International Journal of Pure and Applied Mathematics
Volume 72 No. 2 2011, 145-157

ON NORMAL RADICALS

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Abstract: In this paper we introduce the notions like Morita context, Normal radicals, Strong radicals and Hypernilpotent radicals and characterize the same. Finally, we prove some useful results on primitive semirings.

AMS Subject Classification: 16Y60
Key Words: semirings, ideal, radical class of semiring, normal radical, Jacobson radical

1. Introduction

In this paper we introduce the notion of Morita context, Normal radicals and characterize the same. In the second section we give the definition of Morita context, Normal radical and some examples of Normal radicals and some properties of Normal radical and Morita context.

In the third section we introduce the notion of primitive semiring and establish some characterizations of Jacobson radical and primitive semirings.

Throughout this paper there are many open questions in Section 2 and 3.

Received: June 2, 2011

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We are very much sure that one can enhance the classical structure theory for non-commutative semirings after answering these questions.

Proposition 1. *Every semiring R can be embedded as an ideal into a semiring R^* with unity element. The semiring R^* is referred to as Dorroh extension R.*

Proof. On the set

$$R^* = \{(a, n) \mid a \in R, n \in \mathbb{Z}\}.$$

Define addition $(a_1, n_1) + (a_2, n_2) = (a_1 + a_2, n_1 + n_2)$ and $(a, n)(b, m) = (ab + ma + nb, mn)$.

Then it is easy to verify that R^* is a semiring and

$$\begin{aligned}(a, n)(0, 1) &= (0 + a + 0, n \cdot 1)\\ &= (a, n),\ \text{for all } (a, n) \in R^*,\end{aligned}$$

it means R^* is a semiring with unity element $(0, 1)$.

Define $\theta \colon R \to (R, 0)$ given by $\theta(a) = (a, 0)$. Then clearly θ is a one-one and onto homomorphism. So $R \cong (R, 0) \triangleleft R^*$. Hence $R \triangleleft R^*$, where R^* is a semiring with unity element $(0, 1)$. □

2. Normal Radicals

Let R and S be semirings $V = {}_RV_S$ and $W = {}_SW_R$ an $R - S$ bisemimodule and an $S - R$ bisemimodule respectively. The quadruple (R, V, W, S) is called a Morita context if the set $\begin{pmatrix} R & V \\ W & S \end{pmatrix}$ of matrices forms a semiring under matrix addition and multiplication, where.

$$\begin{pmatrix} R & V \\ W & S \end{pmatrix} = \{\begin{pmatrix} r & v \\ w & s \end{pmatrix} \mid r \in R, s \in S, v \in V, w \in W\}.$$

This definition will make sense, if we assume the existence of mappings

$$V \times W \to R \ \text{ and } \ W \times V \to S$$

given by

$$(v, w) \hookrightarrow vw \ \text{ and } \ (w, v) \hookrightarrow wv \ \text{ for all } \ v \in V \ \text{ and } \ w \in W$$

such that for any $v, v_1, v_2 \in V, w, w_1, w_2 \in W, r \in R, s \in S$ the following identities are satisfied

$$\begin{aligned}(v_1+v_2)w &= v_1w+v_2w\\ r(vw) &= (rv)w\\ (vs)w &= v(sw)\\ v(w_1+w_2) &= vw_1+vw_2\\ (vw)r &= v(wr)\\ (v_1w)v_2 &= v_1(wv_2).\end{aligned}$$

and their duals

$$\begin{aligned}(w_1+w_2)v &= w_1v+w_2v\\ s(uv) &= (su)v\\ (ur)v &= u(rv)\\ w(v_1+v_2) &= wv_1+wv_2\\ (uv)s &= u(vs)\\ (w_1v)w_2 &= w_1(vw_2).\end{aligned}$$

A radical $\mathcal{R}$ is said to be **normal** if $V\mathcal{R}(S)W \subseteq \mathcal{R}(R)$ for every Morita context (R, V, W, S). Also (S, W, V, R) is a Morita context, therefore the normality of $\mathcal{R}$ can be defined as $W\mathcal{R}(R)V \subseteq \mathcal{R}(S)$ along with the Morita context (S, W, V, R). The Morita context (S, W, V, R) is called dual to (R, V, W, S).

Theorem 2. (see [9]) *Let $\mathcal{R}$ be a radical class of a universal class $\mathbb{U}$ of semirings and $\rho = \rho_{\mathcal{R}}$ the corresponding radical operator. Then, for each ideal I of a semiring $R \in \mathbb{U}$ the radical $\rho(I)$ of I is an ideal of R, which in particular yields $\rho(I) \subseteq \rho(R) \cap I$.*

Example 3. Levitzki radical $\mathcal{L}$, Jacobson radical $\mathcal{J}$ are the Normal radicals.

If A, B, C, D are subsets of R, V, W, S respectively, then we should denote by $\begin{pmatrix} A & B \\ C & D \end{pmatrix}$ the subset of $\begin{pmatrix} R & V \\ W & S \end{pmatrix}$ consisting of elements $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$, where $a \in A, b \in B, c \in C, d \in D$.

Applying the A-D-S-Theorem 2 to Morita context we have the following.

Lemma 4. *If $\begin{pmatrix} R & V \\ W & S \end{pmatrix}$ is a Morita context and $\mathcal{R}$ is any radical, then*

$$\mathcal{R}\begin{pmatrix} R & V \\ W & S \end{pmatrix} = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

with appropriate ideals A, D of R and S respectively and sub-bisemimodules B, C of V, W respectively.

Proof. Let $\begin{pmatrix} R & V \\ W & S \end{pmatrix}$ be Morita context and let R^*, S^* denote usual over-semirings with unity element of R and S respectively. V, W are unitary R^* and S^*-semimodules in the obvious way and $\begin{pmatrix} R^* & V \\ W & S^* \end{pmatrix}$ is a Morita context which contains an ideal the context $\begin{pmatrix} R & V \\ W & S \end{pmatrix}$.

It is clear that for any radical $\mathcal{R}$,

$$\mathcal{R}\begin{pmatrix} R & V \\ W & S \end{pmatrix} \lhd \begin{pmatrix} R & V \\ W & S \end{pmatrix} \lhd \begin{pmatrix} R^* & V \\ W & S^* \end{pmatrix}.$$

So by A-D-S-Theorem 2

$$\mathcal{R}\begin{pmatrix} R & V \\ W & S \end{pmatrix} \lhd \begin{pmatrix} R^* & V \\ W & S^* \end{pmatrix}.$$

Now it is easy to verify that $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\mathcal{R}\begin{pmatrix} R & V \\ W & S \end{pmatrix}$ and $\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\mathcal{R}\begin{pmatrix} R & V \\ W & S \end{pmatrix}$ are of the type $\begin{pmatrix} A & B \\ C & D \end{pmatrix}$. Similarly $\mathcal{R}\begin{pmatrix} R & V \\ W & S \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ and $\mathcal{R}\begin{pmatrix} R & V \\ W & S \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$ are of the type $\begin{pmatrix} A & B \\ C & D \end{pmatrix}$. Hence the lemma. □

Definition 5. A radical $\mathcal{R}$ which satisfies the condition $L \in \mathcal{R} \Rightarrow L \subseteq \mathcal{R}(R)$, for all $L \lhd_l R$ is said to be left strong. Right strong radical defined correspondingly.

Proposition 6. *A radical $\mathcal{R}$ is left strong if and only if $0 \neq L \lhd_l R \in \mathcal{S}_{\mathcal{R}}$, implies that and $L \notin \mathcal{R}$.*

Proof. If part is immediate, Only if part follows if $L \notin \mathcal{R} \Leftrightarrow \mathcal{R}(L) \subsetneqq \mathcal{R}$. □

Lemma 7. *If $\mathcal{R}$ is a normal radical, then $\mathcal{R}$ is left and right strong.*

Proof. Let $L \lhd_l R \in \mathcal{S}_{\mathcal{R}}$ and $L \in \mathcal{R}$. Consider the Morita context (R, L, R^*, L) with the naturally defined multiplication. Since $\mathcal{R}$ is normal, therefore $L\mathcal{R}(L)R^* \subseteq \mathcal{R}(R) = 0$, since $R \in \mathcal{S}_{\mathcal{R}}$. But $L \in \mathcal{R} \Rightarrow \mathcal{R}(L) = L \Rightarrow L^2 \subseteq L^2R^* = 0$. Hence $L \lhd L + LR \in \mathcal{S}_{\mathcal{R}}$. So $L \in \mathcal{S}_{\mathcal{R}} \cap \mathcal{R} \Rightarrow L = 0$, a contradiction, as $0 \neq L \in R \in \mathcal{S}_{\mathcal{R}} \Rightarrow L \notin \mathcal{R}$. □

Definition 8. A radical which contains all zero-semirings (or equivalently nilpotent semirings) is called a Hypernilpotent radical.

Proposition 9. *For a radical $\mathcal{R}$ the following conditions are equivalent;*

1. *$\mathcal{R}$ is Hypernilpotent.*
2. *If $I \lhd R$ and $I^n = 0$, then $I \subseteq \mathcal{R}(R)$.*
3. *If $I \lhd_l R \in \mathcal{S}_{\mathcal{R}}$ and $L^n = 0$, then $L = 0$.*

Lemma 10. *If $\mathcal{R}$ is a Hypernilpotent normal radical, then $\mathcal{R}$ is left and right hereditary.*

Proof. Let $L \lhd_l R \in \mathcal{R}$ and let us consider the Morita context (L, R, L, R). Then we have $R\mathcal{R}(R)L \subseteq \mathcal{R}(L)$. Since $R \in \mathcal{R} \Rightarrow \mathcal{R}(R) = R$. Hence $R^2L \subseteq \mathcal{R}(L)$. But $L^3 \subseteq R^2L \subseteq \mathcal{R}(L) \Rightarrow L^2/\mathcal{R}(L) = 0 \Rightarrow L/\mathcal{R}(L) \subseteq \mathcal{R}(L/\mathcal{R}(L)) = 0 \Rightarrow L = \mathcal{R}(L) \in \mathcal{R} \Rightarrow \mathcal{R}$ is (left) hereditary. Hence $\mathcal{R}(R) = \mathcal{R}(\overline{R}) \cap R$. □

The infinite cyclic semigroup $C(\infty)$ and the zero-semiring $\mathbb{Z}(\infty)$ (includes 0) built on $C(\infty)$. Remember the convention that $\mathbb{Z}$ and $\mathbb{Z}(\infty)$ denote the semiring of non-negative integers and that with zero multiplication on the infinite cyclic semigroup.

Theorem 11. *If $\mathcal{R}$ is a radical in a class of associative semirings, then the following conditions are equivalent.*

1. *$\mathcal{R}$ is a normal radical.*
2. *(a) If a semiring R is an ideal of a semiring $\overline{R}$ such that the factor semiring $\overline{R}/R$ is isomorphic with the semiring of non negative integers, then*

$$\mathcal{R}(R) = \mathcal{R}(\overline{R}) \cap R.$$

   (b) If $e = e^2$ is an idempotent of some semiring R, then

$$\mathcal{R}(eRe) = e\mathcal{R}(R)e.$$

3. *For every Morita context we have $\mathcal{R}\begin{pmatrix} R & V \\ W & S \end{pmatrix} = \begin{pmatrix} \mathcal{R}(R) & B \\ C & \mathcal{R}(S) \end{pmatrix}$ where B, C are the subsemimodules of V, W respectively.*

Proof. 2) $\Rightarrow$ 3) Let $\begin{pmatrix} R & V \\ W & S \end{pmatrix}$ be a Morita context and let R^*, S^* be as in Lemma 4. It is easy to verify that

$$\begin{pmatrix} R^* & V \\ W & S^* \end{pmatrix} \lhd \begin{pmatrix} R^* & V \\ W & S \end{pmatrix} \lhd \begin{pmatrix} R & V \\ W & S \end{pmatrix}.$$

Also by 2(a) $\begin{pmatrix} R & V \\ W & S \end{pmatrix} \big/ \begin{pmatrix} R^* & V \\ W & S \end{pmatrix} \cong \mathbb{Z}$
and $\begin{pmatrix} R^* & V \\ W & S \end{pmatrix} \big/ \begin{pmatrix} R^* & V \\ W & S^* \end{pmatrix} \cong \mathbb{Z}$.
This shows that $\mathcal{R}\begin{pmatrix} R^* & V \\ W & S \end{pmatrix} = \mathcal{R}\begin{pmatrix} R & V \\ W & S \end{pmatrix} \cap \begin{pmatrix} R^* & V \\ W & S \end{pmatrix}$ and $\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \mathcal{R}\begin{pmatrix} R^* & V \\ W & S^* \end{pmatrix} = \mathcal{R}\begin{pmatrix} R^* & V \\ W & S \end{pmatrix} \cap \begin{pmatrix} R^* & V \\ W & S^* \end{pmatrix}$, where A, B, C, D are as in Lemma 4. Let $e = e^2 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ be a matrix from $\begin{pmatrix} R^* & V \\ W & S^* \end{pmatrix}$. Hence by 2(b) and 2(a)

$$\begin{aligned} \begin{pmatrix} A & 0 \\ 0 & 0 \end{pmatrix} &= e\mathcal{R}\begin{pmatrix} R & V \\ W & S \end{pmatrix} e \\ &= e(\mathcal{R}\begin{pmatrix} R^* & V \\ W & S^* \end{pmatrix} \cap \begin{pmatrix} R & V \\ W & S \end{pmatrix})e \\ &= e\mathcal{R}\begin{pmatrix} R^* & V \\ W & S^* \end{pmatrix} e \cap \begin{pmatrix} R & 0 \\ 0 & 0 \end{pmatrix} \end{aligned}$$

$$\begin{aligned} &= \mathcal{R}\begin{pmatrix} R^* & 0 \\ 0 & 0 \end{pmatrix} \cap \begin{pmatrix} R & 0 \\ 0 & 0 \end{pmatrix} \\ &= \begin{pmatrix} \mathcal{R}(R^*) & 0 \\ 0 & 0 \end{pmatrix} \cap \begin{pmatrix} R & 0 \\ 0 & 0 \end{pmatrix} \\ &= \begin{pmatrix} \mathcal{R}(R^*) \cap R & 0 \\ 0 & 0 \end{pmatrix} \\ &= \begin{pmatrix} \mathcal{R}(R) & 0 \\ 0 & 0 \end{pmatrix}. \end{aligned}$$

This gives us $A = \mathcal{R}(R)$. Similarly $B = \mathcal{R}(S)$. Hence

$$\mathcal{R}\begin{pmatrix} R & V \\ W & S \end{pmatrix} = \begin{pmatrix} \mathcal{R}(R) & B \\ C & \mathcal{R}(S) \end{pmatrix},$$

where B, C are subsemimodule of ${}_R V_S$ and ${}_S W_R$ respectively.
3) $\Rightarrow$ 1) Let $\mathcal{R}$ a be radical such that for every Morita context (R, V, W, S) we

have

$$\mathcal{R}\begin{pmatrix} R & V \\ W & S \end{pmatrix} = \begin{pmatrix} \mathcal{R}(R) & * \\ * & \mathcal{R}(S) \end{pmatrix}$$

where the $*$ are for suitable bisemimodule. Since

$$\mathcal{R}\begin{pmatrix} R & V \\ W & S \end{pmatrix} \lhd \begin{pmatrix} R & V \\ W & S \end{pmatrix},$$

we have

$$\begin{pmatrix} 0 & V \\ 0 & 0 \end{pmatrix} \mathcal{R}\begin{pmatrix} R & V \\ W & S \end{pmatrix} \begin{pmatrix} 0 & 0 \\ W & 0 \end{pmatrix} = \begin{pmatrix} 0 & V \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \mathcal{R}(R) & * \\ * & \mathcal{R}(S) \end{pmatrix} \begin{pmatrix} 0 & 0 \\ W & 0 \end{pmatrix}$$

$$= \begin{pmatrix} V & V\mathcal{R}(S) \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ W & 0 \end{pmatrix}$$

$$= \begin{pmatrix} V\mathcal{R}(S)W & 0 \\ 0 & 0 \end{pmatrix} \subset \begin{pmatrix} \mathcal{R}(R) & * \\ * & \mathcal{R}(S) \end{pmatrix}.$$

This shows that $V(\mathcal{R}(S))W \subseteq \mathcal{R}(R)$. Hence $\mathcal{R}$ is a normal radical.

Without involving much technicalities and rest of the things are similar to rings as in [8], we have the following.

1) $\Rightarrow$ 2) If $\mathbb{Z}(\infty) \in \mathcal{R}$, then it is to verify that $\mathcal{R}$ is Hypernilpotent, implies that $\mathcal{R}$ is (left) hereditary, hence $\mathcal{R}(R) = \mathcal{R}(\overline{R}) \cap R$.

Consider $\mathcal{R}(\mathbb{Z}) = (n)$ of $\mathbb{Z}$. If $n \neq 0$, then clearly $\mathcal{R}(\mathbb{Z}) = 0$, a contradiction. Therefore $n = 0$ and $\mathbb{Z} \in \mathcal{S}_{\mathcal{R}}$. By $\overline{R}/R \cong \mathbb{Z} \in \mathcal{S}_{\mathcal{R}}$, implies that $\mathcal{R}(\overline{R}/R) = 0$. This shows that by A-D-S Theorem $\mathcal{R}(\overline{R}) = \mathcal{R}(R)$. Hence 2(a).

2(b) Let us consider the Morita context $(R,\ Re,\ eR,\ eRe)$. Since $\mathcal{R}$ is normal $Re\mathcal{R}(eRe)eR \subseteq \mathcal{R}(R)$ and $eR\mathcal{R}(R)Re \subseteq \mathcal{R}(eRe)$. Since e is unity element in eRe, $\mathcal{R}(eRe) = e^3\mathcal{R}(eRe)e^3 \subseteq eRe\mathcal{R}(eRe)eRe \subseteq e\mathcal{R}(R)e$ and $e\mathcal{R}(R)e = e^2\mathcal{R}(R)e^2 \subseteq eR\mathcal{R}(R)Re \subseteq \mathcal{R}(eRe) \Rightarrow \mathcal{R}(eRe) = e\mathcal{R}(R)e$. □

3. The Jacobson Radical

Definition 12. (see [1]) An element $a \in R$ of a semiring R is said to be left quasiregular if there exists an element $b \in R$ such that $b \circ a = b + a + ba = 0$.

In this case the element b is called a left quasi-inverse of the element R.

A right quasiregular and right quasi-inverse are defined correspondingly.

A semiring R is called quasiregular if each of its elements are left quasiregular.

Note 1. R is quasiregular if and only if $(R, \circ)$ is a semigroup with identity element 0.

Proposition 13. *$(R, \circ)$ is a monoid for any semiring R and 0 is the unity element of the circle operation $\circ$, that is,*

$$(a \circ b) \circ c = a \circ (b \circ c), \quad \forall\ a, b, c \in R,$$

$$0 \circ a = a \circ 0 = a, \quad \forall\ a \in R.$$

Proof. Proof is straightforward ☐

Proposition 14. *Every nilpotent element of a semiring is left quasiregular but not conversely.*

Definition 15. The element r of the semiring R is said to be right semiregular if there exist elements $r_1, r_2 \in R$ such that $r + r_1 + rr_1 = r_2 + rr_2$.

Definition 16. The right ideal I is said to be right semiregular if for every pair of elements $r_1, r_2 \in I$ there exist elements $s_1, s_2 \in I$ such that $r_1 + s_1 + r_1s_1 + r_2s_2 = r_2 + s_2 + r_2s_1 + r_1s_2$.

Lemma 17. (see [3]) *If I and I^* are right semiregular ideals then $I + I^*$ is a right semiregular ideal.*

Theorem 18. (see [3]) *The sum S of all the right semiregular ideals of a semiring R is a right semiregular two-sided ideal.*

Definition 19. The right Jacobson radical S of a semiring R is the sum of all right semiregular ideals of R.

In a corresponding manner, we obtain the left Jacobson radical S^* of R as the sum of all left semiregular ideals of R. An ideal of R is said to be semiregular if it is both right and left semiregular.

Theorem 20. (see [3]) *The right Jacobson radical S of a semiring R is a left semiregular ideal.*

We may now refer the Jacobson radical $\mathcal{J}(R)$ of a semiring R.

Proposition 21. *For any semiring R, $\mathcal{J}(R)$ is a subtractive ideal (k-ideal).*

Proposition 22. *Let R be a cancellative and semisubtractive semiring. If $\overline{m}$ is a maximal ideal in R^e, then $\overline{m}^c$ is a maximal ideal in R.*

Proposition 23. *If R is a cancellative and semisubtractive semiring, then $\mathcal{J}(R)$ is the intersection of maximal k-ideals of R.*

The semiregularity in a semiring is equivalent to the quasiregularity in a ring, i.e., either $a+b-ab=0$ or $a+b+ab=0$. But the later part is used as a definition for the quasiregularity in [1]. Therefore we have the following.

1. The Jacobson radical class $\mathcal{J}$ is the class of all quasi-regular semirings.

2. The Jacobson radical class $\mathcal{J}$ of a semiring R is the sum of all quasi-regular ideals of R equivalently the sum of semiregular ideals of a semiring R.

3. Jacobson radical $\mathcal{J}$ is hereditary.

4. The $\mathcal{J}$ contains properly Köthes nil radical class $\mathcal{N}$.

Definition 24. A left R-semimodule is said to be simple if it has no non-zero proper subsemimodules.

Definition 25. A semiring $R \neq 0$ is called left primitive (primitive) if R contains a maximal left k-ideal L such that $xR \subseteq L$, implies that $x=0$.

Definition 26. An ideal P of a semiring R is said to be semiprimitive if P is the annihilator of the factor semimodule of some semimodular semimaximal left ideal L of the given semiring.

Remark 27. Every semiprimitive ideal is a k-ideal.

It clear from the definition of primitivity that if the maximal left k-ideal L happens to be a two-sided ideal, then necessarily $L=0$, and the primitive semiring is a division semiring. In particular, a commutative semiring is primitive if and only if it is a semifield. The primitive semirings are non-commutative generalizations of semifield.

Reformulating the definition of a primitive semiring. A semiring R is said to be primitive if and only if R possesses a maximal left k-ideal L such that

$$(L:R)_R = \{x \in R \mid xR \subseteq L\} = 0$$

if and only if the annihilator $(0:R/L)_R$ of the simple R-semimodule R/L is 0. To see this,

$$\begin{aligned}(0:R/L)_R &= \{x \in R \mid x(R/L) = 0\}\\ &= \{x \in R \mid xR/L = L\}\\ &= \{x \in R \mid xR \subseteq L\}\\ &= (L:R)_R\}.\end{aligned}$$

So, if

$$\begin{aligned}(L : R)_R = 0 &\Rightarrow (0 : R/L)_R = 0\\ &\Rightarrow (0 : R/L)_R = 0\\ &\Rightarrow R/L \textit{ is simple},\end{aligned}$$

and conversely.

Proposition 28. *The factor semiring $R/\mathtt{m}$ is simple if and only if $\mathtt{m}$ is the maximal k-ideal in R.*

Proposition 29. *Let R be an additively cancellative and semisubtractive semiring. R is primitive if and only if there exists an additively cancellative simple faithful R-semimodule.*

Proof. Suppose that R is a primitive semiring. Therefore R/L is a faithful simple R-semimodule where L is a maximal left k-ideal of R as required in the definition.

Let M be an additively cancellative and faithful simple R-semimodule. For any fixed $m \in M, m \neq 0$, the mapping $f\colon R \to M$ given by $f(a) = am$, for all $a \in R$ is an onto steady morphism. Therefore $R/kerf \cong M$. Since M is simple, $kerf = L$ is a maximal k-ideal in R.

Moreover M is faithful therefore $(0\colon R/L)_R = 0$. Hence R is primitive. □

Theorem 30. (see [3]) *Semiradical of a semiring contains a Jacobson radical $\mathcal{J}(R)$ of R.*

Theorem 31. (see [3]) *Semiradical of a semiring is the intersection of semimodular semimaximal (maximal) ideals in R.*

Theorem 32. (see [3]) *Semiradical of a semiring is the intersection of semiprimitive (primitive) ideals in R.*

Theorem 33. *For an additively cancellative and semisubtractive semiring R, the $\mathcal{J}(R)$ is the intersection of semimodular semimaximal (maximal) ideals in R*

Theorem 34. *For an additively cancellative and semisubtractive semiring R, the $\mathcal{J}(R)$ is the intersection of semiprimitive (primitive) ideals in R.*

The Jacobson radical of a semiring has many further useful characterizations.

Definition 35. A semiring R is said to be semiprimitive if the intersection of all primitive ideals of R is 0.

In a view of Theorem 33-34 semiprimitivity is just an another name of semisimplicity for an additively cancellative and semisubtractive semiring R.

Theorem 36. *The $\mathcal{J}(R)$ of a semiring R is given as the intersection*

$$\mathcal{J}(R) = \cap\{a \in R/\ aM = 0,\ for\ any\ irreducible\ R-semimodule\ M\}.$$

Theorem 37. *Every primitive semiring is a prime semiring.*

Proof. Suppose that R is a primitive semiring, and M is a faithful irreducible R-semimodule. Claim that R is prime.

Let $I \neq 0$ and $K \neq 0$ be ideals in R such that $IK = 0$. Since M is faithful irreducible, therefore $KM \neq 0$, implies that $KM = M$. Also $IM \neq 0$, $\Rightarrow IM = M$. But $M = IM = IKM = 0$, a contradiction. Hence R is a prime semiring. □

Definition 38. A semiring R with unity is said to be a division semiring if every non-zero element of it has multiplicative inverse.

Lemma 39. *If M and N are simple R-semimodules, then a non-zero steady R-homomorphism $f\colon M \to N$ is an isomorphism.*

Proof. Clearly $ker f$ and $Im f$ are subsemimodules of M and N respectively. Since M is simple, $ker f = 0$, shows that f is one-one. Now $Im f = N$, since N is simple and $f \neq 0$, implies that f is onto. Hence f is an isomorphism. □

Let R be a primitive semiring and let M be a faithful irreducible R-semimodule. Since M is simple, therefore a steady R-semimodule homomorphism is either zero or an automorphism.

Lemma 40. *If $M \in R$-smod is simple, then the set of all steady R-homomorphisms $\#End_R(M)$ is a division semiring.*

Proof. If $M = N$, then proof follows immediately by lemma 39. □

It should be noted that the endomorphism semiring of an additively cancellative, semisubtractive, irreducible and faithful R-semimodule M is a division semiring D, $D = End(M)$.

Henceforth we are assuming all semirings (semimodules) are additively cancellative and semisubtractive.

Thus the endomorphism semiring of an irreducible and faithful R-semimodule M is a division semiring D, $D = End(M)$.

Consider the endomorphism of the additive semigroup M^+, that is, $End(M^+)$. We have clearly $D \subseteq End(M^+)$ and any element $a \in R$, the left multiplication $f_a(x) = ax$, for all $x \in M$, is in $End(M^+)$ and

$$\begin{aligned} f_a(x) = f_b(x) &\Rightarrow ax = bx \\ &\Rightarrow [ax, bx] = 0 \\ &\Rightarrow [a, b][x, 0] = 0 \\ &\Rightarrow [a, b] = 0 \\ &\Rightarrow a = b. \end{aligned}$$

Remark 41. Note that in this case if M is R-faithful, then M^e is R^e faithful.

Thus R can be embedded into $End(M^+)$ under the map defined by $a \hookrightarrow f_a$. Moreover for all elements $d \in D$ and $a \in R$, we have $df_a = f_a d$, implies that $d \in \mathbb{Z}_R(M^+)$, the center of R in $End(M^+)$. In fact $D = \mathbb{Z}_R(M^+)$. Thus an irreducible R-semimodule may be viewed as semivector space over the division semiring $D = \mathbb{Z}_R(M^+)$ and the semiring R as a subsemiring of linear transformations of the semivector space M. Therefore in a view of proposition 28 we have the following.

Theorem 42. *Every primitive semiring R is isomorphic to a subsemiring of all linear transformations of semivector space over a division semiring.*

Acknowledgments

The second author greatly acknowledges the support from the NBHM, DAE, Mumbai.

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