gamma_n]$$
$$\leq \alpha_n\lambda_n(1+l_n)(1+l'_n)[\xi_n\|R^n x_n - p\| + \eta_n\|h(x_n)-p\| + \zeta_n\|w_n-p\|]$$
$$+\|x_n-p\|[k'\alpha_n\mu_n(1+l_n) + k\beta_n] + M[\alpha_n\theta_n(1+l_n)+\gamma_n]$$
$$\leq \alpha_n\lambda_n(1+l_n)(1+l'_n)[\xi_n\Big(\frac{1+k''_n}{1-k''_n}\Big)\|x_n-p\| + \eta_n\|h(x_n)-h(p)\| + \zeta_n M]$$
$$+\|x_n-p\|[k'\alpha_n\mu_n(1+l_n) + k\beta_n] + M[\alpha_n\theta_n(1+l_n)+\gamma_n]$$
$$\leq \alpha_n\lambda_n(1+l_n)(1+l'_n)[\xi_n\Big(1+\frac{2k''_n}{1-k''_n}\Big)\|x_n-p\| + \eta_n k''\|x_n-p\| + \zeta_n M]$$
$$+\|x_n-p\|[k'\alpha_n\mu_n(1+l_n) + k\beta_n] + M[\alpha_n\theta_n(1+l_n)+\gamma_n]$$
$$\leq \alpha_n\lambda_n(1+l_n)(1+l'_n)[\xi_n(1+l''_n)\|x_n-p\| + \eta_n k''\|x_n-p\| + \zeta_n M]$$
$$+\|x_n-p\|[k'\alpha_n\mu_n(1+l_n) + k\beta_n] + M[\alpha_n\theta_n(1+l_n)+\gamma_n]$$
$$\leq \alpha_n\lambda_n\xi_n(1+l_n)(1+l'_n)(1+l''_n)\|x_n-p\|$$
$$+k''\alpha_n\lambda_n\eta_n(1+l_n)(1+l'_n)\|x_n-p\| + (1+l_n)(1+l'_n)\alpha_n\lambda_n\zeta_n M$$
$$+[(1+l_n)k'\alpha_n\mu_n + k\beta_n]\|x_n-p\| + M[\alpha_n\theta_n(1+l_n)) + \gamma_n]$$
$$\leq [\alpha_n\lambda_n\xi_n(1+l_n)(1+l'_n)(1+l''_n) + k''\alpha_n\lambda_n\eta_n(1+l_n)(1+l'_n)$$
$$+(1+l_n)k'\alpha_n\mu_n + k\beta_n]\|x_n-p\|$$
$$+[(1+l_n)(1+l'_n)\alpha_n\lambda_n\zeta_n + \alpha_n\theta_n(1+l_n)+\gamma_n]M$$
$$\leq [\alpha_n\lambda_n\xi_n + k''\alpha_n\lambda_n\eta_n + k'\alpha_n\mu_n + k\beta_n](1+l_n)(1+l'_n)(1+l''_n)\|x_n-p\|$$
$$+[(1+l_n)(1+l'_n)\alpha_n\lambda_n\zeta_n + (1+l_n)\alpha_n\theta_n + \gamma_n]M$$
$$\leq (1+l_n)(1+l'_n)(1+l''_n)\|x_n-p\|$$
$$+[(1+l_n)(1+l'_n)\alpha_n\lambda_n\zeta_n + (1+l_n)\alpha_n\theta_n + \gamma_n]M$$

That is,

$$\|x_{n+1}-p\| \leq (1+l_n)(1+l'_n)(1+l''_n)\|x_n-p\| + [(1+l_n)(1+l'_n)\zeta_n + (1+l_n)\theta_n + \gamma_n]M$$

Since $\{\kappa_n\}$ is bounded sequence,then there exists $h>0$ such that $\kappa_n \geq h, h \geq 1$.

There fore ,

$$\|x_{n+1}-p\| \leq (1+l_n)(1+l'_n)(1+l''_n)\|x_n-p\|+[(1+l_n)(1+l'_n)\zeta_n+(1+l_n)\theta_n+\gamma_n]M$$

$$\|x_{n+1}-p\| \leq (1+\delta_n)\|x_n-p\| + b_n M$$

where $\delta_n = (1+l_n)(1+l'_n)(1+l''_n)$ and $b_n = [(1+l_n)(1+l'_n)\zeta_n + (1+l_n)\theta_n + \gamma_n]$.

Using the facts that $\Sigma_{n=1}^{\infty}\gamma_n < \infty, \Sigma_{n=1}^{\infty}\upsilon_n < \infty, \Sigma_{n=1}^{\infty}\zeta_n < \infty$,and $\Sigma_{n=1}^{\infty}\kappa_n < \infty$ and applying Lemmas 1.4 and 1.5, $\lim_{n\to\infty}\|x_{n+1}-p\|$ exists.

Theorem 2.3. *Let C be a nonempty closed convex subset of a real Banach space E.Let S, T and R be generalized quasi-nonexpansive mappings with a sequence $\{\kappa_n\}_{n=1}^{\infty}$ in $[0,1)$ such that $\Sigma_{n=1}^{\infty}\kappa_n < \infty$. Let $F \neq \phi$, and let $\{x_n\}$ be the sequence defined in Lemma 2.2.Then $\{x_n\}$ converges strongly to some common fixed point of S, T and R if and only if $\lim inf_{n\to\infty} d(x_n, F) = 0$, where $d(x, F) = inf_{p\in F}\|x-p\|$.*

Proof. In the proof of Lemma 2.2, we obtained that

$$\|x_{n+1}-p\| \leq (1+l_n)(1+l'_n)(1+l''_n)\|x_n-p\|+[(1+l_n)(1+l'_n)\zeta_n+(1+l_n)\theta_n+\gamma_n]M$$

which implies

$$d(x_{n+1}, F) \leq (1+l_n)(1+l'_n)(1+l''_n)d(x_n, F)+[(1+l_n)(1+l'_n)\zeta_n+(1+l_n)\theta_n+\gamma_n]M$$

We can prove by an argument similar to that in the proof of Lemma 2.2, that $\lim_{n\to\infty} d(x_n, F)$ exists for any $p \in F$. But $\lim inf_{n\to\infty} d(x_n, F) = 0$, by hypothesis,hence we have $\lim_{n\to\infty} d(x_n, F) = 0$.

Now,for any $q \in F$,and any integer $m \geq 1$,we have

$$\|x_{n+m} - x_n\| \leq \|x_{n+m} - q\| + \|x_n - q\|$$

which implies that

$$\|x_{n+m} - x_n\| \leq d(x_{n+m}, F) + d(x_n, F) \tag{9}$$

Letting $n \to \infty$ on on both sides of(9), we get

$$\lim_{n\to\infty}\|x_{n+m} - x_n\| = 0$$

Hence $\{x_n\}$ is a cauchy sequence.Since C is a closed subset of a Banach space E, then,$\{x_n\}$ converges to some $q \in C$.Since $\lim_{n\to\infty} d(x_n, F) = 0$,then $d(q, F) = 0$.

Now we prove that $F = \{p : p \in F(S) \cap F(T) \cap F(R)\}$is closed.So let,$\{p_n\}$ be an arbitrary sequence of element of F such that $p_n \to p$. We show that $p \in F$,i.e, show that $Sp = Tp = Rp = p$.For this purpose, consider the following

$$\|Sp - p\| \leq \|Sp - p_n\| + \|p_n - p\|$$

$$\leq (1 + l_n)(1 + l'_n)(1 + l''_n)\|p - p_n\| + \|p_n - p\|$$

which, as $n \to \infty$, gives

$$\|Sp - p\| \leq 0$$

Thus $Sp = p$. Similarly $Tp = p$ and $Rp = p$. By closedness of F,then $d(q, F) = 0$ yields that $q \in F$,which completes the proof.

Corollary 2.4. *Let C be a nonempty closed convex subset of a real Banach space E. Let S, T and R be asymptotically quasi-nonexpansive mappings with a sequence $\{\kappa_n\}_{n=1}^{\infty}$ in $[0, 1)$ such that $\Sigma_{n=1}^{\infty}\kappa_n < \infty$. Let $F \neq \phi$, and let $\{x_n\}$ be the sequence defined in Lemma 2.2. Then $\{x_n\}$ converges strongly to some common fixed point of S, T and R if and only if* $\lim inf_{n\to\infty} d(x_n, F) = 0$*, where $d(x, F) = inf_{p\in F}\|x - p\|$.*

Corollary 2.5. *Let C be a nonempty closed convex subset of a real Banach space E. Let S, T and R be asymptotically nonexpansive mappings with a sequence $\{\kappa_n\}_{n=1}^{\infty}$ in $[0, 1)$ such that $\Sigma_{n=1}^{\infty}\kappa_n < \infty$. Let $F \neq \phi$, and let $\{x_n\}$ be the sequence defined in Lemma 2.2. Then $\{x_n\}$ converges strongly to some common fixed point of S, T and R if and only if* $\lim inf_{n\to\infty} d(x_n, F) = 0$*, where $d(x, F) = inf_{p\in F}\|x - p\|$.*

References

[1] Y.J. Cho, H. Zhou, G. Guo, Weak and strong convergence theorems for three step iterations with errors for asymptotically nonexpansive mappings, *Comput. Math. Appl.*, **47** (2004), 707-717.

[2] H. Fukhar-ud-Din, S.H. Khan, Convergence of two-step iterative with errors for two asymptotically nonexpansive mappings, *Int. J. Math. Anal. Sci.*, **37** (2004), 1956-1971.

[3] K. Goebel, W.A. Kirk, A fixed point theorem for asymptotically nonexpansive mappings, *Proc. Amer. Math. Soc.*, **35** (1972), 171-174.

[4] J.U. Jeong, S.H. Kim, Weak and strong convergence of the Ishikawa iteration process with errors for two asymptotically nonexpansive mappings, *Appl. Math. Comp.*, **181** (2006), 1394-1401.

[5] L.S. Liu, Ishikawa and Mann iterative processes with errors for nonlinear strongly accretive mappings in Banach space, *J. Math. Anal. Appl.*, **194**, No. 1 (1995), 114-125.

[6] Y.G. Xu, Ishikawa and Mann iterative processes with errors for nonliner strongly accretive operator equations, *J. Math. Anal. Appl.*, **224** (1998), 1-101.

[7] B.L. Xu, M.A. Noor, Fixed point iterations for asymptotically nonexpansive mappings in Banach spaces, *J. Math. Anal. Appl.*, **267** (2002), 444-453.

[8] Y.C. Lin, Three-step iterative convergence theorems with errors in Banach space, *Taiwanese J. Math.*, **10**, No. 1 (2006), 75-86.

[9] Q.H. Liu. Iterative sequences for asymptotically quasi nonexpansive mappings, *J. Math. Anal. Appl.*, **259** (2001), 1-8.

[10] J. Schu, Weak and strong convergence of fixed points of asymptotically nonexpansive mappings, *Bull. Austral. Math. Soc.* **43** (1991), 153-159.

[11] H.Y. Lan, Common fixed point iterative processes with errors for generalized asymptotically quasi-nonexpansive mappings, *Comput. Math. Appl.*, **52** (2006), 1403-1412.

[12] K.K. Tan, H.K. Xu, Approximating fixed points of nonexpansive mappings by the Ishikawa iteration process, *J. Math. Anal. Appl.*, **178** (1993), 301-308.