

EXCHANGEABILITY HYPOTHESIS AND INITIAL
PREMIUM FEASIBILITY IN XL REINSURANCE
WITH REINSTATEMENTS

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Abstract: This paper studies excess of loss reinsurance with reinstatements in the case in which the aggregate claims are generated by a discrete distribution, in the framework of risk adjusted premium principle. By regarding to comonotonic exchangeability, a generalized definition of initial premium is proposed and some regularity properties characterizing it are presented, both with reference to conditions on underlying distortion functions both with respect to composing functions. The attention is then focused on conditions ensuring feasibility of generalized initial premiums with reference to the limit on the payment of each claim.

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1. Introduction

In actuarial literature, the study of excess of loss reinsurance with reinstatements has become a flourishing theme, particularly with reference to the classical evaluation of pure premiums which is based on the collective model of risk theory. Given that this option requires the knowledge of the claim size distribution in the classical evaluation of pure premiums and that, generally, only few characteristics of aggregate claims can be explained, the interest for general properties characterizing the involved premiums represents a noteworthy question. In this paper the excess of loss reinsurance model we examine refers to the proposals of Campana [1] and Campana and Ferretti [3] where the original methodology developed by Sundt [10] to price excess of loss reinsurance with reinstatements for pure premiums and the basic hypothesis made by Walhin and Paris [11] to compute the initial premium P under the Proportional Hazard transform premium principle have been generalized. In this way, the attention has been moved to the study of risk adjusted premiums which belong to the class of distortion risk measures defined by Wang [13].

Recently Goovaerts et al. (see [6], [7]), Wu and Zhou (see [14]) propose a generalization of Yaari's risk measure [15] by relaxing his proposal. Comonotonic additivity has been generalized to comonotonic exchangeability in [6], while the link between comonotonic additivity and independent additivity has been addressed in [7]. Additivity for a finite number of comonotonic risks is substituted by countable additivity and countable exchangeability in [14] in order to characterize generalized distortion premium principles. As comonotonic additivity is the most essential property of distortion integral (and Choquet integral), comonotonic exchangeability by generalizing comonotonic additivity suggests a new definition of distortion premium principle (and Choquet price principle), that is the *generalized risk adjusted expected value* $W_g^h(X)$ of a risk X with decumulative distribution function $\bar{F}_X(x) = \mathcal{P}(X > x)$ such as follows

$$W_g^h(X) = h \left(\int_0^\infty g(\bar{F}_X(x)) dx \right)$$

where the function g is a distortion, h is a continuous non-decreasing function on R^+ . The related premium principle is called *generalized risk adjusted premium principle*.

In the model we recently studied (see [3]) the main role is played by the initial risk adjusted premium P , which has been defined as solution of the equilibrium condition in which the value of the total premium income $\delta(P)$ equals the distorted expected value of the aggregate claims S : by applying

the same scheme to each layer $L_X(im, (i + 1)m)$, we proved that the local equilibrium condition for each reinstatements ensures the global equilibrium condition. Note that the same result is true with reference to two distortion functions g_j ($j = 1, 2$), that are not necessarily the same: in this way it is possible to make a distinction between the attitude towards uncertainty of the two decision makers, i.e. the insurer and the reinsurer.

The interest on the local equilibrium condition is here strengthened by setting the definition of the initial premiums P_{0i} ($i = 1, \dots, K$) for which the local equilibrium condition is satisfied in each layer: in other words, now we turn our attention to the analysis of the local equilibrium condition and in order to consider the case of generalized risk adjusted premium principle we define and study the following *generalized local equilibrium condition*

$$W_{g_1}^{h_1}(P_i) = W_{g_2}^{h_2}(L_X(im, (i + 1)m))$$

where the functions g_j are distortion functions and h_j are continuous non-decreasing functions on R^+ ($j = 1, 2$). Accordingly, the definition of *generalized initial premium* naturally follows: it is solution, write P_{0i} ($i = 1, \dots, K$), of the previous generalized local equilibrium condition.

As it is well-known, the reinsurance companies often assess treaties under the assumption that there are only total losses. This happens, for example, when they use the rate on line method to price catastrophe reinsurance. Then it follows that the aggregate claims are generated by a discrete distribution and the premium for the i -th reinstatement is supposed to be a two-point random variable distributed as $c_i P B_{p_i}$ where B_{p_i} denotes a Bernoulli random variable such that $\mathcal{P}[B_{p_i} = 1] = p_i = 1 - \mathcal{P}[B_{p_i} = 0]$. In this context, it is possible to write an explicit generalized local equilibrium condition on which the generalized initial premium is defined. In this way, setting our analysis in the recent framework of generalized risk adjusted premiums proposed by Wu and Zhou [14], we study properties exhibited by the proposed generalized initial premium, both with respect to equilibrium condition (Section 3), both with respect to feasibility with reference to the limit on the payment of each claim (Section 4). Finally Section 5 ends the paper with some concluding remarks.

2. Excess of Loss Reinsurance with Reinstatements: Basic Settings and Preliminary Results

In order to present the excess of loss reinsurance model we study, some notations, abbreviations and conventions used throughout the paper are in the following presented (for more details, see [3]).

An excess of loss reinsurance for the layer m in excess of d , written m xs d , is a reinsurance treaty which covers the part of each claim that exceeds the deductible d ; there is a limit on the payment of each claim, which is set equal to m ; equivalently, the reinsurer covers for each claim of size Y the amount

$$L_Y(d, d + m) = \min\{(Y - d)_+, m\} \quad (1)$$

where $(Y - d)_+ = Y - d$ if $Y > d$, otherwise $(Y - d)_+ = 0$.

Given an insurance portfolio, where N denotes the number of claims occurred in the portfolio during the reference year and Y_i is the i -th claim size ($i = 1, 2, \dots, N$), the aggregate claims to the layer result to be the random sum given by

$$X = \sum_{i=1}^N L_{Y_i}(d, d + m) \quad (2)$$

where $X = 0$ when $N = 0$.

An excess of loss reinsurance, or for short an XL reinsurance, for the layer m xs d with aggregate deductible D and aggregate limit M covers only the part of X that exceeds D but with a limit M , that is:

$$L_X(D, D + M) = \min\{(X - D)_+, M\}. \quad (3)$$

This cover is known as an XL reinsurance for the layer m xs d with aggregate layer M xs D .

In the actuarial literature it is generally assumed that the aggregate limit M is given as a whole multiple of the limit m , i.e. $M = (K + 1)m$: in this case we say that there is a limit in the number of the losses covered by the reinsurer. This reinsurance cover is called an XL reinsurance for the layer m xs d with aggregate deductible D and K reinstatements and provides total cover for the following amount

$$L_X(D, D + (K + 1)m) = \min\{(X - D)_+, (K + 1)m\}. \quad (4)$$

The initial premium P covers the original layer, that is

$$L_X(D, D + m) = \min\{(X - D)_+, m\} \quad (5)$$

and in some way, it can be considered as the 0-th reinstatement.

The condition that the reinstatement is paid pro rata implies that the premium for the i -th reinstatement is a random variable given by

$$\frac{c_i P}{m} L_X(D + (i - 1)m, D + im) \quad (6)$$

where $0 \leq c_i \leq 1$ denotes the i -th percentage of reinstatement. When it is supposed that $c_i = 0$ the reinstatement is free, otherwise it is paid.

After this preliminary settings, it is possible to define the related total premium income, which results to be a random variable, say $\delta(P)$, which is so given

$$\delta(P) = P \left(1 + \frac{1}{m} \sum_{i=0}^{K-1} c_{i+1} L_X(D + im, D + (i + 1)m) \right). \tag{7}$$

The aggregate claims S paid by the reinsurer for this XL reinsurance treaty, namely

$$S = L_X(D, D + (K + 1)m) \tag{8}$$

is the other meaningful random variable of the treaty and it satisfies the relation

$$S = \sum_{i=0}^K L_X(D + im, D + (i + 1)m). \tag{9}$$

In [3] we recently studied the initial risk adjusted premium P as solution of the equilibrium condition expressing the fact that the expected values of the total premium income $\delta(P)$ and of the aggregate claims S are equal with reference to two not necessarily coincident distortion functions, that is

$$W_{g_1}(\delta(P)) = W_{g_2}(S) \tag{10}$$

where

$$W_{g_j}(X) = \int_0^\infty g_j(\bar{F}_X(x)) dx \tag{11}$$

where the computations are made with reference to the distortion risk measure introduced by Wang [13].

The functions g_j ($j = 1, 2$) are distortion function, i.e. non-decreasing functions $g_j : [0, 1] \rightarrow [0, 1]$ such that $g_j(0) = 0$ and $g_j(1) = 1$; $\bar{F}_X(x) = \mathcal{P}(X > x)$ is the decumulative distribution function, or tail function, of X .

Examples of risk measures belonging to this class are the well-known *quantile risk measure* and the *Tail Value-at-Risk*. When it is considered a power g function, i.e. $g(x) = x^{1/\rho}$, $\rho \geq 1$, the corresponding risk measure results to be the *PH-transform risk measure* proposed by Wang [13], that is the same choice subsequently made by Walhin and Paris [11]. As it is well-known, distortion risk measures satisfy the following properties (see Wang [13] and Dhaene et al. [5]):

P1. Monotonicity

$$W_g(X) \leq W_g(Y) \tag{12}$$

for any two random variables X and Y where $X \leq Y$ with probability 1.

P2. Positive homogeneity

$$W_g(aX) = aW_g(X) \quad \text{for any } a \in \mathbb{R}^+. \tag{13}$$

P3. Translation invariance

$$W_g(X + b) = W_g(X) + b \quad \text{for any } b \in \mathbb{R}. \tag{14}$$

P4. Additivity for comonotonic risks

$$W_g(S^c) = \sum_{i=1}^n W_g(X_i) \tag{15}$$

where S^c is the sum of the components of the random vector \mathbf{X}^c with the same marginal distributions of \mathbf{X} and with the comonotonic dependence structure.

When a distortion measure is concave, then the related distortion risk measure preserves stop-loss order and it is also sub-additive. Very well-known examples of concave distortion risk measures are the *Tail Value-at-Risk* and the *PH-transform risk measure*, whereas *quantile risk measure* is not a concave risk measure.

In the context of distortion risk measures, the initial premium P is well-defined: as a matter of fact, starting from equation (10) we obtained (see [4]) the following explicit expression

$$P = \frac{\sum_{i=0}^K W_{g_2}(L_X(im, (i + 1)m))}{1 + \frac{1}{m} \sum_{i=0}^{K-1} c_{i+1} W_{g_1}(L_X(im, (i + 1)m))}. \tag{16}$$

We observed that (10) gives a sort of global equilibrium in which the distortion risk measure associated with the aggregate claims to the layer S must be equal to the distortion risk measure associated with the total premium income $\delta(P)$. By applying the same scheme to each layer, we obtained the following equation that the premium P_i , for all $i \in \{0, 1, \dots, K\}$, must satisfy

$$W_{g_1}(P_i) = W_{g_2}(L_X(im, (i + 1)m)). \tag{17}$$

By property *P2*, it is

$$W_{g_1}(P_0) = P_0 = W_{g_2}(L_X(0, m)). \tag{18}$$

Starting from the relation on the premium for the i -th reinstatement

$$P_i = \frac{c_i P_0}{m} L_X(D + (i - 1)m, D + im), \tag{19}$$

it follows

$$\frac{c_i P_0}{m} W_{g_1}(L_X((i - 1)m, im)) = W_{g_2}(L_X(im, (i + 1)m)). \tag{20}$$

Moreover, it is

$$c_i = \frac{m W_{g_2}(L_X(im, (i + 1)m))}{P_0 W_{g_1}(L_X((i - 1)m, im))} \tag{21}$$

where the initial premium P_0 given by (18). Starting from formula (21) involving the reinstatements percentages c_i , we proved that $P = P_0$ both in the case of same distortion functions $g_1 = g_2$ (see [1]) both in the case of not necessarily equal distortion functions g_j ($j = 1, 2$) (see [4]).

3. Exchangeability and Generalized Initial Premiums

We actually refer to a non-empty collection of risks \mathcal{L} on a probability space $(\Omega, \mathcal{F}, \mathcal{P})$ where Ω represents all possible states of nature at the end of a period, \mathcal{F} is a σ -algebra of subsets of Ω , \mathcal{P} is the probability measure defined on measurable space (Ω, \mathcal{F}) . The risk set \mathcal{L} is supposed to satisfy the following requirements: $\min\{X, a\} - \min\{X, b\}$, aX and $X + a$ are all in \mathcal{L} for any $X \in \mathcal{L}$ and for any $a, b \in R$. Insurance premium principles and risk measures can be characterized with reference to a set of axioms, that is to a set of desirable properties that they should have for actuarial practice. Given that, sometimes, the proposed sets of axioms are rather arbitrarily chosen and not so in accordance with real life insurance problems, some Authors have focused their attention to different characterizations of premium principles and/or risk measures. Recently, Goovaerts et al. (see [6], [7]), Wu and Zhou [14] propose a generalization of Yaari's risk measure [15] by relaxing his proposal. In fact, comonotonic additivity is generalized to comonotonic exchangeability in [6], while the link between comonotonic additivity and independent additivity is addressed in [7]. Additivity for a finite number of comonotonic risks is substituted by countable additivity and countable exchangeability in [14], in order to characterize generalized distortion premium principles. More precisely, with reference to a premium principle $H : \mathcal{L} \rightarrow R^+$ Goovaerts et al. [6] refer to the notions of exchangeability and of comonotonic exchangeability:

A1. Exchangeability

$$H(\mathbf{X}^c + \mathbf{Y}^c) = H(\mathbf{X}^{*c} + \mathbf{Y}^c) \text{ if } H(\mathbf{X}) = H(\mathbf{X}^*) \tag{22}$$

where the random vectors \mathbf{X}^c , \mathbf{X}^{*c} and \mathbf{Y}^c admit the same marginal distributions of \mathbf{X} , \mathbf{X}^* and \mathbf{Y} , respectively, with the comonotonic dependence structure;

A2. Comonotonic exchangeability

$$H\left(\sum_{i=1}^{\infty} X_i + \sum_{i=1}^{\infty} Y_i\right) = H\left(\sum_{i=1}^{\infty} \bar{X}_i + \sum_{i=1}^{\infty} Y_i\right) \tag{23}$$

for comonotonic random vectors \mathbf{X} and \mathbf{Y} , $\bar{\mathbf{X}}$ and \mathbf{Y} , such that $H(X_i) = H(\bar{X}_i)$ and $\sum_{i=1}^{\infty} X_i, \sum_{i=1}^{\infty} Y_i, \sum_{i=1}^{\infty} \bar{X}_i$ are all in \mathcal{L} .

As comonotonic additivity is the most essential property of distortion integral (and Choquet integral), by generalizing comonotonic additivity comonotonic exchangeability suggests a new definition of distortion premium principle, write W_g^h , (and Choquet price principle), such as follows

$$W_g^h(X) = h\left(\int_0^{\infty} g(\bar{F}_X(x))dx\right) \tag{24}$$

which is called *generalized risk adjusted premium principle*; the function g is a distortion function and h is a continuous non-decreasing function on R^+ .

Wu and Zhou (see [14]) prove that a premium principle H is a generalized distortion premium principle if and only if H satisfies axiom *A2*, monotonicity in stochastic order and finiteness.

Now we turn our attention to the analysis of local equilibrium condition (17): in order to consider the case of generalized risk adjusted premium principle, we define and study the following *generalized local equilibrium condition*

$$W_{g_1}^{h_1}(P_i) = W_{g_2}^{h_2}(L_X(im, (i + 1)m)) \tag{25}$$

where the functions g_j are distortion functions and h_j are continuous non-decreasing functions on R^+ ($j = 1, 2$). Accordingly, the definition of *generalized initial premium* naturally follows: it is solution, write P_{0i} ($i = 1, \dots, K$), of the generalized local equilibrium condition (25).

With reference to actuarial practice, when the reinsurance companies assess treaties under the assumption that there are only total losses, (for example when it is used the rate on line method to price catastrophe reinsurance), the aggregate claims are generated by a discrete distribution. Here we suppose that the premium for the i -th reinstatement (6) is a two-point random variable

distributed as $c_i P B_{p_i}$ where B_{p_i} denotes a Bernoulli random variable such that $\mathcal{P}[B_{p_i} = 1] = p_i = 1 - \mathcal{P}[B_{p_i} = 0]$.

In the following we focus our attention on properties exhibited by the generalized initial premium P_{0i} , with reference to some assumptions made on the functions involved, namely the distortion functions g_j and the composing functions h_j , ($j = 1, 2$). We can state the following result.

Theorem 1. *Let g_j ($j = 1, 2$) be \mathcal{C}^2 -differentiable distortion functions and let h_j be \mathcal{C}^2 -differentiable and increasing functions on R^+ ($j = 1, 2$), where $g_1(p_i) \neq 0$ and $h'_1 \left[\frac{c_i x}{m} g_1(p_i) \right] \neq 0$ ($i = 1, \dots, K$). Let $\mathcal{A} = (\overline{p}_i, \overline{p}_{i+1})$ and $\mathcal{B} = (\widetilde{p}_i, \widetilde{p}_{i+1})$ be such that $g'_1(\overline{p}_i) = 0$, $g'_2(\overline{p}_{i+1}) = 0$ and $g'_1(\widetilde{p}_i) = 0$, $h'_2[g_2(\widetilde{p}_{i+1})] = 0$ and $g'_2(\widetilde{p}_{i+1}) \neq 0$, ($i = 1, \dots, K$). If g_1 is strictly concave (convex) and g_2 is strictly convex (concave), then \mathcal{A} is a local minimum (maximum) point of the generalized initial premium P_{0i} ($i = 1, \dots, K$); if g_1 is strictly concave (convex) and h_2 is strictly convex (concave), then \mathcal{B} is a local minimum (maximum) point of the generalized initial premium P_{0i} ($i = 1, \dots, K$).*

Proof. Let us consider the following function

$$\mathcal{Z}_{g_1 g_2}^{h_1 h_2}(x, p_i, p_{i+1}) = h_1 \left[\frac{c_i x}{m} g_1(p_i) \right] - h_2 [g_2(p_{i+1})]. \tag{26}$$

The generalized initial premium P_{0i} results to be solution of the following equation

$$\mathcal{Z}_{g_1 g_2}^{h_1 h_2}(P_{0i}, p_i, p_{i+1}) = 0. \tag{27}$$

Given that by hypothesis g_j ($j = 1, 2$) are continuously differentiable distortion functions and h_j are continuously differentiable and non-decreasing functions on R^+ ($j = 1, 2$), where $g_1(p_i) \neq 0$ and $h'_1 \left[\frac{c_i x}{m} g_1(p_i) \right] \neq 0$, the generalized initial premium P_{0i} results to be a continuously differentiable function of (p_i, p_{i+1}) . In detail, it is

$$\begin{aligned} \frac{\partial P_{0i}}{\partial p_i}(p_i, p_{i+1}) &= - \left[\frac{\partial \mathcal{Z}_{g_1 g_2}^{h_1 h_2}(x, p_i, p_{i+1})}{\partial p_i} \right] / \left[\frac{\partial \mathcal{Z}_{g_1 g_2}^{h_1 h_2}(x, p_i, p_{i+1})}{\partial x} \right] \\ &= -x \frac{g'_1(p_i)}{g_1(p_i)} \\ \frac{\partial P_{0i}}{\partial p_{i+1}}(p_i, p_{i+1}) &= - \left[\frac{\partial \mathcal{Z}_{g_1 g_2}^{h_1 h_2}(x, p_i, p_{i+1})}{\partial p_{i+1}} \right] / \left[\frac{\partial \mathcal{Z}_{g_1 g_2}^{h_1 h_2}(x, p_i, p_{i+1})}{\partial x} \right] \\ &= \frac{m}{c_i} \frac{h'_2[g_2(p_{i+1})]}{h'_1 \left[\frac{c_i x}{m} g_1(p_i) \right]} \frac{g'_2(p_{i+1})}{g_1(p_i)} \end{aligned}$$

where $x = P_{0i}$.

Furthermore, setting

$$\begin{aligned} A &= h'_1 \left(x \frac{c_i}{m} g_1(p_i) \right), \\ B &= h'_2 (g_2(p_{i+1})), \\ C &= h''_1 \left(x \frac{c_i}{m} g_1(p_i) \right), \\ D &= h''_2 (g_2(p_{i+1})), \\ E &= g'_2(p_{i+1}), \\ F &= g''_2(p_{i+1}), \end{aligned}$$

we have

$$\begin{aligned} \frac{\partial^2 P_{0i}}{\partial p_i^2}(p_i, p_{i+1}) &= x \frac{2[g'_1(p_i)]^2 - g''_1(p_i)g_1(p_i)}{[g_1(p_i)]^2}, \\ \frac{\partial^2 P_{0i}}{\partial p_i \partial p_{i+1}}(p_i, p_{i+1}) &= -\frac{m}{c_i} \frac{h'_2[g_2(p_{i+1})]}{h'_1[\frac{c_i x}{m} g_1(p_i)]} \frac{g'_1(p_i)g'_2(p_{i+1})}{[g_1(p_i)]^2}, \\ \frac{\partial^2 P_{0i}}{\partial p_{i+1}^2}(p_i, p_{i+1}) &= \frac{m}{c_i} \frac{A^2(DE^2 + BF) - CB^2E^2}{g_1(p_i)A^3}, \end{aligned}$$

where $x = P_{0i}$.

Then the related Hessian matrix H_Z of the function (26) admits the following structure

$$H_Z = \begin{pmatrix} x \frac{2[g'_1(p_i)]^2 - g''_1(p_i)g_1(p_i)}{[g_1(p_i)]^2} & -\frac{m}{c_i} \frac{BEg'_1(p_i)}{A[g_1(p_i)]^2} \\ -\frac{m}{c_i} \frac{BEg'_1(p_i)}{A[g_1(p_i)]^2} & \frac{m}{c_i} \frac{A^2(DE^2 + BF) - CB^2E^2}{g_1(p_i)A^3} \end{pmatrix}. \tag{28}$$

Stationarity condition, that is

$$\nabla P_{0i}(p_i, p_{i+1}) = \left(-x \frac{g'_1(p_i)}{g_1(p_i)}, \frac{m}{c_i} \frac{h'_2[g_2(p_{i+1})]}{h'_1[\frac{c_i x}{m} g_1(p_i)]} \frac{g'_2(p_{i+1})}{g_1(p_i)} \right) = (0, 0) \tag{29}$$

is equivalent to the following sets of conditions

$$\left\{ \begin{aligned} g'_1(p_i) &= 0, \\ g'_2(p_{i+1}) &= 0, \end{aligned} \right. \quad \text{or} \quad \left\{ \begin{aligned} g'_1(p_i) &= 0, \\ h'_2(g_2(p_{i+1})) &= 0. \end{aligned} \right. \tag{30}$$

The results stated in the proposition follow by direct inspection of the Hessian matrix on the points $\mathcal{A} = (\overline{p_i}, \overline{p_{i+1}})$ and $\mathcal{B} = (\widetilde{p_i}, \widetilde{p_{i+1}})$, where \mathcal{A} and \mathcal{B} are such that $g'_1(\overline{p_i}) = 0$, $g'_2(\overline{p_{i+1}}) = 0$ and $g'_1(\widetilde{p_i}) = 0$, $h'_2[g_2(\widetilde{p_{i+1}})] = 0$ and $g'_2(\widetilde{p_{i+1}}) \neq 0$. \square

Remark. Note that the generalized initial premium P_{0i} is monotone function of the probabilities: as a function of the probability p_i it is decreasing, while as a function of the probability p_{i+1} it is increasing.

In the next proposition the conditions that are sufficient to characterize \mathcal{A} and \mathcal{B} as saddle points of generalized initial premium P_{0i} are presented.

Theorem 2. *Let g_j ($j = 1, 2$) be \mathcal{C}^2 -differentiable distortion functions and let h_j be \mathcal{C}^2 -differentiable and increasing functions on R^+ ($j = 1, 2$), where $g_1(p_i) \neq 0$ and $h_1'[\frac{c_i x}{m} g_1(p_i)] \neq 0$ ($i = 1, \dots, K$). Let $\mathcal{A} = (\overline{p}_i, \overline{p}_{i+1})$ and $\mathcal{B} = (\widetilde{p}_i, \widetilde{p}_{i+1})$ be such that $g_1'(\overline{p}_i) = 0$, $g_2'(\overline{p}_{i+1}) = 0$ and $g_1'(\widetilde{p}_i) = 0$, $h_2'[g_2(\widetilde{p}_{i+1})] = 0$ and $g_2'(\widetilde{p}_{i+1}) \neq 0$, ($i = 1, \dots, K$). If g_1 and g_2 are both strictly concave (convex), then \mathcal{A} is a saddle point of the generalized initial premium P_{0i} ; if g_1 and h_2 are both strictly concave (convex), then \mathcal{B} is a saddle point of the generalized initial premium P_{0i} .*

Proof. The proof is a direct effect of the conditions involving the functions g_1 and g_2 , g_1 and h_2 and the Hessian matrix H_Z . □

Remark. Note the asymmetric role played by conditions on distortion function g_1 and on composing function h_2 : strictly concavity (convexity) of g_1 and h_2 are sufficient to characterize \mathcal{B} as saddle point of the generalized initial premium P_{0i} . There is not an *equivalent* result involving concavity and/or convexity both of the distortion function g_2 , both of the composing function h_1 . In this way, we can speak of asymmetry.

4. Feasibility of Initial Premiums

In excess of loss reinsurance with reinstatements, it is generally supposed that reinstatement percentages are fixed in advance. In this direction may be useful formula (21) involving the reinstatements percentages c_i : in fact in this way, it is possible to guarantee the local equilibrium for each reinstatement. In connection with the initial premium defined by local equilibrium condition (17), in [4] we studied acceptability in the market of the reinsurance treaty, with respect to the limit on the payment of each claim. More precisely, we analyzed the set of inequalities

$$P_{0i} \leq m \tag{31}$$

for each $i = 1, \dots, K$, in search of sufficient conditions ensuring their validity. The premiums P_{0i} ($i = 1, \dots, K$), for which the inequalities are verified, are called feasible premiums.

Starting from the definition of the initial premium P_{0i} ($i = 1, \dots, K$) for which the local equilibrium condition is satisfied in each layer, that is

$$P_{0i} = \frac{mW_{g_2}(L_X(im, (i + 1)m))}{c_iW_{g_1}(L_X((i - 1)m, im))} = \frac{mg_2(p_{i+1})}{c_i g_1(p_i)} \tag{32}$$

we directed our attention to the analysis of the relation between two consecutive values of the initial premium, that is to the comparison of P_{0i} to P_{0i+1} , so as to obtain the following existence result.

Theorem 3. *Given an XL reinsurance with K reinstatements and distortion functions g_1 and g_2 , where g_2 is continuous and the percentages of reinstatement are decreasing, there exists a finite sequence of probabilities p_i ($i = 1, \dots, K + 1$), with $g_2(p_{K+1}) \leq g_1(p_K)$ or $g_2(p_2) \leq g_1(p_1)$, such that $P_{0i} \leq m$, for all $i = 1, \dots, K$.*

Following the steps of the proof of the theorem, it is possible to construct the finite sequence of probabilities that are mentioned in the statement.

By analyzing the same question in the more general setting of generalized initial premiums which satisfy the generalized local equilibrium condition (25), we have the following sufficiency result.

Theorem 4. *Let g_j ($j = 1, 2$) be \mathcal{C}^2 -differentiable distortion functions and let h_j be \mathcal{C}^2 -differentiable and increasing functions on R^+ ($j = 1, 2$), where $g_1(p_i) \neq 0$ and $h'_1 \left[\frac{cx}{m} g_1(p_i) \right] \neq 0$ ($i = 1, \dots, K$). Let $(\overline{p}_i, \overline{p}_{i+1})$ and $(\widetilde{p}_i, \widetilde{p}_{i+1})$ be such that $g'_1(\overline{p}_i) = 0$, $g'_2(\overline{p}_{i+1}) = 0$ and $g'_1(\widetilde{p}_i) = 0$, $h'_2[g_2(\widetilde{p}_{i+1})] = 0$ and $g'_2(\widetilde{p}_{i+1}) \neq 0$, ($i = 1, \dots, K$). If g_1 is strictly convex, g_2 is strictly concave and $P_{0i}(\overline{p}_i, \overline{p}_{i+1}) \leq m$, ($i = 1, \dots, K$) then $P_{0i}(p_i, p_{i+1}) \leq m$ for all $i = 1, \dots, K$; if g_1 is strictly convex, h_2 is strictly concave and $P_{0i}(\widetilde{p}_i, \widetilde{p}_{i+1}) \leq m$, then $P_{0i}(p_i, p_{i+1}) \leq m$ for all $i = 1, \dots, K$.*

Proof. Under the above conditions, by Theorem 1, $P_{0i}(\overline{p}_i, \overline{p}_{i+1})$ and $P_{0i}(\widetilde{p}_i, \widetilde{p}_{i+1})$ represent the maximum values assumed by P_{0i} . The inequality involving the maximum values of the generalized initial premium and the limit on the payment of each claim ensures the feasibility of each generalized initial premium. □

Remark. The very stringent conditions on probabilities $(\overline{p}_i, \overline{p}_{i+1})$ and $(\widetilde{p}_i, \widetilde{p}_{i+1})$ and on functions g_j ($j = 1, 2$) and h_2 are sufficient to ensure that *any* generalized initial premium $P_{0i}(p_i, p_{i+1})$ is feasible with respect to the limit m .

5. Concluding Remarks

In Actuarial Literature excess of loss reinsurance with reinstatement has been essentially studied in the framework of collective model of risk theory for which the classical evaluation of pure premiums requires the knowledge of the claim size distribution. Generally, in actuarial practice, there is incomplete information: in fact only few characteristics of the aggregate claims can be computed. In this situation the interest for general properties characterizing the involved premiums is flourishing. In the considered model, when the reinstatements are paid the total premium income becomes a random variable which is correlated to aggregate claims: incomplete information on aggregate claims distribution amplifies the interest for general properties characterizing the involved premiums. We set our analysis in the recent framework of generalized risk adjusted premiums: we focus our study on properties exhibited by generalized risk initial premiums, a new proposed definition of initial premium. Given that the reinsurance companies often assess treaties under the conjecture that there are only total losses, we suppose that the premium for each reinstatement is a two-point random variable, a particularly interesting hypothesis. Compared with the existing results on initial premium, the proposed characterizations of the generalized initial premium are useful in order to study feasibility of initial premium with respect to the limit on the payment of each claim. The regularity results of the generalized initial premium hint that further research may be addressed to the analysis of optimal reinsurance policy.

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