International Journal of Pure and Applied Mathematics
Volume 72 No. 3 2011, 259-278

CONSTELLATION PROBLEMS AND INTEGRABILITY FOR LINEAR ODE'S WITH CONSTANT COEFFICIENTS

Benno Fuchssteiner[1 §], Kai Gehrs[2]

[1]Rheinsbergerstrasse 50,
D 10435, Berlin, GERMANY
e-mail: bf@fuchssteiner.de

[2]Department of Mathematics
University of Paderborn
D-33100, Paderborn, GERMANY
e-mail: Kai.Gehrs@math.uni-paderborn.de

Abstract: In this note we show that for each system of linear differential equations with constant coefficients – wether or not the characteristic polynomial of the coefficient matrix can be solved algorithmically and symbolically – a suitable set of constants of motion can be constructed in a purely algorithmic way. The constants of motion are given as integrals over rational functions involving the matrix elements and the components of the solution vectors. Thus, these systems are integrable in the same way as systems considered in integrability theory in the situation of the Liouville-Arnold theorem for Hamiltonian systems.

AMS Subject Classification: 34-XX, 37K05
Key Words: linear differential equations, constants of motion, integrability

1. Constellations and Integrability

We[1] consider ordinary differential equations (ODEs) or partial differential equations (PDEs)

$$u_t = K(u), \tag{1.1}$$

where $u(t)$ runs on some suitable manifold (finite dimensional in case of an

Received: October 30, 2009

§Correspondence author

[1]We are indebted to Walter Oevel for the helpful discussions we had on the subject.

ODE and infinite dimensional in case of a PDE) and $K(u)$ is a C^∞-vector field on that manifold. The manifold under consideration is called *phase space*. Often one is focused on what is called an *explicit solution*, i.e. a solution of the form $u = u(t, u(0))$, where $u(0)$ represents the initial data. However, in many applications one is not really interested in this explicit solution, but more in other problems.

For example, whether or not the orbit – under certain conditions on the initial data – meets a certain target point, or whether a certain quantity converges to zero, and so on. A practical example of that kind is when initial data have to be chosen such that a ballistic object can reach a prescribed point with a prescribed velocity. Solving this problem with explicit solutions requires the determination of an inverse function (which cannot always be done algorithmically).

Thus the general formal problem is: Choose in a set of initial data (eventually determined by one or several constraints) a value such that another set given by other constraints intersects with the orbit of the solution.

For the sake of simplicity, we restrict our considerations on finite dimensional phase spaces, say on phase spaces of dimension $N \in \mathbb{N}$.

1.1. The Formal Constellation Problem: *Given the dynamic system*

$$u_t = K(u) \tag{1.1}$$

and furthermore a submanifold F (called initial constellation*) of codimension k and another submanifold Z (called* target constellation*) of codimension $N-k$. Choose a starting point $u(0) \in F$ such that the orbit $u(t, u(0))$ has a common point with Z.*

For the solution of this problem having the solution of the differential equation (1.1) in explicit form $u = u(t)$, is more of a nuisance than a benefit. This because t has to be expressed first in terms of other quantities on the constellation manifolds, and then this immaterial quantity has to be eliminated from the solution formulas.

We could solve the problem more easily if the solution, at least locally, were of the form

$$t = f(u), \quad g(u) = 0, \tag{1.2}$$

with $f(u)$ a suitable scalar function and $g(u)$ a multi-component function (of codimension 1).

This form of solution is what we call *integrable*. Surprisingly, an integrable solution form is not even known in the case of a linear system with constant coefficients.

Let us make this notion of integrability more precise (for more details we refer to the forthcoming paper [1]). First, we rewrite (1.1) in component form

$$\begin{pmatrix} u_1(t) \\ u_2(t) \\ \vdots \\ u_{N-1}(t) \\ u_N(t) \end{pmatrix}_t = \begin{pmatrix} k_1(u_1,\dots,u_N) \\ k_2(u_1,\dots,u_N) \\ \vdots \\ k_{N-1}(u_1,\dots,u_N) \\ k_N(u_1,\dots,u_N) \end{pmatrix}, \tag{1.3}$$

where the u_i denote the components of u and the k_i the components of the vector field K. We consider the smallest algebra $\mathcal{A}$ containing the components of the general manifold point u, and a predefined and fixed set of functions which contains at least the functions $k_1,\dots,k_N$.

Definition 1.2. Equation (1.3) then is said to be *strictly integrable* whenever there are new coordinates $F_1,\dots,F_N$ in phase space, which can be obtained by quadratures[2] of quotients of elements in $\mathcal{A}$, such that the flow given by the equation (1.3) leaves the N coordinates $F_2,\dots,F_N$ invariant and lets the first coordinate F_1 grow linearly with t.

In contrast to this *strict integrability*, we speak of *integrability* if in addition to the elements from $\mathcal{A}$ also implicit functions, involving such elements, are involved in the construction of the coordinate functions $F_1,\dots,F_N$.

2. An Integrability Result

We review and specialize principal results which will be exhibited in more detailin [1]. For this we now assume that the manifold M is some N-dimensional vector space V. We recall that when Γ, K are C^∞-vector fields, then by $[\![\Gamma, K]\!]$ we denote the *commutator of the vector fields* K *and* Γ which is defined as

$$\begin{aligned}[\![\Gamma, K]\!](u) &:= K'(u)[\Gamma(u)] - \Gamma'(u)[K(u)] \\ &:= \frac{\partial}{\partial\epsilon}_{|\epsilon=0}\big(K(u+\epsilon\,\Gamma(u)) - \Gamma(u+\epsilon\,K(u))\big).\end{aligned} \tag{2.1}$$

The derivatives $K'(u)[\Gamma(u)]$ and $\Gamma'(u)[K(u)]$ are referred to as *directional derivatives* (see e.g. [7], Section 4.1 on variational calculus).

[2] i.e. the gradients of the coordinate functions $F_1,\dots,F_N$ are quotients of elements in $\mathcal{A}$.

Consider again the evolution equation (1.1), where for consistency of the subsequent notation we now write K_1 instead of K (K_1 in contrast to k_1 being an N-component vector field)

$$u_t = K_1(u). \tag{2.2}$$

Let $K_2(u), \ldots, K_N(u)$ be symmetry generators for (2.2), i.e. C^∞-vector fields such that

$$[\![K_1(u), K_i(u)]\!] = 0$$

for all $1 \leq i \leq N$. In addition, we assume that the $K_2, \ldots, K_N$ commute among each other, i.e. $[\![K_i(u), K_j(u)]\!] = 0$ for all $2 \leq i, j \leq N$.

A point $\hat{u} \in M$ is said to be *non-degenerate* if $\{K_1(\hat{u}), K_2(\hat{u}), \ldots, K_N(\hat{u})\}$ is a set of linear independent vectors in the tangent space at $\hat{u}$. In that case, because of continuity all points in some suitable star shaped neighborhood $\mathcal{V}(\hat{u})$ of $\hat{u}$ are non-degenerate as well.

The following holds (see [1]):

2.1. Integrability Result: *Let $\mathcal{V}(\hat{u})$ be a star shaped neighborhood of $\hat{u}$ consisting of non-degenerate points, then equation (2.2) is strictly integrable in $\mathcal{V}(\hat{u})$, and for the gradients of the coordinate functions $F_1, \cdots, F_N$ we may take the rows of the inverse of the matrix defined by $[K_1(u), K_2(u), \ldots, K_N(u)]$. This inverse exists for all points in $\mathcal{V}(\hat{u})$, since we assumed non-degeneracy.*

Let us demonstrate how from here on, the coordinate functions are constructed explicitly. We parameterize the line between the center $\hat{u}$ of $\mathcal{V}(\hat{u})$ and some arbitrary point $u \in \mathcal{V}(\hat{u})$ via

$$U(\lambda, u) = \hat{u} + \lambda(u - \hat{u}) \tag{2.3}$$

for $0 \leq \lambda \leq 1$. Since the $\{K_1(U(\lambda, u)), K_2(U(\lambda, u)), \ldots, K_N(U(\lambda, u))\}$ are linear independent (i.e. a basis of V), we find unique scalar coefficients $\beta_i(\lambda, u)$, $0 \leq i \leq N$, such that

$$u - \hat{u} = \sum_{i=0}^{N} \beta_i(\lambda, u)\, K_i(U(\lambda, u)). \tag{2.4}$$

Define

$$F_j(u) = \int_0^1 \beta_j(\lambda, u)\, d\lambda \tag{2.5}$$

for all $0 \leq j \leq N$. Then

Lemma 2.2. *The solution curves in $\mathcal{V}(\hat{u})$ of the evolution equation (2.2) for initial value $\vec{u}(0)$ are characterized in implicit form by*

$$\begin{pmatrix} F_1(u(t)) \\ F_2(u(t)) \\ \vdots \\ F_N(u(t)) \end{pmatrix} = \begin{pmatrix} F_1(u(0)) + t \\ F_2(u(0)) \\ \vdots \\ F_N(u(0)) \end{pmatrix}, \tag{2.6}$$

where the $F_i(u(t))$, $0 \le i \le N$, are defined as in (2.5).

Proof. Consider the matrix with the K_i's as columns

$$A(\lambda, u) := [K_1(U(\lambda, u)), \cdots, K_N(U(\lambda, u))], \tag{2.7}$$

then from the integrability result above, we know that its inverse has as rows the gradients of the suitable coordinates

$$A(\lambda, u)^{-1} = \begin{pmatrix} \nabla F_1(U(\lambda, u)) \\ \nabla F_2(U(\lambda, u)) \\ \vdots \\ \nabla F_N(U(\lambda, u)) \end{pmatrix}. \tag{2.8}$$

We know that we can recover the differences of the F_i at the endpoints of the lines from $\hat{u}$ to u by integration along these lines

$$F_i(u) - F_i(\hat{u}) = \int_0^1 \nabla F_i(U(\lambda, u)) \frac{dU(\lambda, u)}{d\lambda} d\lambda. \tag{2.9}$$

On the other hand we obtain

$$A(\lambda, u) \left(A(\lambda, u)^{-1} \frac{dU(\lambda, u)}{d\lambda} \right) = \frac{dU(\lambda, u)}{d\lambda} = u - \hat{u}. \tag{2.10}$$

Because of (2.7) a comparison of this with (2.4) yields

$$\left(A(\lambda, u)^{-1} \frac{dU(\lambda, u)}{d\lambda} \right) = \begin{pmatrix} \beta_1(\lambda, u) \\ \beta_2(\lambda, u) \\ \vdots \\ \beta_N(\lambda, u) \end{pmatrix}. \tag{2.11}$$

By picking out the i-th line we obtain

$$\nabla F_i(U(\lambda, u)) \frac{dU(\lambda, u)}{d\lambda} = \beta_i(\lambda, u), \tag{2.12}$$

and the right-hand-side of (2.9) must be equal to (2.5). $\square$

Remark 2.3. Of course, the implicit characterization of orbits as given by (2.6) also holds outside of $\mathcal{V}(\hat{u})$, namely in any simply connected open subset where the F_i are continuously defined and do have the rows of $[K_1(u), K_1(u), \cdots, K_N(u)]^{-1}$ as gradients.

One should observe that in the coordinate space given by the F_i the flow under consideration is linear. Hence integrability implies linearization, however, not every linearization yields integrability in the strict sense. This statement seems surprising, because in soliton theory, i.e. in integrability or "pseudo"-integrability in case of PDE's, both these notions are often considered as being equivalent (see the discussion in [12]).

Lemma 2.2 gives the solutions of (2.2) in the form as requested by (1.2) when we define

$$f(u) := F_0(u(t)) - F_0(u(0)), \quad g(u) := \begin{pmatrix} F_1(u(t)) - F_1(u(0)) \\ \vdots \\ F_N(u(t)) - F_N(u(0)) \end{pmatrix}.$$

Remark 2.4. This integrability result presented here, should not be mixed up with Lie's celebrated result (see e.g. [10] p. 86 or [7], Theorem 2.64, p. 155). Lie's result deals with the case of Lie point symmetries whereas the present result deals with flows in N-dimensional space, which corresponds to the case of Lie–Baecklund symmetries. The difference to Lie's theorem becomes clear by the fact that generators for Lie point symmetries define two-dimensional flows.

Note that to invert $[K_1(u), \ldots, K_N(u)]$ means to compute the inverse of a matrix with symbolic coefficients. Using standard Gaussian elimination to compute $[K_1(u), \ldots, K_N(u)]^{-1}$ usually is not efficient enough for larger values of N, since the performance of standard Gaussian elimination on symbolic matrices suffers from increasing expression swell. This expression swell arises from the division by pivot elements during the elimination process. Costly normalization strategies have to be applied to the matrix components on the one hand to keep the components as small and simple as possible and on the other hand we must be able to decide, which elements may serve as pivot elements in the elimination process (i.e. one has to decide, whether a complex symbolic expression simplifies to zero or not).

Hence, in a practical implementation of the above proposed strategy it is essential to use specialized algorithms for the inversion of symbolic matrices to keep up efficiency. Such an algorithm, which works fine in practice and in the examples we considered, is proposed by Sasaki and Murao in [8][3]. However for

[3] An implementation of these methods is already available in the standard distribution of

the purpose of this paper an even more efficient algorithm for inverting matrices with symbolic data had to be developed, this is published in [2].

3. Application to Linear ODEs

We apply the symmetry results stated in the preceding section for systems of linear ODEs. It is surprising, that in this well-investigated case, something new can be presented. Therefore, in order that the sceptic reader may feel more at ease, we shall later on prove for this case all results explicitly (however by use of infinite sums).

3.1. Systems of ODEs with Constant Coefficients

Let $M_1, \ldots, M_N \in \mathbb{R}^{N\times N}$ be $(N \times N)$-matrices such that

$$[\![M_i, M_j]\!] = M_j\, M_i - M_i\, M_j = 0$$

for all $1 \leq i, j \leq N$.

Standard Situation: As *standard situation* we denote the case, when M is an $(N \times N)$-matrix, the equation under consideration is the ODE

$$u(t)_t = Mu(t) \tag{3.1}$$

and the M_i are

$$\begin{aligned} M_1 &= M \\ M_2 &= M^2 \\ &\vdots \\ M_{N-1} &= M^{N-1} \\ M_N &= I_N, \end{aligned}$$

where I_N denotes the $(N \times N)$-identity matrix. Clearly these matrices do commute. However, when the matrix is not invertible, then additional arguments are needed.

Consider as manifold M an N-dimensional vector space V, where these matrices act, and here in particular the vector fields

$$\vec{u} \to K_i(\vec{u}) := M_i\, \vec{u}$$

---

computer algebra systems like MuPAD.

for $1 \leq i \leq N$, where

$$\vec{u} = \begin{pmatrix} u_1 \\ \vdots \\ u_N \end{pmatrix}$$

denotes the manifold variable. I.e. we are interested in the evolution equations

$$ {}^i\vec{u}_t = M_i \ {}^i\vec{u}, \tag{3.2}$$

where $1 \leq i \leq N$. Here we have chosen a notation where a superscript on the left distinguishes between different vectors and a subscript on the right between the components of a vector. Of course, formally a solution for (3.2) is – by use of the exponential of a matrix – easily found:

$$ {}^i\vec{u}(t) = \exp(tM_i) \ {}^i\vec{u}(0). \tag{3.3}$$

However, when the characteristic polynomial of the M_i cannot be solved algorithmically, then this does not lead to explicit and algorithmically accessible formulas for the flow in the original coordinate system.

The vector fields K_k clearly commute, since the matrices do commute. Now, we form a matrix as in (2.7) and as before we call an element $\vec{u} \in V$ *non-degenerate*, if this matrix

$$\begin{aligned} A_{\vec{u}} &= [\ K_1(\vec{u}), \cdots, K_N(\vec{u})\ ] \\ &= [\ M_1\,\vec{u} \quad \cdots \quad M_N\,\vec{u}\ ] \end{aligned}$$

has rank N, i.e. is an invertible matrix.

For a non-degenerate $\vec{u}$ the vector fields $K_1(\vec{u}), \ldots, K_N(\vec{u})$ are linearly independent.

The systems (3.2) we want to solve for a non-degenerate initial condition[4] explicitly, even in case of matrices where the characteristic polynomials cannot be solved algorithmically. In order to achieve this, we are willing to allow for a new coordinate system as well as for expressions involving integrals.

One should observe that when ${}^i\vec{u}(0)$ is non-degenerate, then all ${}^i\vec{u}(t)$ are non-degenerate. This is true, because the matrix

$$A_{{}^i\vec{u}(t)} = [\ M_1\ {}^i\vec{u}(t), \cdots, M_N\ {}^i\vec{u}(t)\ ]$$

is obtained from

$$A_{{}^i\vec{u}(0)} = [\ M_1\ {}^i\vec{u}(0), \cdots, M_N\ {}^i\vec{u}(0)\ ]$$

[4]In a while we will see what to do in cases of degenerate initial conditions.

by application of the exponential of M_i, clearly an invertible matrix.

Now, we consider the quantities introduced in (2.4) and (2.5). By this we obtain for $\lambda \in [0,1]$ with regard to fixed $\vec{u}(0)$ and general $\vec{u}$

$$\vec{\beta}(\lambda, \vec{u}) = (A_{\vec{u}(0)} + \lambda(A_{\vec{u}-\vec{u}(0)}))^{-1}(\vec{u} - \vec{u}(0)), \tag{3.4}$$

where we put the different β_j as entries into a vector $\vec{\beta}$. In the same way we can put the F_j as entries into a vector:

$$\vec{F}(\vec{u}) = \int_0^1 \vec{\beta}(\lambda, \vec{u})\, d\lambda. \tag{3.5}$$

This formula means that we have taken the line integral from $\vec{u}(0)$ to $\vec{u}$ along the straight line connecting them. Since for these $\beta's$ line integrals are path independent (they are closed covector fields), we could take any other path in the same simply connected component of V, connecting these two points, thus avoiding eventual singularities on that line. For getting simple and compact formulas and for demonstrating the method, we have chosen the most simple path. However, when poles arise, then it is essential in which way the path around the poles is chosen, since different pathes then differ by the residuum of the pole. This multi-valued behavior occurs when the orbits of the flows are periodic or in general when the assignment of t to curve points is a multi-layered function.

Remark 3.1. Another way, by choosing a different path, for computing the $\vec{F}(\vec{u})$ would be the following: Let $G_{ij}(u_1, u_2, \cdots, u_N)$ be the the element in line i and row j of the inverse of the matrix

$$A_{\vec{u}} = [\ M_1\,\vec{u}, \cdots, M_N\,\vec{u}\ ],$$

where u_1 to u_n denote the components of $\vec{u}$ and $u_1(0)$ to $u_n(0)$ those of $\vec{u}(0)$. Then

$$\begin{aligned} F_i(\vec{u}) = &\int_{u_1(0)}^{u_1} G_{i,1}(u_1, u_2(0), u_3(0), \cdots, u_N(0))\, du_1 \\ &+ \int_{u_2(0)}^{u_2} G_{i,2}(u_1, u_2, u_3(0), \cdots, u_N(0))\, du_2 \\ &+ \int_{u_3(0)}^{u_3} G_{i,3}(u_1, u_2, u_3, u_4(0), \cdots, u_N(0))\, du_3 \\ &\vdots \end{aligned} \tag{3.6}$$

$$+\int_{u_N(0)}^{u_N} G_{i,N}(u_1, u_2, u_3, \cdots, u_N)\, du_N \ .$$

However, from the computational point of view (3.5) seems more advantageous since a costly integration by use of (3.6) has to be performed N^2-times, whereas in (3.5) we only need N integrations.

Anyway, the solution, locally reads in our new coordinates

$$\vec{F}({}^{i}\vec{u}(t)) = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ t \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \tag{3.7}$$

where apart form the i-th row only zeros occur. Locally, the system now is linear, and all the quantities – up to one integration over rational functions – can be written down explicitly.

Remark 3.2. One should observe that the quantities given by (3.5) are constants of motion even in the case when other initial values are chosen. This is a consequence from the fact that they are the potentials of invariant covector fields.

Hence a full set of constants of motions can be determined algorithmically for every linear system with constant coefficients, whether or not the characteristic polynomial of the coefficient matrix can be solved algorithmically.

So, when no singularities occur, these constants may be used in order to describe solutions for different initial values. For example, if the initial value for (3.2) is $\vec{\tilde{u}}(0)$ instead of $\vec{u}(0)$, then for obtaining a solution formula the left-hand-side $\vec{F}({}^{i}\vec{u}(t))$ in (3.7) has to be replaced by $\vec{F}({}^{i}\vec{u}(t)) - \vec{F}(\vec{\tilde{u}}(0))$.

Remark 3.3. When we are willing to use, as in the case when we consider exponentials of matrices, analytic functions with matrix entries (defined by infinite sums) then we can express our solutions in more closed form:

$$\vec{F}(\vec{u}) = \ln(A_{\vec{u}(0)}^{-1} A_{\vec{u}})(A_{\vec{u}} - A_{\vec{u}(0)})^{-1}(\vec{u} - \vec{u}(0)). \tag{3.8}$$

Proof. Assuming that all series to be considered are converging and all matrices are invertible we find for matrices B, C

$$(B + \lambda C)^{-1} = \big(B(I + \lambda B^{-1}C)\big)^{-1}$$

$$
\begin{aligned}
&= \left(I + \lambda B^{-1}C\right)^{-1} B^{-1} \\
&= \big(\big(I - \lambda(B^{-1}C) + \lambda^2(B^{-1}C)^2 \\
&\quad -\lambda^3(B^{-1}C)^3 + \cdots\big)\,(B^{-1}C)\big)\,(C^{-1}) \\
&= \big((B^{-1}C) - \lambda(B^{-1}C)^2 + \lambda^2(B^{-1}C)^3 - \lambda^3(B^{-1}C)^4 + \cdots\big)\,(C^{-1}) \\
&= \left(\frac{d}{d\lambda}\ln(I + \lambda B^{-1}C)\right)(C^{-1}),
\end{aligned}
$$

where I denotes the identity matrix (of suitable format) and where in the third line a geometric series expansion was used and where the logarithm is defined by

$$\ln(I + \lambda M) = \lambda M - \frac{1}{2}\lambda^2 M^2 + \frac{1}{3}\lambda^3 M^3 - \frac{1}{4}\lambda^4 M^4 + \cdots\ .$$

Hence,

$$\int_0^1 (B + \lambda C)^{-1} d\lambda = \ln(I + B^{-1}C)C^{-1}. \tag{3.9}$$

Now substituting $B = A_{\vec{u}(0)}$ and $C = A_{\vec{u}-\vec{u}(0)} = A_{\vec{u}} - A_{\vec{u}(0)}$ we obtain the desired result. $\square$

Using this result we are now able to present a "direct" proof of (3.7) within the formal calculus of analytic matrix functions.

Remark 3.4. If

$$\vec{u}(t)_t = M_i \vec{u}(t)$$

or equivalently

$$\vec{u}(t) = \exp(tM_i)\vec{u}(0)$$

then

$$\vec{F}(\vec{u}(t)) = t\,\vec{e}_i := \begin{pmatrix} 0 \\ \vdots \\ 0 \\ t \ \ \text{(the } i\text{-th line)} \\ 0 \\ \vdots \\ 0 \end{pmatrix}. \tag{3.10}$$

Proof. We need the following simple and basic facts:

1. By power series expansion we see for matrices B, X

$$\ln(B(I+X)B^{-1}) = B\ln(I+X)B^{-1},$$

because the inner products $B^{-1}B$ do cancel.

2. By matrix calculus we have

$$(\exp(tM)B - B)^{-1} = B^{-1}(\exp(tM) - I)^{-1}.$$

3. By definition of the matrix A we obtain:

$$A_{u(0)} \begin{pmatrix} 0 \\ \vdots \\ 0 \\ t \text{ (i-th line)} \\ 0 \\ \vdots \\ 0 \end{pmatrix} = t\, M_i u(0).$$

Now by use of Remark 3.3 and straightforward computation we obtain

$$\begin{aligned}
\vec{F}(\vec{u}(t)) &= \ln(A^{-1}_{\vec{u}(0)}\, A_{\vec{u}(t)})\, A^{-1}_{\vec{u}(t)-\vec{u}(0)}\, (\vec{u}(t) - \vec{u}(0)) \\
&= \ln(A^{-1}_{\vec{u}(0)}\, e^{tM_1}\, A_{\vec{u}(0)})\, (e^{tM_1}\, A_{\vec{u}(0)} - A_{\vec{u}(0)})^{-1}(e^{tM_1}\, \vec{u}(0) - \vec{u}(0)) \\
&= A^{-1}_{\vec{u}(0)}\, t\, M_1\, \vec{u}(0) \\
&= t\, \vec{e}_1,
\end{aligned}$$

where 1. and 2. were used by going from the second to the third line, and 3. by going from the third to the fourth. $\square$

3.2. First Example

Let us illustrate our solution formulas by presenting a simple example, an example chosen such that the solution process can be compared to the classical one:

$$\vec{u}_t = \begin{pmatrix} 1 & 2 \\ 7 & 5 \end{pmatrix} \vec{u}. \tag{3.11}$$

As initial value $\vec{u}(0)$ we choose

$$\vec{u}(0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

Then the matrix (2.7) is in this case

$$A(\lambda, u) = \begin{pmatrix} u_1 - \lambda\,(u_1 - 1)\,, & u_1 + 2\; u_2 - 2\,\lambda\,u_2 - \lambda\,(u_1 -\; 1) \\ u_2 - \lambda\,u_2, & 7\,u_1 + 5\,u_2 -\; 5\,\lambda\,u_2 - 7\,\lambda\,(u_1 - 1) \end{pmatrix},$$

and as inverse matrix (2.8) we obtain

$$\begin{aligned} &A(\lambda, u)^{-1} \\ &= d_{\lambda,u}^{-1} \begin{pmatrix} 7\,\lambda + 7\,u_1 + 5\,u_2 - 7\,\lambda\,u_1 - 5\,\lambda\,u_2, & \lambda\,u_1 - u_1 -\; 2\,u_2 - \lambda + 2\,\lambda\,u_2 \\ \lambda\,u_2 - u_2, & \lambda + u_1 - \lambda\; u_1 \end{pmatrix}, \end{aligned}$$

where the denominator is

$$\begin{aligned} d_{\lambda,u} &= 7\,\lambda^2\,u_1^2 + 4\,\lambda^2\,u_1\,u_2 - 14\,\lambda^2\,u_1 - 2\,\lambda^2\,u_2^2 - 4\,\lambda^2\,u_2 \\ &\quad + 7\,\lambda^2 - 14\,\lambda\;\, u_1^2 - 8\,\lambda\,u_1\,u_2 + 14\,\lambda\;\, u_1 + 4\,\lambda\,u_2^2 + 4\,\lambda\,u_2 \\ &\quad + 7\,u_1^2 + 4\,u_1\,u_2 - 2\,u_2^2. \end{aligned}$$

The solution then is

$$\begin{aligned} &0 = \\ &\int_0^1 -\frac{u_2\;(\lambda + u_1 + 2u_2 - \lambda u_1 - 2\lambda u_2) - (u_1 - 1)\,(7\lambda + 7u_1 + 5u_2 - 7\lambda u_1 - 5\lambda u_2)}{d_{\lambda,u}} d\lambda \end{aligned}$$

$$t = \int_0^1 \frac{u_2\,(\lambda + u_1 - \lambda u_1) - (u_2 - \lambda u_2)\,(u_1 - 1)}{d_{\lambda,u}} d\lambda.$$

Since the denominator can easily be decomposed into linear factors in λ, the integration can be carried out, and we receive a solution in a coordinate system which in physics would be called a system of *action-angle variables*. The integral on the right-hand-side of the first line would be an *action variable* or a *conserved quantity* (since this quantity does not change on the orbit) and the integral in the second line an angle variable (because it increases linearly with the flow just like the angle in case of the harmonic oscillator). Truely, the solution does look a little bit unfamiliar compared to the classical solution. However, it can be obtained in a purely algorithmic way, and that does not change, when the characteristic polynomial cannot be solved in an algorithmic way. Of course, with higher order, the expression swell will be considerable, a problem which can be handled by using modern computer algebra.

3.3. The Degenerate Case

We are treating the standard situation with $(N \times N)$-matrix M and initial value $\vec{u}(0)$ such that

$$[M\vec{u}(0), M^2\vec{u}(0), \cdots, M^{N-1}\vec{u}(0), \vec{u}(0)] \tag{3.12}$$

is not invertible, i.e. we are dealing with a degenerate case. In this case we choose the smallest k such that

$$\vec{u}(0), M\vec{u}(0), \cdots, M^k\vec{u}(0)$$

are linearly dependent, i.e. for suitable scalars α_j we have

$$M^k\vec{u}(0) = \sum_{j=0}^{k-1} \alpha_j M^j \vec{u}(0). \tag{3.13}$$

Then we restrict our consideration to the M-invariant k-dimensional subspace spanned by

$$\vec{u}(0), M\vec{u}(0), \cdots, M^{k-1}\vec{u}(0). \tag{3.14}$$

These vectors are linearly independent and we choose them as basis for this subspace. The matrix $\tilde{M}$ given by the action of M in that space then has, with respect to this basis, the special form

$$\tilde{M} = \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 & \alpha_0 \\ 1 & 0 & 0 & \cdots & 0 & \alpha_1 \\ 0 & 1 & 0 & \cdots & 0 & \alpha_2 \\ \vdots & \ddots & \ddots & \ddots & & \vdots \\ \vdots & & \ddots & \ddots & 0 & \vdots \\ 0 & 0 & 0 & \cdots & 1 & \alpha_{k-1} \end{pmatrix}. \tag{3.15}$$

Instead of the original system we then solve

$$\vec{v}_t = \tilde{M}\vec{v} \tag{3.16}$$

with initial value

$$\vec{v}(0) = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \tag{3.17}$$

whose solution corresponds uniquely to the solution of the original system and which clearly is a non-degenerate problem. Computing from the solution of this modified problem the solution of the original one amounts to elementary linear algebra.

3.4. Second Example

We have seen that the degenerate case is more simple than the non-degenerate one, insofar as it allows a reduction to a lower dimensional flow. We illustrate this with a simple example. Again an example where the solution process can be compared to the classical one. We consider the flow:

$$\vec{u}_t = \begin{pmatrix} 0 & 2 & -1 \\ -\frac{1}{2} & 1 & 2 \\ -\frac{2}{3} & \frac{2}{3} & 2 \end{pmatrix} \vec{u}. \tag{3.18}$$

We want to find a suitable initial value such that we have to deal with a degenerate case. Such an initial value, for example, could be found by taking any point where a given conserved quantity is stationary, i.e. its gradient vanishes. Since all gradients of conserved quantities (at least in the non-degenerate case) can be expressed by linear combinations of rows $2, \ldots, n$ of the matrix (3.12), a suitable condition would be

$$\mathrm{row}_2 + \mathrm{row}_3 = 0. \tag{3.19}$$

This requirement however, looks like complete nonsense since we look for a vector where (3.12) is **not** invertible, and then, clearly this inverse does not exist. But this obstacle can be circumvented by taking the rows of the matrix of minors instead (i.e. the inverse multiplied by the determinant in the non-degenerate case). This matrix cannot only be computed but any sophisticated method for inverting matrices with symbolic entries will look for an effective computation of this matrix anyway. So let us compute this matrix for the given system. We obtain (up to a factor 18):

$$M^{-1} = \begin{pmatrix} m_{11} & m_{12} & m_{13} \\ m_{21} & m_{22} & m_{23} \\ m_{31} & m_{32} & m_{33} \end{pmatrix}, \tag{3.20}$$

where

$$m_{11} = -\,12\,u_2^2 - 84\,u_2\,u_3 + 30\,u_1\,u_2 + 117\,u_3^2 - 33\,u_1\,u_3, \tag{3.21}$$

$$
\begin{aligned}
m_{21} =& 12\,u_2^2 + 18\,u_2\,u_3 - 12\,u_1\,u_2 - 36\,u_3^2 + 9\,u_1\,u_3,\\
m_{31} =& -7\,u_1^2 + 2\,u_1\,u_2 + 30\,u_1\,u_3 - 4\,u_2^2 + 6\,u_2\,u_3 - 18\,u_3^2,\\
m_{12} =& -30\,u_1^2 + 114\,u_1\,u_3 + 12\,u_2\,u_1 - 36\,u_3^2 - 24\,u_2\,u_3,\\
m_{22} =& 12\,u_1^2 - 36\,u_1\,u_3 - 12\,u_2\,u_1 - 18\,u_3^2 + 36\,u_2\,u_3,\\
m_{32} =& 4\,u_1^2 + 40\,u_1\,u_2 - 66\,u_1\,u_3 - 8\,u_2^2 - 132\,u_2\,u_3 + 180\,u_3^2,\\
m_{13} =& 33\,u_1^2 - 30\,u_1\,u_2 - 117\,u_3\,u_1 + 24\,u_2^2 + 36\,u_3\,u_2,\\
m_{23} =& -9\,u_1^2 + 18\,u_1\,u_2 + 36\,u_3\,u_1 - 36\,u_2^2 + 18\,u_3\,u_2,\\
m_{33} =& -3\,u_1^2 - 48\,u_1\,u_2 + 63\,u_1\,u_3 + 24\,u_2^2 + 126\,u_2\,u_3 - 189\,u_3^2.
\end{aligned}
$$

Condition[5] 3.19 then leads to

$$
\begin{aligned}
0 =& -7\,{u_1}^2 - 10\,u_1\,u_2 + 39\,u_1\,u_3 + 8\,{u_2}^2 + 24\,u_2\,u_3 - 54\,{u_3}^2,\\
0 =& 16\,{u_1}^2 + 28\,u_1\,u_2 - 102\,u_1\,u_3 - 8\,{u_2}^2 - 96\,u_2\,u_3 + 162\,{u_3}^2,\\
0 =& -12\,{u_1}^2 - 30\,u_1\,u_2 + 99\,u_1\,u_3 - 12\,{u_2}^2 + 144\,u_2\,u_3 - 189\,{u_3}^2.
\end{aligned}
\tag{3.22}
$$

One possible solution, among others, is:

$$
u_1 = -1,\ u_2 = \frac{1}{2},\ u_3 = 0. \tag{3.23}
$$

Hence as an initial value $\vec{u}(0)$ for a degenerate case we can choose

$$
\vec{u}(0) = \begin{pmatrix} -1 \\ \frac{1}{2} \\ 0 \end{pmatrix}.
$$

Then obviously

$$
M^2\,\vec{u}(0) = \vec{u}(0) + 2\,M\,\vec{u}(0), \tag{3.24}
$$

which corresponds to the fact that the characteristic polynomial of M can be factored by

$$
P(x) = x^2 - 2\,x - 1. \tag{3.25}
$$

[5]However this condition, although it certainly yields a subspace invariant under the flow, does not look like a stationarity condition since the rows are not gradients. Only divided by the determinant they were gradients, but the determinant becomes zero. But, indeed, a detailed analysis of conserved quantities shows that for any conserved quantity the stationarity condition can be expressed by a linear dependence among the rows of the matrix of minors.

From (3.24) we deduce that a basis of the space $\tilde{V}$ (spanned by $\vec{u}(0), M\ \vec{u}(0)$) is given by

$$\left\{ \begin{pmatrix} -1 \\ \frac{1}{2} \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \right\} \tag{3.26}$$

and $\tilde{V}$ is invariant under (3.18). Therefore we may consider the flow as flow on (3.26), i.e. our vectors under consideration should be

$$\begin{pmatrix} v_1 \\ v_2 \end{pmatrix}. \tag{3.27}$$

Which then reads

$$\vec{v}_t = \begin{pmatrix} 0 & 1 \\ 1 & 2 \end{pmatrix} \vec{v}, \tag{3.28}$$

and the initial value $\vec{v}(0)$ now is

$$\vec{v}(0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix},$$

which clearly is non-degenerate.

It should be observed that in this case we did not construct a full set of constants of motion, only sufficiently many for characterizing all flows on the space spanned by (3.26). If a full set is needed then we have to consider other starting points and to treat the null space of M separately.

3.5. A Single ODE System with Constant Coefficients

Another but similar class of equations integrable by the same methods is that of homogeneous linear differential equations with constant coefficients. Consider

$$u^{(n)}(t) + m_{n-1}\, u^{(n-1)}(t) + \ldots + m_1\, u'(t) + m_0\, u(t) = 0, \tag{3.29}$$

$m_i \in \mathbb{R}$, $0 \le i \le n-1$, $u(t)$ some real valued function. In phase space notation we obtain

$$\underbrace{\begin{pmatrix} u(t) \\ u'(t) \\ \vdots \\ u^{(n-1)}(t) \end{pmatrix}_t}_{\vec{u}(t)_t}$$

$$= \underbrace{\begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & 0 & 0 & & 1 \\ -m_0 & -m_1 & -m_2 & \cdots & -m_{n-1} \end{pmatrix}}_{=M} \underbrace{\begin{pmatrix} u(t) \\ u'(t) \\ \vdots \\ u^{(n-1)} \end{pmatrix}}_{=\vec{u}(t)} \tag{3.30}$$

and we are back at what we considered a few lines ago. Here we always have a non-degenerate situation.

3.6. Concluding Remarks

Often the degenerate case does not occur. For example if the characteristic polynomial of the matrix M is irreducible over some field F then if the initial values are chosen in that field, then the case must be non-degenerate. This because the degeneracy condition determines an invariant subspace for M and the characteristic polynomial for the matrix restricted to that subspace must be a genuine factor of the characteristic polynomial for M itself.

It has certain advantages in case of matrices with non-vanishing traces, i.e. $\text{trace}(M) \neq 0$, to replace the angle variable by another variable evolving in a well defined way with time, and where the integration can always be carried out. This is in particular advisable when the case is degenerate. To explain this consider again the matrix

$$A(\vec{u}) = [M\vec{u}, M^2\vec{u}, \cdots, M^{N-1}\vec{u}, \vec{u}] \tag{3.31}$$

and let

$$L_1(\vec{u}), L_2(\vec{u}), \cdots, L_n(\vec{u})$$

be the rows of the matrix formed by its minors. i.e. in the non-degenerate case its inverse up to a factor given by the determinant. Then, also these rows are not necessarily gradients, the following linear combination

$$L = \text{trace}(M)L_1(\vec{u}) + \text{trace}(M^2)L_2(\vec{u}) + \cdots + \text{trace}(M^{n-1})L_{n-1}(\vec{u}) \\ + nL_n(\vec{u}) \tag{3.32}$$

is a gradient because its potential obviously is the determinant of (3.31). This is easily seen from the fact that when the determinant $\det(A(\vec{u}(t)))$ is taken on an orbit of

$$\vec{u}_t = M^k\vec{u} \tag{3.33}$$

then its time dependence is given by

$$\det(A(\vec{u}(t))) = \det(A(\vec{u}(0))) \exp(\mathrm{trace}(M^k)\, t).$$

Hence

$$\langle \nabla \det(A(\vec{u})), M^k \vec{u} \rangle = \mathrm{trace}(M^k) \det(A(\vec{u})) = \langle L, M^k \vec{u} \rangle.$$

This proves the claim for the non-degenerate case, for the degenerate case a somewhat more involved argument is given, by representing the determinant as a suitable sum over its minors.

In case of irreducible characteristic polynomials of M the integrals (2.9) need to be evaluated by numerical methods. Therefore one might get the impression that for the determination of the level surfaces of the conserved quantities given by these integrals the usual computation by exponentials of matrices might be more efficient (see [5], [6]). However in that situation a mixture of both representations of solutions is advisable. It is suggested that first an approximation of the level surfaces should be approximately computed by using exponentials of matrices, then the error of this computation should be controlled by the numeric evaluation of (2.9), then by use of the explicit gradients these errors should be corrected approximately.

References

[1] B. Fuchssteiner, Symmetries for ordinary differential equations, *Preprint*.

[2] B. Fuchssteiner, Modified Gauss algorithm for matrices with symbolic entries, *ACM Communications in Computer Algebra*, **42**, No. 3 (September 2008).

[3] F.R. Gantmacher, *The Theory of Matrices*, Chelsea, New York (1959).

[4] K. Gehrs, *Algorithmic Methods for Ordinary Differential Equations*, PhD Thesis, University of Paderborn (2006).

[5] C. Moler, C. Van Loan, Nineteen Dubious ways to compute the exponential of a matrix, *SIAM Review*, **20**, No. 4 (1978), pp. 44; http://www.jstor.org/view/00361445/di973481/97p0252j/0.

[6] C. Moler, C. Van Loan, Nineteen Dubious ways to compute the exponential of a matrix, *Twenty-Five Years Later, SIAM Review*, **45**, No. 1 (2003), pp. 44; http://www.cs.cornell.edu/cv/ResearchPDF/19ways+.pdf.

[7] P.J. Olver, *Applications of Lie Groups to Differential Equations*, Springer Verlag, New York-Berlin-Heidelberg-Tokyo (1986).

[8] T. Sasaki, H. Murao, Efficient Gaussian elimination method for symbolic determinants and linear systems, *ACM Transactions on Mathematic Software*, **8**, No. 3 (1982), 277-289.

[9] M. Schreiber, *Differential Forms – A Heuristic Introduction*, Springer Verlag, New York-Heidelberg-Berlin, Universitext (1977).

[10] H. Stephani, *Differential Equations – Their Solutions Using Symmetries*, Cambridge University Press, Cambridge-New York-Port Chester-Melbourne-Sidney (1989).

[11] W. Walter, *Gewöhnliche Differentiagleichungen*, Springer Verlag, New York-Heidelberg-Berlin (1996).

[12] V.E. Zakharov, (Ed.), *What is Integrability?*, Springer Verlag, Berlin-Heidelberg-New York, Springer Series in Nonlinear Dynamics (1991), 115-184.