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SOME PROPERTIES OF IDEAL (α)-CONVERGENCE IN (ℓ)-GROUPS

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Abstract: We study some relations between (α)-convergence and ideal (α)-convergence for (ℓ)-group-valued sequences of functions. We give some conditions for continuity of the limit function and pose some open problems.

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1. Introduction

The concept of (α)-convergence or continuous convergence of real-valued function sequences has been known in the literature since the beginning of last century (see [4, 6, 8]). This notion was recently further studied in [1, 3, 9]. In [1] the notion of ideal (α)-convergence was introduced and some of its fundamental properties were established. In [3] a similar investigation was carried out within the class of (ℓ)-groups paying special attention to the powerful notion of ideal exhaustiveness. In this paper we continue the investigation of this kind in the context of (ℓ)-groups and prove mainly some results about continuity of the limit functions and relations among (α)-convergence and ideal (α)-convergence. Finally, we pose some open problems.

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2. Preliminaries

We now recall the basic concepts, which will be useful in the sequel.

Definitions 2.1. (a) An (ℓ)-group is said to be *Dedekind complete* iff every subset $R_1 \subset R$, $R_1 \neq \emptyset$ bounded from above has supremum in R.

(b) A bounded double sequence $(a_{i,j})_{i,j}$ in R is called (D)-*sequence* or *regulator* iff for all $i, j \in \mathbb{N}$ we have $a_{i,j} \geq a_{i,j+1}$ and $\wedge_j a_{i,j} = 0$ for all $i \in \mathbb{N}$. A sequence $(x_n)_n$ in R is said to be (D)-*convergent* to $x \in R$ (and we write $(D)\lim_n x_n = x$) iff there exists a (D)-sequence $(a_{i,j})_{i,j}$ in R, such that to every $\varphi \in \mathbb{N}^{\mathbb{N}}$ there corresponds an $n_0 \in \mathbb{N}$ such that $|x_n - x| \leq \bigvee_{i=1}^{\infty} a_{i,\varphi(i)}$ for all $n \in \mathbb{N}$, $n \geq n_0$.

(c) A family of sets $\mathcal{I} \subset \mathcal{P}(\mathbb{N})$, where $\mathbb{N}$ is the set of the natural numbers, is called an *ideal* of $\mathbb{N}$ iff $A \cup B \in \mathcal{I}$ whenever $A, B \in \mathcal{I}$ and for each $A \in \mathcal{I}$ and $B \subset A$ we get $B \in \mathcal{I}$. An ideal is said to be *non-trivial* iff $\mathcal{I} \neq \emptyset$ and $\mathbb{N} \notin \mathcal{I}$. A non-trivial ideal $\mathcal{I}$ is said to be *admissible* iff it contains all singletons.

(d) Let $\mathcal{I}$ be an admissible ideal of $\mathbb{N}$. A sequence $(x_n)_n$ in R $(D\mathcal{I})$-*converges to* $x \in R$ iff there is a (D)-sequence $(a_{i,j})_{i,j}$ such that

$$\Big\{n \in \mathbb{N} : |x_n - x| \not\leq \bigvee_{i=1}^{\infty} a_{i,\varphi(i)}\Big\} \in \mathcal{I}$$

for all $\varphi \in \mathbb{N}^{\mathbb{N}}$. Note that (D)-convergence implies $(D\mathcal{I})$-convergence, while the converse is in general not true (see also [3]).

(e) Let (X, d) be a metric space. A sequence $(x_n)_n$ in X is said to be $\mathcal{I}$-*convergent* to $x \in X$ (and we write $x = (\mathcal{I})\lim_n x_n$) iff for every $\varepsilon > 0$ we get $\{n \in \mathbb{N} : d(x_n, x) > \varepsilon\} \in \mathcal{I}$.

(f) We say that $(f_n)_n$ $(\mathcal{I}\alpha)$-*converges to* $f : X \to R$ iff for every $x \in X$ there exists a regulator $(a_{i,j})_{i,j}$ such that for each sequence $(x_n)_n$ in X with $(\mathcal{I})\lim_n x_n = x$ we get $(D\mathcal{I})\lim_n f_n(x_n) = f(x)$ with respect to the regulator $(a_{i,j})_{i,j}$. We say that $(f_n)_n$ (α)-*converges to* $f : X \to R$ iff it $(\mathcal{I}_{\text{fin}}\alpha)$-converges to f.

(g) The sequence $(f_n)_n$ is said to be *globally* $(\mathcal{I}\alpha)$-*convergent to* $f : X \to R$ iff a (D)-sequence $(a_{i,j})_{i,j}$ can be found, such that for any $x \in X$ and for each sequence $(x_n)_n$ in X with $(\mathcal{I})\lim_n x_n = x$ we have $(D\mathcal{I})\lim_n f_n(x_n) = f(x)$ with respect to the (D)-sequence $(a_{i,j})_{i,j}$.

(h) A function sequence $f_n : X \to R$, $n \in \mathbb{N}$, is called $\mathcal{I}$-*exhaustive at* $x \in X$ iff there exists a regulator $(a_{i,j})_{i,j}$ (depending on x) such that to every $\varphi \in \mathbb{N}^{\mathbb{N}}$

there correspond a $\delta > 0$ and an element $A \in \mathcal{I}$ (depending on φ and x) such that

$$|f_n(z) - f_n(x)| \le \bigvee_{i=1}^{\infty} a_{i,\varphi(i)}$$

whenever $d(z, x) < \delta$ and $n \in \mathbb{N} \setminus A$. We say that $(f_n)_n$ is *exhaustive* iff it is $\mathcal{I}_{\rm fin}$-exhaustive.

(i) Let $f : X \to R$ be a function and $x \in X$. Then f is said to be *continuous at* x iff there exists a regulator $(a_{i,j})_{i,j}$ (depending on x) such that to every $\varphi \in \mathbb{N}^{\mathbb{N}}$ there corresponds a $\delta > 0$ (depending on φ and x) such that $|f(x) - f(z)| \le \bigvee_{i=1}^{\infty} a_{i,\varphi(i)}$ whenever $d(x, z) < \delta$. A function $f : X \to R$ is *continuous on* X iff f is continuous at every point $x \in X$.

(j) A function $f : X \to R$ is called *globally continuous on* X iff there exists a regulator $(a_{i,j})_{i,j}$ such that for every $\varphi \in \mathbb{N}^{\mathbb{N}}$ and $x \in X$ there is a $\delta > 0$ (depending on φ and x) with $|f(x) - f(z)| \le \bigvee_{i=1}^{\infty} a_{i,\varphi(i)}$ whenever $d(x, z) < \delta$.

3. The Main Results

From now on let R be a Dedekind complete (ℓ)-group and $\mathcal{I}$ be an admissible ideal of $\mathbb{N}$. We prove that, for R-valued functions, (α)-convergence implies ($\mathcal{I}\alpha$)-convergence. We extend [9, Theorem 1] to the context of (ℓ)-groups.

Theorem 3.1. *If $f_n : X \to R$, $n \in \mathbb{N}$, (α)-converges to $f : X \to R$, then $(f_n)_n$ ($\mathcal{I}\alpha$)-converges to f too.*

Proof. Let $(f_n)_n$ be (α)-convergent to f. By [3, Theorem 3.3], we get that $(f_n)_n$ is pointwise (D)-convergent to f and exhaustive. Since $\mathcal{I}$ is admissible, this implies that $(f_n)_n$ is pointwise ($D\mathcal{I}$)-convergent (see also [2]) and $\mathcal{I}$-exhaustive. Again by [3, Theorem 3.3], we obtain that $(f_n)_n$ ($\mathcal{I}\alpha$)-converges to f, that is the thesis. □

Example 3.2. Note that, even for real-valued functions, when $\mathcal{I} \ne \mathcal{I}_{\rm fin}$, ($\mathcal{I}\alpha$)-convergence does not imply (α)-convergence, as the following example shows (see also [9]).

Let Y be a set with at least two distinct elements y_1 and y_2 and $H \in \mathcal{I}$ be an infinite set. Since $\mathcal{I} \ne \mathcal{I}_{\rm fin}$, then H does exist. Set $f_n(x) = y_1$ for all $x \in X$ and $n \in \mathbb{N} \setminus H$, and $f_n(x) = y_2$ for every $x \in X$ and $n \in H$. Put $f(x) = y_1$

for each $x \in X$. For every sequence $(x_n)_n$ in X with $x_0 = (\mathcal{I})\lim_n x_n$ we get $(\mathcal{I})\lim_n f_n(x_n) = y_1 = f(x_0)$, but $\lim_n f_n(x_n)$ does not exist in the usual sense. Thus the sequence $(f_n)_n$ $(\mathcal{I}\alpha)$-converges to f, but not (α)-converges to f.

Analogously as Theorem 3.1, using [3, Theorem 3.4], it is possible to prove the following:

Theorem 3.3. *If $f_n : X \to R$, $n \in \mathbb{N}$, is globally (α)-convergent to $f : X \to R$, then $(f_n)_n$ is globally $(\mathcal{I}\alpha)$-convergent to f too.*

We now prove the following technical result, which will be useful in the sequel.

Theorem 3.4. *Let $x_0 \in X$ and $(z_k)_k$ be a sequence of points of X such that $\lim_k z_k = x_0$ in the ordinary sense. Let $g_n : X \to R$, $n \in \mathbb{N}$, be such that $(D\mathcal{I})\lim_n g_n(x_n) = g(x_0)$ for every sequence $(x_n)_n$ in X, $\mathcal{I}$-converging to x_0 (with respect to a regulator, depending on x_0 but independent of $(x_n)_n$) and $(D\mathcal{I})\lim_n g_n(z_k) = g(z_k)$ for all $k \in \mathbb{N}$ with respect to a common regulator. Then $(D)\lim_k g(z_k) = g(x_0)$.*

Proof. Let $(a_{i,j})_{i,j}$ be a (D)-sequence associated to $(D\mathcal{I})$-convergence of $(g_n(x_n))_n$ to $g(x_0)$ and $(b_{i,j})_{i,j}$ be a regulator related with $(D\mathcal{I})$-convergence of $(g_n(z_k))_n$ to $g(z_k)$, $k \in \mathbb{N}$. Set $c_{i,j} = 2(a_{i,j} + b_{i,j})$, $i, j \in \mathbb{N}$: it is not difficult to check that $(c_{i,j})_{i,j}$ is a regulator too.

We now prove that the sequence $(g(z_k))_k$ (D)-converges to $g(x_0)$ with respect to the regulator $(c_{i,j})_{i,j}$. Otherwise there exist a $\varphi \in \mathbb{N}^{\mathbb{N}}$ and a subsequence $(z_{q_k})_k$ of $(z_k)_k$ with

$$|g(z_{q_k}) - g(x_0)| \not\leq \bigvee_{i=1}^{\infty} c_{i,\varphi(i)}. \tag{1}$$

Without loss of generality, we can suppose $q_k = k$ for all $k \in \mathbb{N}$. We claim that a subsequence $(x_n)_n$ of $(z_k)_k$ can be constructed, such that $(\mathcal{I})\lim_n x_n = x_0$, but $(g_n(x_n))_n$ does not $(D\mathcal{I})$-converge to $g(x_0)$ with respect to the regulator $(a_{i,j})_{i,j}$, getting a contradiction and so proving the theorem. We firstly consider the case $\bigcup_k B_k \in \mathcal{I}$, where for every $k \in \mathbb{N}$, $B_k = \left\{n \in \mathbb{N} : |g_n(z_k) - g(z_k)| \not\leq \bigvee_{i=1}^{\infty} b_{i,\varphi(i)}\right\}$, and φ is the one of the relation (1). Note that $B_k \in \mathcal{I}$ for each k by hypothesis and, if $n \notin B_k$, then $|g_n(z_k) - g(x_0)| \not\leq \bigvee_{i=1}^{\infty} a_{i,\varphi(i)}$: otherwise we have

$$|g(z_k) - g(x_0)| \leq |g_n(z_k) - g(x_0)| + |g_n(z_k) - g(z_k)| \leq \bigvee_{i=1}^{\infty} c_{i,\varphi(i)}, \tag{2}$$

which contradicts (1). Hence $|g_n(z_n) - g(x_0)| \not\leq \bigvee_{i=1}^{\infty} a_{i,\varphi(i)}$ for all $n \notin \bigcup_k B_k$. Set now $x_n = z_n$, $n \in \mathbb{N}$. By hypothesis, $\lim_n x_n = x_0$, and thus we get also $(\mathcal{I}) \lim_n x_n = x_0$ (see [7]). Thus the sequence $(g_n(x_n))_n$ does not $(D\mathcal{I})$-converge to $g(x_0)$ with respect to the regulator $(a_{i,j})_{i,j}$, and hence we get the claim, at least when $\bigcup_k B_k \in \mathcal{I}$.

If $\bigcup_k B_k \notin \mathcal{I}$, then, proceeding analogously as in [9, Lemma 1] and by setting $B_0 = \emptyset$, there is a strictly increasing sequence $(k_m)_m$ in $\mathbb{N}$ with $C_m = B_{k_m} \setminus \left(\bigcup_{i=1}^{k_m-1} B_i\right) \neq \emptyset$. Note that $C_m \in \mathcal{I}$ for all $m \in \mathbb{N}$, the C_m's are pairwise disjoint and $\bigcup_m C_m = \bigcup_k B_k \notin \mathcal{I}$. Let us define $x_n = z_{k_m-1}$ for all $n \in C_m$, $m \in \mathbb{N}$, and $x_n = x_0$ for $n \notin \bigcup_m C_m$. Since the C_m's are pairwise disjoint, this construction makes sense. If $m \in \mathbb{N}$ and $n \in C_m$, then $n \notin B_{k_m-1}$, and so we get $|g_n(x_n) - g(x_n)| = |g_n(z_{k_m-1}) - g(z_{k_m-1})| \leq \bigvee_{i=1}^{\infty} b_{i,\varphi(i)}$. From this, arguing analogously as in the first case, we deduce that $|g_n(x_n) - g(x_0)| \not\leq \bigvee_{i=1}^{\infty} a_{i,\varphi(i)}$. Thus $|g_n(x_n) - g(x_0)| \not\leq \bigvee_{i=1}^{\infty} a_{i,\varphi(i)}$ for all $n \in \bigcup_m C_m$, and hence the sequence $(g_n(x_n))_n$ does not $\mathcal{I}$-converge to $g(x_0)$ with respect to the regulator $(a_{i,j})_{i,j}$.

If we prove that $(\mathcal{I}) \lim_n x_n = x_0$, then we get the requested. Fix arbitrarily $\varepsilon > 0$. If $n \in \mathbb{N}$ is such that $d(x_0, x_n) > \varepsilon$, then obviously $x_n \neq x_0$, and by construction there exists a unique $m = m(n) \in \mathbb{N}$ such that $n \in C_{m(n)}$ and hence $x_n = z_{k_{m(n)}-1}$. This may happen only when $z_{k_{m(n)}-1} \neq x_0$. In any case, we get

$$\{n \in \mathbb{N} : d(x_0, x_n) > \varepsilon\} \subset \{n \in \mathbb{N} : d(x_0, z_{k_{m(n)}-1}) > \varepsilon\}. \tag{3}$$

Note that, since $\lim_n z_n = x_0$ in the usual sense, then $\lim_m z_{k_m} = x_0$ in the ordinary sense too, and hence $d(x_0, z_{k_m-1}) > \varepsilon$ only for finitely many m's. From this and (3) it follows that the set $\{n \in \mathbb{N} : d(x_0, x_n) > \varepsilon\}$ is contained in a finite union of C_m's and so belongs to $\mathcal{I}$. Hence, $(\mathcal{I}) \lim_n x_n = x_0$. This ends the proof of the theorem. □

We now give some conditions for continuity of the limit function.

Corollary 3.5. *If $f_n : X \to R$, $n \in \mathbb{N}$ is pointwise $(D\mathcal{I})$-convergent with respect to a common regulator and $(\mathcal{I}\alpha)$-convergent to $f : X \to R$, then f is continuous.*

Proof. Choose arbitrarily $x_0 \in X$. We first claim that, if $(z_k)_k$ is a sequence in X, convergent to x_0 in the usual sense, then we have $(D)\lim_k f(z_k) = f(x_0)$ with respect to a regulator $(c_{i,j})_{i,j}$, independent of the choice of $(z_k)_k$.

Let $(z_k)_k$ be as above. By $(\mathcal{I}\alpha)$-convergence of $(f_n)_n$ to f, we have

$$(D\mathcal{I}) \lim_n f_n(z_k) = f(x_0)$$

with respect to a regulator $(a_{i,j})_{i,j}$, depending on x_0 and independent of the choice of $(z_k)_k$. Let $(b_{i,j})_{i,j}$ be a regulator associated to $(D\mathcal{I})$-pointwise convergence of $(f_n)_n$ to f: by hypothesis, $(b_{i,j})_{i,j}$ does not depend on the chosen sequence $(z_k)_k$. Thus, the hypotheses of Theorem 3.4 are fulfilled, and so, if $c_{i,j} = 2(a_{i,j} + b_{i,j})$, $i, j \in \mathbb{N}$, then $(c_{i,j})_{i,j}$ is the requested regulator.

We now prove that f is continuous at x_0 with respect to the (D)-sequence $(c_{i,j})_{i,j}$. Otherwise there exists $\varphi \in \mathbb{N}^{\mathbb{N}}$ such that to every $k \in \mathbb{N}$ there is $z_k \in X$ with $d(z_k, x_0) \leq 1/k$ but $|f(z_k) - f(x_0)| \not\leq \bigvee_{i=1}^{\infty} c_{i,\varphi(i)}$ for all $k \in \mathbb{N}$. So $(z_k)_k$ converges to x_0 in the usual sense, but $(f(z_k))_k$ does not converge to $f(x_0)$ with respect to $(c_{i,j})_{i,j}$, a contradiction. Thus we get continuity of f, by arbitrariness of $x_0 \in X$. $\square$

Similarly as Corollary 3.5, it is possible to prove the following

Corollary 3.6. *If $f_n : X \to R$, $n \in \mathbb{N}$ is globally $(\mathcal{I}\alpha)$-convergent to $f : X \to R$, then f is globally continuous.*

Open problems: (a) Find conditions under which $(\mathcal{I}\alpha)$-convergence implies (α)-convergence, when $\mathcal{I} \neq \mathcal{I}_{\text{fin}}$.

(b) Find more general sufficient conditions to obtain continuity of the limit function.

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