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ON THE GONALITY SEQUENCE OF A STABLE CURVE

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Abstract: Here we study the existence of codimension one subvarieties of $\overline{\mathcal{M}}_g$ parametrizing curves with very low gonality with respect to balanced line bundles (i.e. the line bundles with multidegrees satisfying certain inequalities introduced by D. Gieseker and L. Caporaso). We also study the gonality sequence of certain hyperelliptic stable curves.

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1. Introduction

Here we study a problem on the Brill-Noether theory of stable curves with respect to balanced line bundles in the sense of D. Gieseker and L. Caporaso ([5], [1], [2]) Many differences where known (see [3] and [4] for the failure of Clifford's inequality without other assumptions). To show the differences we recall the classical case in the less well-known case of an integral, but possibly singular, projective curve. Let X be an integral projective curve defined over an algebraically closed field $\mathbb{K}$. Set $g := p_a(X)$. For each integer $r \geq 1$ let d_r be the minimal integer t such that there is a rank one torsion free sheaf F on X such that $\deg(F) = t$ and $h^0(X, F) \geq r+1$. The integer d_1 is the usual gonality of X. The integer k_r is called the r-gonality of X. Obviously $d_r = r + p_a(X)$ if $r > g$ and $d_g = 2g-2$. For integral curves any F computing k_r is spanned and $h^0(X, F) = r + 1$ (easy).

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Let X be a genus g semistable curve. Let L be a degree d line bundle on X and C a proper subcurve of X. The pair (L, C) satisfies the Basic Inequality if and only if

$$|(2g-2)\deg(L|C) - d \cdot \deg(\omega_X|C)| \leq (g-1) \cdot \sharp(C \cap \overline{X \backslash C}) \qquad (1)$$

L is called *semibalanced* ([2], Definition 4.6, [7], Definition 1.1) or satisfying the *Basic Inequality* ([1], p. 611) if (1) is satisfied for all proper subcurves C of X. Now assume that X is quasistable. The line bundle L is called *balanced* if it is semibalanced and $\deg(L|E) = 1$ for every exceptional component E of X, i.e. for every irreducible component E of X such that $E \cong \mathbb{P}^1$ and $\sharp(E \cap \overline{X \backslash E}) = 2$. If L is a balanced line bundle on a quasistable curve Y and $h^0(Y, L) > 0$, then the total degree $\deg(L)$ of L is non-negative (part (a) of [4], Theorem 2.1, and Serre duality).

Let X be a stable curve. For each integer $r > 0$ let $d_r(X)$ be the minimal degree of a balanced line bundle L on a quasistable model Y of X such that $h^0(Y, L) \geq r + 1$. If we only admit line bundles on X, then we get a corresponding integer $d'_r(X)$. Now assume that X is singular. For each integer s such that $1 \leq s \leq \sharp(\mathrm{Sing}(X))$ let $d''_r(X)_s$ be the minimal degree of a line bundle L on a quasistable model Y of X such that Y has s exceptional components and $h^0(Y, L) \geq r + 1$. Set $d''_r(X) := \min_{1 \leq s \leq \sharp(\mathrm{Sing}(X))} d''_r(X)$. Here we give an example of a stable curve for which the sequence $d''_r(X)$ is not strictly increasing (Proposition 1).

Here we take $X = X_1 \cup X_2$ with X_1 and X_2 smooth and irreducible and $\sharp(X_1 \cap X_2) = 1$, say $X_1 \cap X_2 = \{P\}$. Set $g := p_a(X)$, $q := p_a(X_1)$ and $\{P\} := X_1 \cap X_2$. Thus $p_a(X_1) = g - q$. We always assume $p_a(X_1) \geq p_a(X_2)$, i.e. $q \geq g/2$. Since X is stable, then $q < g$. Call $\Delta(g, q)$ the set of all such curves. The closure $\overline{\Delta(g, q)}$ of $\Delta(g, q)$ in $\overline{\mathcal{M}}_g$ is an irreducible divisor. For any $X = X_1 \cup X_2 \in \Delta(g, q)$ and any $L \in \mathrm{Pic}(X)$ its multidegree is the pair $(\deg(L|X_1), \deg(L|X_2)) \in \mathbb{Z}^{\oplus 2}$. We have $\mathrm{Pic}(X) \cong \mathrm{Pic}(X_1) \times \mathrm{Pic}(X_2)$, i.e. any $L \in \mathrm{Pic}(X)$ is uniquely determined by a pair (R_1, R_2) with $R_i \in \mathrm{Pic}(X_i)$ (here $R_i = L|X_i$). Call Y_i the irreducible component of Y such that $\pi(Y_i) = X_i$. Any $L \in \mathrm{Pic}(Y)$ is unique determined by the triple $(L|Y_1, \deg(L|E), L|Y_2)$. Let $(\deg(L|Y_1), \deg(L|E), \deg(L|Y_2)) \in \mathbb{Z}^{\oplus 3}$ be its multidegree.

In Section 2 we prove the following results.

Theorem 1. *Assume $g \geq 6$ and even.*

(a) $\overline{\Delta(g, g/2)}$ is the only irreducible divisor of $\overline{\mathcal{M}}_g$ such that a general $U \in \Gamma$ has a quasistable model with a degree 1 balanced line bundle.

(b) Fix $X \in \Delta(g,q)$. *X has no balanced line bundles L such that* $\deg(L) = 1$ *and* $h^0(X,L) \geq 2$. *The quasistable model Y of X has a line bundle R such that* $\deg(R) = 1$ *and* $h^0(Y,R) \geq 2$ *if and only if* $q = g/2$. *If* $q = g/2$, *then the line bundle R as above is unique and* $R = (\mathcal{O}_{Y_1}, 1, \mathcal{O}_{Y_2})$.

Theorem 2. *Assume* $g \geq 5$ *and odd. There is no irreducible hypersurface* $\Gamma \subset \overline{\mathcal{M}}_g$ *such that a general* $U \in \Gamma$ *has a quasistable model with a degree* 1 *balanced line bundle.*

Theorem 3. *Fix* $X = X_1 \cup X_2 \in \Delta(g,q)$. *Call L the line bundle on X such that* $\deg(L|X_i) \cong \mathcal{O}_{X_i}(P)$ *for all* $i \in \{1,2\}$. *We have* $h^0(X,L) = 2$ *and L is the only line bundle R on X with multidegree* $(1,1)$ *and* $h^0(X,R) \geq 2$. *L is balanced if and only if* $g/2 \leq q \leq (3g-1)/4$.

In the last section we study the gonality sequences $d_r(X)$, $d_r'(X)$ and $d_r''(X)$ for a curve $X = X_1 \cup X_2 \in \Delta(g/2, g/2)$ when X_1 and X_2 are hyperelliptic and P is a Weierstrass point both of X_1 and X_2. Such a curve is in the closure in $\overline{\mathcal{M}}_g$ of the hyperelliptic locus of $\mathcal{M}_g$, because P is a Weierstrass point both of X_1 and of X_2 ([6], part b) of Theorem 5).

2. The Proofs

Lemma 1. *Let* $Y = Y_1 \cup E \cup Y_2$ *be the quasistable curve associated to some* $X \in \Delta(g,q)$. *A line bundle L of multidegree* $(0,1,0)$ *is balanced if and only if* $g = 2q$.

Proof. L is balanced if and only if $|2q-1| \leq g-1$, $|2g-2q-1| \leq g-1$, $|2g-2-(2q-1)| \leq g-1$ and $|2g-2-(2g-2q-1)| \leq g-1$. Since $q \geq g/2$, the first inequality gives $q = g/2$. If $g = 2q$, then all the inequalities are satisfied. □

Proof of Theorem 1 and 2. Of course, no integral curve with positive genus has gonality one (even considering rank 1 torsion free sheaves). Hence it is sufficient to look at reducible curves. Since Theorem 2 and part (a) of Theorem 1 only concern hypersurfaces of $\overline{\mathcal{M}}_g$ contained in $\overline{\mathcal{M}}_g \setminus \mathcal{M}_g$, it is sufficient to use Lemma 3 and that on an integral projective curve T the sheaf $\mathcal{O}_T$ is the only line bundle A such that $\deg(A) \leq 0$ and $h^0(T,A) > 0$. □

Lemma 2. *Fix* $X \in \Delta(g,q)$. *For any* $R \in \mathrm{Pic}(X)$ *we have*

$$\max\{0, h^0(X_1, R|X_1) + h^0(X_2, R|X_2) - 1, 0\} \leq h^0(X,R) \leq h^0(X_1, R|X_1) + h^0(X_2, R|X_2)$$

if and only if P is a base point of both $R|X_1$ and $R|X_2$.

Proof. Use the Mayer-Vietoris exact sequence on X:

$$0 \to R \to R|X_1 \oplus R|X_2 \to R|\{P\} \to 0 \tag{2}$$

(here we use that $\{P\}$ is the scheme-theoretic intersection of X_1 and X_2). □

Of course, in part (ii) of Lemma 2 we may exchange the role of X_1 and X_2.

Proof of Theorem 3. Since $q > 0$ and $g - q > 0$, Lemma 2 gives that L is the only line bundle on X with multidegree $(1, 1)$ and $h^0(X, R) \geq 2$ and that indeed $h^0(X, L) = 2$. Recall that $q \geq g/2$. L is balanced if and only if the following inequalities are satisfied: $|2g - 2 - 2(2q - 1)| \leq g - 1$ and $|2g - 2 - 2(2g - 2q - 1)| \leq g - 1$, i.e. if and only if $|4g - 8q| \leq 2g - 2$, i.e. if and only if $q \leq (3g - 1)/4$. □

3. A Hyperelliptic Curve

In this section we study the following example $X = X_1 \cup X_2$. Set $q := g/2$.

Example 1. Fix $X = X_1 \cup X_2 \in \Delta(2q, q)$. Assume that both X_1 and X_2 are hyperelliptic. Set $\{P\} := X_1 \cap X_2$ and assume that P is a Weierstrass point of X_i for each i.

Lemma 3. *Fix $R \in \mathrm{Pic}(X)$ and call (d_1, d_2) its multidegree. R is balanced if and only if $|d_1 - d_2| \leq 1$.*

Proof. R is balanced if and only if $|d_1(4q - 2) - (d_1 + d_2)(2q - 1)| \leq 2q - 1$ and $|d_2(4q - 2) - (d_1 + d_2)(2q - 1)| \leq 2q - 1$, i.e. if and only if $|2d_1 - (d_1 + d_2)| \leq 1$ and $|2d_2 - (d_1 + d_2)| \leq 1$, i.e. if and only if $|d_1 - d_2| \leq 1$. □

Call M_i, $i = 1, 2$, the hyperelliptic line bundle on X_i. Since P is a Weierstrass point, the hyperelliptic involution of X_1 and X_2 induces an involution with a reducible plane conic as its image. Let $\phi : X \to \mathbb{P}^2$ be the morphism induced by gluing together the hyperelliptic involutions of X_1 and X_2. Thus M is a spanned line bundle of bidegree $(2, 2)$ and $M|X_i \cong M_i$, $i = 1, 2$. From Lemma 2 we get $h^0(X, M^{\otimes e}) = 2e + 1$ for all $e \in \{1, \ldots, q + 1\}$.

From Lemmas 2 and 3 and the cohomology of line bundles on hyperelliptic curves we get the following statement.

Lemma 4. *Fix an integer e such that $1 \le e \le q$.*

(i) $d'_{2e}(X) = 4e$ and $d'_{2e}(X)$ is computed only by the spanned line bundle $M^{\otimes e}$.

(ii) $d'_{2e-1}(X) = 4e-2$ and $d'_{2e-1}(X)$ is computed only by the line bundle L such that $L|X_i \cong M_i^{\otimes(e-1}(P)$ for all $i \in \{1,2\}$, which is not spanned.

Since $\mathrm{Pic}(Y) \cong \mathrm{Pic}(Y_1) \times \mathrm{Pic}(\mathbb{P}^1) \times \mathrm{Pic}(Y_2)$, we may describe each line bundle L on Y by a triple (L_1, t, L_2) with $L_i \in \mathrm{Pic}(Y_i)$ and $t \in \mathbb{Z}$. For any integer e let $N(e)$ denote the line bundle $(M_1^{\otimes e}, 1, M_2^{\otimes e})$.

Lemma 5. *Take $L \in \mathrm{Pic}(Y)$ with multidegree $(c_1, 1, c_2)$. L is balanced if and only if $c_1 = c_2$.*

Proof. The pair (L, Y_1) satisfies the Basic Inequality if and only if $|c_1 - (c_1+c_2+1)(2q-1)/(4q-2)| \le 1/2$, i.e. if and only if $|c_1 - 1 - c_2| \le 1$. The pair $(L, Y_1 \cup E)$ satisfies the Basic Inequality if and only if $c_1 + 1 - (c_1 + c_2 + 1)(2q-1)/(4q-2)| \le 1/2$, i.e. if and only if $|c_1 + 1 - c_2| \le 1$. These two inequalities are satisfied if and only if $c_1 = c_2$. Taking Y_2 instead of Y_1 we get the lemma. □

From Lemmas 4 and 2 we get the following statement.

Proposition 1. *Fix an integer e such that $1 \le e \le q$. Then $d_{2e+1}(X) = d_{2e}(X) + 1$*

Lemma 6. *Let L be a line bundle on Y such that $\deg(L|E) = 1$. Then $h^0(Y, L) = h^0(Y_1, L|Y_1) + h^0(Y_2, L|Y_2)$.*

Proof. Use the Mayer-Vietoris exact sequence

$$0 \to L \to (L|T_1 \cup Y_2) \oplus L|E \to L|(E \cap (Y_1 \cup Y_2)) \to 0 \tag{3}$$

and that the restriction map $H^0(E, L|E) \to H^0(E \cap (Y_1 \cup Y_2), L|(E \cap (Y_1 \cup Y_2)))$ is bijective. □

From Lemmas 5 and 6 and the cohomology of line bundles on hyperelliptic curves we get the following statement.

Proposition 2. *Fix an integer e such that $0 \le e \le q$. Then $d''_{2e+1}(X) = d''_{2e}(X) = 4e+1$ and both $d''_{2e+1}(X)$ and d''_{2e} are computed only by the line bundle $N(e)$.*

The curve X described in Example 1 is a flat limit of a family of smooth hyperelliptic curves, because P is a Weierstrass point both of X_1 and of X_2 ([6], part b) of Theorem 5).

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