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ON A BI-PARAMETRIC CLASS OF OPTIMAL EIGHTH-ORDER DERIVATIVE-FREE METHODS

F. Soleymani

Young Researchers Club
Islamic Azad University
Zahedan Branch, Zahedan, IRAN

Abstract: This paper proposes a novel class of three-step without memory iterations including two free parameters. The suggested bi-parametric class of methods needs four function evaluations per iteration and it also supports the optimality conjecture of Kung and Traub [6] for constructing multi-point iterations without memory. Our class can be viewed as the generalization of the two-step derivative-free family of Ren et al. [8]. The analytical proof of the proposed derivative-free class is given. And finally, numerical examples are employed to corroborate the underlying theory developed in this contribution.

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Key Words: nonlinear scalar equations, optimality, order of convergence, derivative-free methods, simple root, without memory iterations

1. Introduction

Almost a half century ago, Traub proved in [18] that one-point iterative methods for solving uni-variate nonlinear equations of the form $f(x) = 0$; which require one evaluation of a given function f and one of its first derivative can reach the order of convergence at most 2. Due to this, a great attention was paid to multi-point iterative methods; since they overcome on the theoretical limits of one-point methods concerning the order and efficiency index. Until recently lots of iterations for solving a nonlinear equation $f(x) = 0$ have been proposed [2, 7, 9, 11, 12, 13]. Most of these methods are based on the Newton's method or the Jarratt type iterations. It should be noted that the availability of most

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iterative methods based on the Newton's method depends on an initial guess and behavior of the function $f(x)$ near the root. Moreover, *an explicit form of the derivative $f'(x)$ is necessary* in implementing the Newton's iteration (or generally derivative-involved methods). Though the secant method can overcome this problem, it takes a cost of slower rate of convergence. Hence, more attention is devoted to build high-order multi-point iterative schemes in which there is no trace of derivative evaluations of the given function per cycle [15]. Unlike the existing high-order derivative-involved methods [10, 14, 16], such derivative-free methods are so fruitful in optimization problems.

The purpose of this study is to present some generalizations of the celebrated second-order Steffensen's method [17] which was mooted by the Danish mathematician Johan Frederik Steffensen (1873-1961) as follows

$$x_{n+1} = x_n - \frac{f(x_n)}{f[x_n, w_n]}, \tag{1}$$

wherein $f[x_n, w_n] = \frac{f(w_n)-f(x_n)}{w_n-x_n}$, $w_n = x_n + f(x_n)$ with higher order of convergence and optimal efficiency index.

Kung-Traub's conjecture [6]. Multi-point iterative methods without memory, requiring $d+1$ function evaluations per iteration, have the order of convergence at most 2^d. Multi-point methods which satisfy the Kung-Traub conjecture (still unproved) are usually called optimal methods.

Kung and Traub moreover in [6] provided a class of n-step derivative-involved methods (by using inverse Hermite interpolation) including n evaluations of the function and one of its first derivative per full iteration to reach the convergence rate 2^n. They also have given a n-step derivative-free family of one parameter (by using the inverse interpolation and consuming $n+1$ evaluations of the function) to again achieve the optimal convergence rate 2^n. Hence, in case of $n = 3$, the schemes reach the order eight with $8^{1/4} \approx 1.682$ as the optimal efficiency index.

Long enough time after the pioneer contributions of Kung and Traub, in 2009, Ren et al. [8] provided a new optimal two-step family using divided differences as follows

$$\begin{cases} y_n = x_n - \dfrac{f(x_n)}{f[x_n, w_n]}, \\ z_n = y_n - \dfrac{f(y_n)}{f[x_n, y_n] + f[y_n, w_n] - f[x_n, w_n] + \gamma(y_n - x_n)(y_n - w_n)}, \end{cases} \tag{2}$$

wherein $\gamma \in \mathbb{R}$ to obtain without memory methods without using derivatives per full iteration. In what follows, we consider a three-step scheme by considering

(2) in its two steps

$$\begin{cases} y_n = x_n - \dfrac{f(x_n)}{f[x_n, w_n]}, \\ z_n = y_n - \dfrac{f(y_n)}{f[x_n, y_n] + f[y_n, w_n] - f[x_n, w_n] + \gamma(y_n - x_n)(y_n - w_n)}, \\ x_{n+1} = z_n - \frac{f(z_n)}{f'(z_n)}, \end{cases} \tag{3}$$

and the Newton's iteration in the last step. In addition, we consider $w_n = x_n + \beta f(x_n)$, $\beta \in \mathbb{R} - \{0\}$ which is a generalization of the considered form by Ren et al. in [8]. Accordingly, we study a new class of three-step derivative-free without memory methods for finding the simple roots of nonlinear equations in which the order of convergence eight will be attained by consuming only four pieces of information per full cycle. This shows that any method derived from our class reaches the efficiency index $8^{1/4} \approx 1.682$ and also support the conjecture of Kung and Traub (1974). In providing our class of methods, we make use of weight function approach in the last step to achieve the highest possible order of convergence by the smallest number of function evaluations in the three steps cycle (3). The order of our class is established theoretically. And finally, numerical examples are employed to support the underlying theory developed in this contribution. To see more related papers in this field of study, the readers might refer to [1, 3, 4, 5, 19, 20].

2. A New Derivative-Free Class

It is crystal clear that scheme (3) is an eighth-order family including two-parameters with five evaluations per iteration. Now, the main question is that "is there any way to keep the order up but reduce the number of evaluations while the method be free from derivative evaluation". For this cause, by using three past known data (except the known value in the node w_n), a powerful approximation for $f'(z_n)$ will be obtained. That is, we approximate $f'(z_n)$ in the domain D of the simple zero, by an approximating polynomial of degree three as follows

$$f(t) \approx B(t) = a_0 + a_1(t - x_n) + a_2(t - x_n)^2 + \gamma(t - x_n)^3. \tag{4}$$

At this time, the three unknown quantities in (4) should be attained by satisfying in the interpolating conditions $f(x_n) = B(x_n)$, $f(y_n) = B(y_n)$, $f(z_n) = B(z_n)$. It is obvious that $a_0 = f(x_n)$. Finally, by solving a system of three linear equations with three unknowns, we can approximate $f'(z_n)$ as follows:

$$f'(z_n) \approx B'(z_n) = a_1 + 2a_2(z_n - x_n) + 3\gamma(z_n - x_n)^2 =$$

$$f[x_n, z_n] + f[z_n, y_n] - f[x_n, y_n] + \gamma(z_n - y_n)(z_n - x_n), \tag{5}$$

where $f[x_n, z_n]$, $f[z_n, y_n]$, and $f[x_n, y_n]$ are divided differences of the function f and $\gamma \in \mathbb{R}$. Consequently a novel three-step iterations by using weight function approach including two free parameters can be defined as follows

$$\begin{cases} y_n = x_n - \dfrac{f(x_n)}{f[x_n, w_n]}, w_n = x_n + \beta f(x_n), \beta \in \mathbb{R} - \{0\}, \\ z_n = y_n - \dfrac{f(y_n)}{f[x_n, y_n] + f[y_n, w_n] - f[x_n, w_n] + \gamma(y_n - x_n)(y_n - w_n)}, \gamma \in \mathbb{R}, \\ x_{n+1} = z_n - \dfrac{f(z_n) \times [H(\xi) + L(o) + G(\pi) + K(\tau) + P(\rho)]}{f[x_n, z_n] + f[z_n, y_n] - f[x_n, y_n] + \gamma(z_n - y_n)(z_n - x_n)}, \end{cases} \tag{6}$$

in which we have four evaluations of the function per iteration and five real valued weight functions $H(\xi)$, $L(o)$, $G(\pi)$, $K(\tau)$, $P(\rho)$ with $\xi = \frac{f(y)}{f(x)}$, $o = \frac{f(y)}{f(w)}$, $\pi = \frac{f(z)}{f(y)}$, $\tau = \frac{f(z)}{f(w)}$ and $\rho = \frac{f(z)}{f(x)}$ (without the index n). We shall see that its order of convergence reaches to the optimal case, i.e. 8, with only four evaluations per full iteration, which means that the proposed class of derivative-free methods possesses the high efficiency index 1.682 and can be viewed as the generalization of (2).

Theorem 1. *Let $\alpha \in D$ be a simple zero of a sufficiently differentiable function $f : D \subset \mathbb{R} \to \mathbb{R}$ for an open interval D, which includes x_0 as the initial approximation of α. Then, the class of derivative-free methods (6) is of optimal order eighth when*

$$\begin{cases} H(0) = 1, H'(0) = H''(0) = H^3(0) = 0, \ \textit{and} \ |H^{(4)}(0)| \leq \infty, \\ L(0) = L'(0) = L''(0) = 0, L^{(3)}(0) = -(6 + 6\beta f[x_n, w_n]), \ \textit{and} \ |L^{(4)}(0)| \leq \infty, \\ G(0) = G'(0) = 0, \ \textit{and} \ |G''(0)| \leq \infty, \\ K(0) = 0, K'(0) = 1, \ \textit{and} \ |K''(0)| \leq \infty, \\ P(0) = P'(0) = 0, \ \textit{and} \ |P''(0)| \leq \infty. \end{cases} \tag{7}$$

Proof. To find the asymptotic error constant of (6)-(7) where $c_j = \frac{f^{(j)}(\alpha)}{j!}$, $j \geq 1$, we expand any terms of (6) around the simple root α in the nth iterate. Thus, we write $f(x_n) = c_1 e_n + c_2 e_n^2 + c_3 e_n^3 + c_4 e_n^4 + c_5 e_n^5 + c_6 e_n^6 + c_7 e_n^7 + c_8 e_n^8 + O(e_n^9)$.

Hence, we obtain

$$x_n - \frac{f(x_n)}{f[x_n, w_n]} - \alpha = c_2(\frac{1}{c_1} + \beta)e_n^2$$

$$+\frac{(c_1c_3(1+c_1\beta)(2+c_1\beta) - c_2^2(2+c_1\beta(2+c_1\beta)))e_n^3}{c_1^2}$$

$$+\frac{1}{c_1^3}(c_1^2c_4(1+c_1\beta)(3+c_1\beta(3+c_1\beta)) + c_2^3(4+c_1\beta(5+c_1\beta(3+c_1\beta)))$$

$$-c_1c_2c_3(7+c_1\beta(10+c_1\beta(7+2c_1\beta)))e_n^4) + \cdots + O(e_n^9). \quad (8)$$

In the same vein, we have

$$z_n - \alpha = \frac{c_2(1+c_1\beta)^2\left(c_2^2 + c_1(-c_3+\gamma)\right)e_n^4}{c_1^3} + \frac{1}{c_1^4}(1+c_1\beta)\left(-c_1^2c_2c_4(1+c_1\beta)\right.$$

$$(2+c_1\beta) - 2c_2^4(2+c_1\beta(2+c_1\beta)) - c_1^2c_3(1+c_1\beta)(2+c_1\beta)(c_3-\gamma)+$$

$$2c_1c_2^2(c_3(4+c_1\beta(5+2c_1\beta)) - (2+c_1\beta(2+c_1\beta))\gamma))e_n^5 + \cdots + O(e_n^9). \quad (9)$$

At this time the Taylor's series expansion of $f(z_n)$ around the root is needed. Hence, we find that

$$f(z_n) = (c_2(1+c_1\beta)^2(c_2^2+c_1(-c_3+\gamma))e_n^4)/c_1^2 + \cdots$$
$$+ 1/c_1^5(-2c_2^6(10+c_1\beta(20+c_1\beta(20+c_1\beta(14+c_1\beta(7+2c_1\beta)))))$$
$$- c_1^2c_2^3c_4(40+c_1\beta(102+c_1\beta(120+c_1\beta(88+c_1\beta(40+9c_1\beta)))))$$
$$- c_1^3c_2(1+c_1\beta)(c_1c_6(1+c_1\beta)(2+c_1\beta)(2+c_1\beta(2+c_1\beta))$$
$$- 2c_3c_4(26+c_1\beta(51+c_1\beta(48+c_1\beta(24+5c_1\beta))))$$
$$+ c_4(18+c_1\beta(34+c_1\beta(33+c_1\beta(17+4c_1\beta))))\gamma)$$
$$+ 2c_1c_2^4(c_3(4+c_1\beta(5+2c_1\beta))(10+c_1\beta(10+c_1\beta(7+4c_1\beta)))$$
$$- (10+c_1\beta(20+c_1\beta(20+c_1\beta(14+c_1\beta(7+2c_1\beta)))))\gamma)$$
$$+ c_1^2c_2^2(c_1c_5(1+c_1\beta)(16+c_1\beta(31+c_1\beta(29+4c_1\beta(4+c_1\beta))))$$
$$- c_3^2(80+c_1\beta(206+c_1\beta(246+c_1\beta(182+5c_1\beta(16+3c_1\beta)))))$$
$$+ c_3(46+c_1\beta(112+c_1\beta(129+c_1\beta(94+c_1\beta(43+9c_1\beta)))))\gamma$$
$$+ 3(2+c_1\beta(2+c_1\beta))(\gamma+c_1\beta\gamma)^2) + c_1^3(1+c_1\beta)(c_3^3(3+c_1\beta)(4$$
$$+ c_1\beta(7+2c_1\beta(3+c_1\beta))) - c_3^2(10+c_1\beta(20+c_1\beta(21+c_1\beta(11+2c_1\beta))))\gamma$$
$$- c_1(1+c_1\beta)(2+c_1\beta)(c_4^2(3+c_1\beta(3+c_1\beta)) - c_5(2+c_1\beta(2+c_1\beta))\gamma)$$
$$- c_3(1+c_1\beta)(2+c_1\beta)(c_1c_5(5+c_1\beta(5+2c_1\beta))$$

$$+ (1 + c_1\beta)\gamma^2)))e_n^7 + O(e_n^8).$$

Considering this Taylor's series expansion, (7) and (9) will result in the follow up final error equation

$$\begin{aligned} e_{n+1} = -\frac{1}{24c_1^7}\left(c_2(1 + c_1\beta)^2\left(c_2^2 + c_1(-c_3 + \gamma)\right)\left(-24c_1^2c_2c_4(1 + c_1\beta)^2\right.\right. \\ + 12c_1^2(1 + c_1\beta)^2c_3 - \gamma)^2G''(0) + 24c_1c_2^2(1 + c_1\beta)(-c_3(1 \\ + c_1\beta)(-2 + G''(0)) + \gamma(-4 + c_1\beta(-3 + G''(0)) + G''(0))) + c_2^4(12(1 \\ + c_1\beta)(6 + G''(0) + c_1\beta(2 + G''(0))) + (1 + c_1\beta)^4H^{(4)}(0) + L^{(4)}(0))))e_n^8 \\ + O(e_n^9). \end{aligned} \quad (10)$$

This shows that the iterative class of derivative-free without memory methods (6)-(7) is of optimal order eight. The proof is complete. □

Now any desired three-step without memory bi-parametric family of iterations without using any derivative of the function per iteration could be attained from our proposed class (6)-(7). Note that the weight functions should be chosen in (6) such that they satisfy in (7) for making the order optimal. Here first to reduce the computational load of the novel methods from our derivative-free class, we choose $\beta = 1$ and $\gamma = 0$. Then, we can generate the following iteration based on (6)-(7)

$$\begin{cases} y_n = x_n - \dfrac{f(x_n)}{f[x_n, w_n]}, w_n = x_n + f(x_n), \\ z_n = y_n - \dfrac{f(y_n)}{f[x_n, y_n] + f[y_n, w_n] - f[x_n, w_n]}, \\ x_{n+1} = z_n - \\ \dfrac{f(z_n) \times [1 + (\frac{f(y_n)}{f(x_n)})^4 - (1 + f[x_n, w_n])(\frac{f(y_n)}{f(w_n)})^3 - (\frac{f(z_n)}{f(y_n)})^2 + \frac{f(z_n)}{f(w_n)} + (\frac{f(z_n)}{f(x_n)})^2]}{f[x_n, z_n] + f[z_n, y_n] - f[x_n, y_n]}, \end{cases} \quad (11)$$

wherein its error equation is $e_{n+1} = x_n - \alpha = (1/c_1^7)((1+c_1)^3c_2(-c_2^2+c_1c_3)((3+c_1(3+c_1(3+c_1)))c_2^4+4c_1(1+c_1)c_2^2c_3 - c_1^2(1+c_1)c_3^2 - c_1^2(1+c_1)c_2c_4)e_n^8) + O(e_n^9)$. As an another very efficient optimal eighth-order derivative-free method of our class, we can have the follow-up scheme

$$\begin{cases} y_n = x_n - \dfrac{f(x_n)}{f[x_n, w_n]}, w_n = x_n + f(x_n), \\ z_n = y_n - \dfrac{f(y_n)}{f[x_n, y_n] + f[y_n, w_n] - f[x_n, w_n]}, \\ x_{n+1} = z_n - \\ \dfrac{f(z_n) \times [1 + (\frac{f(y_n)}{f(x_n)})^5 - (1 + f[x_n, w_n])(\frac{f(y_n)}{f(w_n)})^3 + (\frac{f(z_n)}{f(y_n)})^3 + \frac{f(z_n)}{f(w_n)} + (\frac{f(z_n)}{f(x_n)})^2]}{f[x_n, z_n] + f[z_n, y_n] - f[x_n, y_n]}, \end{cases} \quad (12)$$

with

$$e_{n+1} = x_n - \alpha = \frac{-(1+c_1)^3 c_2^2 \left(c_2^2 - c_1 c_3\right)\left((3+c_1)c_2^3 + 2c_1(1+c_1)c_2c_3 - c_1^2(1+c_1)c_4\right) e_n^8}{c_1^7} + O(e_n^9), \quad (13)$$

as its simple error equation. Such obtained optimal high-order derivative-free methods are very useful in real-world applications, such as the problems in Physics, Chemistry and optimization. As an illustration in physics, from position and velocity coordinates for several given instants, it is possible to determine orbital elements for the preliminary orbit. This theoretical trajectory, also known as Keplerian orbit, is defined taking only into account mutual gravitational attraction forces between both bodies, the Earth and the satellite. Nevertheless it should be refined with later observations from ground stations, whose geographic coordinates are previously known. Different methods have been developed for this purpose, constituting a fundamental element in navigation control, tracking and supervision of artificial satellites. Most of these methods need, in their process, to find a solution of a nonlinear function which is more desirable to follow this aim by derivative-free algorithms such as (11) or (12).

3. Numerical Testing

The main objective of this section is to provide a robust comparison between the presented schemes and the already known methods in literature. For numerical reports here, we have used the quadratically scheme of Steffensen (SM2), the fourth-order method given by Ren et al. (2) as (RM4) with $\gamma = 0$, the optimal eighth-order three-step derivative-free family of Kung and Traub [6] with $\beta = 1$ as (KT8-1), and our optimal three-step eighth-order methods (11) and (12).

Test Functions	Roots	Initial Guesses
$f_1(x) = 3x + \sin(x) - e^x$	$\alpha_1 \approx 0.360421702960324$	$x_0 = 0.9$
$f_2(x) = \sin(x) - 0.5$	$\alpha_2 \approx 0.523598775598299$	$x_0 = 0.3$
$f_3(x) = x^2 - e^x - 3x + 2$	$\alpha_3 \approx 0.257530285439861$	$x_0 = 0.4$
$f_4(x) = x^3 + 4x^2 - 10$	$\alpha_4 \approx 1.365230013414097$	$x_0 = 1.37$
$f_5(x) = xe^{-x} - 0.1$	$\alpha_5 \approx 0.111832559158963$	$x_0 = 0.2$
$f_6(x) = x^3 - 10$	$\alpha_6 \approx 2.15443490031884$	$x_0 = 2.16$
$f_7(x) = 10xe^{-x^2} - 1$	$\alpha_7 \approx 1.679630610428450$	$x_0 = 1.4$
$f_8(x) = \cos(x) - x$	$\alpha_8 \approx 0.739085133215161$	$x_0 = 0.3$

Table 1: The test functions considered in this study

The considered nonlinear test functions, their roots and the initial guesses in the neighborhood of the simple roots are furnished in Table 1.

We have used Div. when the iteration diverges for the considered starting point. The results are summarized in Tables 2 and 3 after two and three full iterations respectively. As they show, novel schemes are comparable with all of the methods. All numerical instances were performed by MATLAB 7.6 using 1000 digits floating point arithmetic (VPA:=1000). We have computed the root of each test function for the initial guess x_0 while the iterative schemes were stopped when $|f(x_n)| \leq 10^{-1000}$. As can be seen, the obtained results in Tables 2 and 3 are in harmony with the analytical procedure given in Section 2. And therefore, the contribution in this paper hits the target.

Remark 1. Experimental results for our contributed methods from the three-step derivative-free class (6)-(7) can give better feedbacks, i.e. provide better accuracy than those illustrated in Tables 2 and 3, by choosing very small positive values for $\beta \in \mathbb{R} - \{0\}$. In fact, by choosing very small positive value for β, the error equation will be narrowed. Also note that, if we approximate β by an iteration through the data of the first step per cycle, then with memory iterations from our class will be attained which herein we do not drag the topic into these kind of iterative processes.

4. Concluding Remarks

Multi-point iterative root solvers belong to the class of the most powerful methods for solving nonlinear equations since they overcome on the theoretical limits of one-point methods. Although the construction of these methods has occurred in the 1960s, their rapid development have started again in the first decade of

Test Functions	SM2	RM4	KT8-1	(11)	(12)
$\|f_1(x_2)\|$	0.6e-1	0.2e-7	Div.	0.3e-13	0.2e-14
$\|f_2(x_2)\|$	0.9e-4	0.1e-14	0.2e-57	0.2e-56	0.4e-63
$\|f_3(x_2)\|$	0.3e-4	0.4e-20	0.7e-82	0.2e-84	0.4e-82
$\|f_4(x_2)\|$	0.5e-5	0.1e-28	0.4e-115	0.3e-101	0.2e-114
$\|f_5(x_2)\|$	0.4e-3	0.6e-15	0.6e-49	0.8e-49	0.5e-53
$\|f_6(x_2)\|$	0.4e-5	0.7e-29	0.2e-115	0.1e-102	0.1e-115
$\|f_7(x_2)\|$	0.1	0.1e-4	0.5e-9	0.1e-23	0.2e-21
$\|f_8(x_2)\|$	0.1e-3	0.1e-15	0.1e-58	0.2e-61	0.1e-64

Table 2: Results of comparisons for different methods after two full iterations

Test Functions	SM2	RM4	KT8-1	(11)	(12)
$\|f_1(x_3)\|$	0.3e-2	0.2e-31	Div.	0.2e-109	0.1e-119
$\|f_2(x_3)\|$	0.5e-8	0.1e-59	0.9e-462	0.3e-454	0.3e-508
$\|f_3(x_3)\|$	0.7e-10	0.2e-84	0.1e-663	0.1e-684	0.1e-665
$\|f_4(x_3)\|$	0.1e-10	0.2e-117	0.3e-927	0.6e-815	0.6e-922
$\|f_5(x_3)\|$	0.3e-6	0.6e-60	0.7e-391	0.4e-390	0.1e-423
$\|f_6(x_3)\|$	0.8e-11	0.1e-118	0.7e-930	0.5e-826	0.5e-931
$\|f_7(x_3)\|$	0.1e-1	0.9e-20	0.1e-75	0.4e-192	0.1e-175
$\|f_8(x_3)\|$	0.1e-8	0.3e-66	0.1e-476	0.8e-492	0.2e-524

Table 3: Results of comparisons for different methods after three full iterations

the 21-st century. The most important class of multi-point methods are optimal methods which attain the convergence order 2^n using $n+1$ evaluations per iteration. On the other hand, the need in derivative-free algorithms is nowadays felt by many researches in the field of applied sciences. Due to these facts, this paper suggests a class of three-step bi-parametric iterations in which the order of convergence is as high as possible by using as small as possible of function evaluations and it also free from derivative evaluations. Some typical cases from our class were given and their stability and consistency was checked through numerical works. Our next aim could be in development of an optimal sixteenth-order class based on the proposed class of methods in this paper.

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