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# AN EXTENDED HARDY-HILBERT TYPE INEQUALITY WITH WEIGHTS AND ITS APPLICATIONS

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**Abstract:** By introducing a power-exponent function with the form $ax^{\alpha+\beta x}+by^{\alpha+\beta y}$ and two pairs of conjugate exponents, an extended Hardy-Hilbert's integral inequality was established. As applications, some extensions of Hardy-Littlewood's integral inequality are also given.

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**Key Words:** Power-exponent function, weight function, integral inequality, conjugate exponent

## 1. Introduction

The well known Hardy-Hilbert's integral inequality is

$$\int_0^{\infty}\int_0^{\infty}\frac{f(x)g(y)}{x+y}\mathrm{d}x\mathrm{d}y \le \frac{\pi}{\sin\frac{\pi}{p}}\Big(\int_0^{\infty} f^p(x)\mathrm{d}x\Big)^{\frac{1}{p}}\Big(\int_0^{\infty} g^q(y)\mathrm{d}y\Big)^{\frac{1}{q}}, \tag{1.1}$$

where $p>1$, $q=p/(p-1)$, $f(x)$, $g(x)\ge 0$, $f(x)\in L^p(0,\infty)$, $g(x)\in L^q(0,\infty)$ and the constant factor $\frac{\pi}{\sin(\pi/p)}$ is the best possible [1]. As we know that the Hardy-Hilbert's integral inequality has widely applications in analysis and oth-

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ers, see Mitrinović [2]. Therefore, various extensions for (1.1) were obtained by many authors, see, for example, [3-8] and the references cited therein.

By introducing a parameter $\alpha$, Kuang [3] generalized (1.1) as following:

$$\int_a^b \int_a^b \frac{f(x)g(y)}{x^\alpha + y^\alpha} dxdy \le \left(\omega(\alpha,p,q) \int_a^b x^{1-\alpha} f^p(x)dx\right)^{\frac{1}{p}} \left(\omega(\alpha,q,p) \int_a^b x^{1-\alpha} g^q(x)dx\right)^{\frac{1}{q}},$$

where

$$\omega(\alpha,\ p,\ q) = \frac{\pi}{\alpha\ \sin\frac{\pi}{p\alpha}} - \int_0^{\frac{a}{b}} \frac{u^{\alpha-2+\frac{1}{q}}}{1+u^\alpha} du.$$

In a different direction, Yang [4] extended (1.1) to the following inequality:

$$\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{x^\alpha + y^\alpha} \mathrm{d}x\mathrm{d}y < \frac{\pi}{\alpha \sin\frac{\pi}{p}} \Big(\int_0^\infty x^{(p-1)(1-\alpha)} f^p(x)\mathrm{d}x\Big)^{\frac{1}{p}} \Big(\int_0^\infty x^{(q-1)(1-\alpha)} g^q(x)\mathrm{d}x\Big)^{\frac{1}{q}},$$

where the constant $\frac{\pi}{\alpha\sin(\pi/p)}$ is the best possible. For other related results, see also [5, 6].

Very recently, people have tried to the above results to more extension. In 2005, by introducing power-exponent function, Yang et al [7] claim the following:

$$\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{x^{1+x} + y^{1+y}} \mathrm{d}x\mathrm{d}y \le \frac{\pi}{\sin\frac{\pi}{p}} \Big(\int_0^\infty \omega(p,x) f^p(x)\mathrm{d}x\Big)^{\frac{1}{p}} \Big(\int_0^\infty \omega(q,x) g^q(x)\mathrm{d}x\Big)^{\frac{1}{q}}, \quad (1.2)$$

where the constant $\frac{\pi}{\sin(\pi/p)}$ is the best possible, and

$$\omega(r,x) = (x^x(1+x+x\ln x))^{1-r}, \quad r>1,\ x>0. \quad (1.3)$$

In 2006, by adding two coefficients in denominator, Jia and Gao[8] obtained the following inequality:

$$\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{ax^{1+x} + by^{1+y}} \mathrm{d}x\mathrm{d}y$$

$$\leq \frac{\pi}{a^{\frac{1}{q}} b^{\frac{1}{p}} \sin\frac{\pi}{p}} \Big( \int_0^{\infty} \omega(p,x) f^p(x) \mathrm{d}x \Big)^{\frac{1}{p}} \Big( \int_0^{\infty} \omega(q,x) g^q(x) \mathrm{d}x \Big)^{\frac{1}{q}}, \quad (1.4)$$

where $\omega(r,x)$ is defined by (1.3), and the constant $\frac{\pi}{a^{1/q} b^{1/p} \sin(\pi/p)}$ is the best possible.

In the present paper, we further extend the results in [7, 8] to more general cases. To this end, by introducing two pairs of conjugate exponents and replace the denominator $x+y$ in (1.1) by the power-exponent function $ax^{\alpha+\beta x}+by^{\alpha+\beta y}$, we give a new extension of Hilbert-type integral which includes (1.2) and (1.4). Moreover we also consider the equivalent forms. As applications, we establish some extensions of the Hardy-Littlewood integral inequality[1].

In order to show our main result, we need the following Lemma.

**Lemma 1.1.** *Let* $\alpha > e^{-2}\beta > 0$, *and*

$$h(x) = \frac{\alpha + \beta x}{x} + \beta \ln x, \qquad x \in (0, +\infty). \quad (1.5)$$

*Then* $h(x) > 0$.

*Proof.* Note that the minimum of $h(x)$ for $x \in (0,+\infty)$ is $\beta(2 + \ln\frac{\alpha}{\beta})$, it follows from $\alpha > e^{-2}\beta > 0$ that (1.5) holds. Thus, the proof is completed. □

## 2. Main Results

For convenience, we define the $\beta$-function $B(\mu,\nu)$ by, cf.[9],

$$B(\mu,\nu) := \int_0^{\infty} \frac{t^{\mu-1}}{(1+t)^{\mu+\nu}}\, dt = B(\nu,\mu), \quad \mu,\nu > 0, \quad (2.1)$$

and

$$u(t) := t^{\alpha+\beta t}, \quad \alpha > 0,\ \beta > 0,\ t \in (0,+\infty), \quad (2.2)$$

$$\theta(r) := \frac{\pi}{\sin\frac{\pi}{r}}, \qquad r > 1. \quad (2.3)$$

Throughout this paper, we assume that

(A1) Let $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, $r > 1$, $\frac{1}{r} + \frac{1}{s} = 1$, $f(x)$, $g(x) \geq 0$, the weight function $h(x)$ defined by (1.5), satisfies

$$\begin{aligned} 0 < ||f||_{p,\,\omega} &:= \Big( \int_0^{\infty} \omega(x)\, f^p(x) \mathrm{d}x \Big)^{\frac{1}{p}} < \infty, \\ 0 < ||g||_{q,\,\varpi} &:= \Big( \int_0^{\infty} \varpi(x)\, g^q(y) \mathrm{d}y \Big)^{\frac{1}{q}} < \infty. \end{aligned} \tag{2.4}$$

where $\omega(x) = u^{-\frac{p}{r}}(x)\, h^{1-p}(x)$, $\varpi(x) = u^{-\frac{q}{s}}(y)\, h^{1-q}(y)$.

(A2) Let $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, $r > 1$, $\frac{1}{r} + \frac{1}{s} = 1$, $f(x) \geq 0$, such that $0 < ||f||_{p,\,\omega} < \infty$.

(A3) Let $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, $r > 1$, $\frac{1}{r} + \frac{1}{s} = 1$, $g(y) \geq 0$, such that $0 < ||g||_{q,\,\varpi} < \infty$.

**Theorem 2.1.** *Assume (A1) holds. Suppose that there exist $\alpha$, $\beta > 0$ such that $\alpha > e^{-2}\beta$. Then for all $a, b > 0$,*

$$\int_0^{\infty}\int_0^{\infty} \frac{f(x)g(y)}{a\,u(x) + b\,u(y)} \mathrm{d}x\mathrm{d}y < a^{-\frac{1}{r}}\, b^{-\frac{1}{s}}\, \theta(r)\, ||f||_{p,\,\omega} ||g||_{q,\,\varpi}\,, \tag{2.5}$$

*where $\theta(r)$ is defined by (2.3), and the constant factor $a^{-\frac{1}{r}} b^{-\frac{1}{s}}\, \theta(r)$ is the best possible.*

*Proof.* Let

$$\begin{aligned} \alpha_1 &= \Big( \frac{1}{a\,u(x) + b\,u(y)} \Big)^{\frac{1}{p}} \frac{(a\,u(x))^{\frac{1}{sq}}}{(b\,u(y))^{\frac{1}{rp}}} \frac{(b\,u'(y))^{\frac{1}{p}}}{(a\,u'(x))^{\frac{1}{q}}}\, f(x)\,, \\ \beta_1 &= \Big( \frac{1}{a\,u(x) + b\,u(y)} \Big)^{\frac{1}{q}} \frac{(b\,u(y))^{\frac{1}{rp}}}{(a\,u(x))^{\frac{1}{sq}}} \frac{(a\,u'(x))^{\frac{1}{q}}}{(b\,u'(y))^{\frac{1}{p}}}\, g(y)\,, \end{aligned}$$

where $u'(x) = \frac{d}{dx}\, u(x)$, and $u'(x) > 0$ (by Lemma 1.1).
Using the weighted Hölder's inequality Kuang [13], we get

$$\begin{aligned} I :&= \int_0^{\infty}\int_0^{\infty} \frac{f(x)g(y)}{a\,u(x) + b\,u(y)} \mathrm{d}x\mathrm{d}y = \int_0^{\infty}\int_0^{\infty} \alpha_1\, \beta_1 \mathrm{d}x\mathrm{d}y \\ &\leq \Big( \int_0^{\infty}\int_0^{\infty} \alpha_1^p\, \mathrm{d}x\mathrm{d}y \Big)^{\frac{1}{p}} \Big( \int_0^{\infty}\int_0^{\infty} \beta_1^q\, \mathrm{d}x\mathrm{d}y \Big)^{\frac{1}{q}}. \end{aligned} \tag{2.6}$$

Note that

$$\begin{aligned}\int_0^\infty\int_0^\infty \alpha_1^p \,\mathrm{d}x\mathrm{d}y &= \int_0^\infty \Big(\int_0^\infty \frac{(b\,u(y))^{-\frac{1}{r}} b\,u'(y)}{a\,u(x)+b\,u(y)}\mathrm{d}y\Big)\frac{(a\,u(x))^{\frac{p-1}{s}}}{(a\,u'(x))^{p-1}} f^p(x)\mathrm{d}x\\ &= \int_0^\infty \Big((a\,u(x))^{\frac{1}{r}}\int_0^\infty \frac{(b\,u(y))^{-\frac{1}{r}} b\,u'(y)}{a\,u(x)+b\,u(y)}\mathrm{d}y\Big)\frac{(a\,u(x))^{\frac{p}{s}-1}}{(a\,u'(x))^{p-1}} f^p(x)\mathrm{d}x\\ &= \int_0^\infty \omega_r(x)\frac{(a\,u(x))^{\frac{p}{s}-1}}{(a\,u'(x))^{p-1}} f^p(x)\mathrm{d}x\ , \qquad (2.7)\end{aligned}$$

where

$$\omega_r(x) = (a\,u(x))^{\frac{1}{r}}\int_0^\infty \frac{(b\,u(y))^{-\frac{1}{r}} b\,u'(y)}{a\,u(x)+b\,u(y)}\mathrm{d}y\ .$$

Let $t = b\,u(y)/(a\,u(x))$. it follows from (2.1) that

$$\omega_r(x) = \int_0^\infty \frac{t^{-\frac{1}{r}}}{1+t}\,dt = B(\frac{1}{r},\ \frac{1}{s}) = \theta(r) = \theta(s). \qquad (2.8)$$

Combining (2.7) and (2.8), then

$$\int_0^\infty\int_0^\infty \alpha_1^p \,\mathrm{d}x\mathrm{d}y = \theta(r)\int_0^\infty \frac{(a\,u(x))^{\frac{p}{s}-1}}{(a\,u'(x))^{p-1}}\ f^p(x)\mathrm{d}x. \qquad (2.9)$$

Similar to the proof of (2.9), we also get

$$\int_0^\infty\int_0^\infty \beta_1^q \,\mathrm{d}x\mathrm{d}y = \theta(s)\int_0^\infty \frac{(b\,u(y))^{\frac{q}{r}-1}}{(b\,u'(y))^{q-1}}\ g^q(y)\mathrm{d}y. \qquad (2.10)$$

Substituting (2.9) and (2.10) into (2.6), it can be obtain

$$I \le \theta(r)\Big(\int_0^\infty \frac{(a\,u(x))^{\frac{p}{s}-1}}{(a\,u'(x))^{p-1}} f^p(x)\mathrm{d}x\Big)^{\frac{1}{p}}\Big(\int_0^\infty \frac{(b\,u(y))^{\frac{q}{r}-1}}{(b\,u'(y))^{q-1}} g^q(y)\mathrm{d}y\Big)^{\frac{1}{q}}. \qquad (2.11)$$

We now claim the equality "=" in (2.11) not hold. If not, then there exist two constants $A$ and $B$ with $A^2+B^2 \neq 0$ such that

$$\begin{aligned}&A\frac{b\,u'(y)}{(b\,u(y))^{\frac{1}{r}}}\frac{(a\,u(x))^{\frac{p-1}{s}}}{(a\,u'(x))^{p-1}} f^p(x)\\ &\qquad = B\frac{a\,u'(x)}{(a\,u(x))^{\frac{1}{s}}}\frac{(b\,u(y))^{\frac{q-1}{r}}}{(b\,u'(y))^{q-1}} g^q(y),\ \ a.e. \text{ in } (0,\infty)\times(0,\infty).\end{aligned}$$

i.e., there exists a constant $c_0$ such that

$$A\,\frac{(a\,u(x))^{\frac{p}{s}}}{(a\,u'(x))^{p}}f^{p}(x)=B\,\frac{(b\,u(y))^{\frac{q}{r}}}{(b\,u'(y))^{q}}g^{q}(y)=c_0,\quad a.\ e.\quad \text{in}\ \ (0,\infty).$$

Without loss of generality, we may suppose $A\neq 0$, then

$$a^{-\frac{p}{r}}\,u^{-\frac{p}{r}}(x)\,h^{1-p}(x)\,f^{p}(x)=\frac{(a\,u(x))^{\frac{p}{s}-1}}{(a\,u'(x))^{p-1}}f^{p}(x)=\frac{c_0(a\,u(x))'}{A\,(a\,u(x))},\quad a.\ e.\ \text{in}\ \ (0,\infty),$$

which contradicts the first form of (2.4). Hence (2.11) takes the form of strict inequality.

Next, we show that the constant factor $\theta(r)$ in (2.11) is the best possible. In fact, let $\varepsilon\in(0,\frac{q}{s})$, define

$$\begin{aligned}
f_\varepsilon(x)&=\begin{cases}0, & x\in(0,1);\\ (a\,u(x))^{-\frac{1}{s}-\frac{\varepsilon}{p}}\,(a\,u(x))', & x\in[1,+\infty),\end{cases}\\
g_\varepsilon(y)&=\begin{cases}0, & y\in(0,1);\\ (b\,u(y))^{-\frac{1}{r}-\frac{\varepsilon}{q}}\,(b\,u(y))', & y\in[1,+\infty).\end{cases}
\end{aligned}\tag{2.12}$$

If the constant factor $\theta(r)$ in (2.11) is not the best possible, then there exists a positive constant $k\in(0,\theta(r))$ such that, by (2.12),

$$\begin{aligned}
I&<k\Big(\int_0^\infty\frac{(a\,u(x))^{\frac{p}{s}-1}}{(a\,u'(x))^{p-1}}\,f_\varepsilon^{p}(x)\mathrm{d}x\Big)^{\frac{1}{p}}\Big(\int_0^\infty\frac{(b\,u(y))^{\frac{q}{r}-1}}{(b\,u'(y))^{q-1}}\,g_\varepsilon^{q}(y)\mathrm{d}y\Big)^{\frac{1}{q}}\\
&=k\Big(\int_1^\infty(a\,u(x))^{-1-\varepsilon}\,a\,u'(x)\mathrm{d}x\Big)^{\frac{1}{p}}\Big(\int_1^\infty(b\,u(y))^{-1-\varepsilon}\,b\,u'(y)\mathrm{d}y\Big)^{\frac{1}{q}}\\
&=a^{-\frac{\varepsilon}{p}}\,b^{-\frac{\varepsilon}{q}}\,\frac{k}{\varepsilon}.
\end{aligned}\tag{2.13}$$

On the other hand, we find

$$\begin{aligned}
I&=\int_0^\infty\int_0^\infty\frac{f_\varepsilon(x)g_\varepsilon(y)}{a\,u(x)+b\,u(y)}\mathrm{d}x\mathrm{d}y\\
&=\int_1^\infty\int_1^\infty\frac{(a\,u(x))^{-\frac{1}{s}-\frac{\varepsilon}{p}}\,a\,u'(x)\ (b\,u(y))^{-\frac{1}{r}-\frac{\varepsilon}{q}}\,b\,u'(y)}{a\,u(x)+b\,u(y)}\mathrm{d}x\mathrm{d}y\\
&=\int_1^\infty(a\,u(x))^{-\frac{1}{s}-\frac{\varepsilon}{p}}\,a\,u'(x)\ \Big(\int_1^\infty\frac{(b\,u(y))^{-\frac{1}{r}-\frac{\varepsilon}{q}}\,b\,u'(y)}{a\,u(x)+b\,u(y)}\mathrm{d}y\Big)\mathrm{d}x\\
&=\int_1^\infty(a\,u(x))^{-1-\varepsilon}\,a\,u'(x)\ \Big(\int_{\frac{b}{au(x)}}^\infty\frac{t^{\frac{1}{s}-1-\frac{\varepsilon}{q}}}{1+t}\,\mathrm{d}t\Big)\mathrm{d}x
\end{aligned}$$

$$
\begin{aligned}
&= \int_1^\infty (a\,u(x))^{-1-\varepsilon}\, a\,u'(x) \Big(\int_0^\infty \frac{t^{\frac{1}{s}-1-\frac{\varepsilon}{q}}}{1+t}\,\mathrm{d}t\Big)\mathrm{d}x \\
&\quad - \int_1^\infty (a\,u(x))^{-1-\varepsilon}\, a\,u'(x) \Big(\int_0^{\frac{b}{au(x)}} \frac{t^{\frac{1}{s}-1-\frac{\varepsilon}{q}}}{1+t}\,\mathrm{d}t\Big)\mathrm{d}x \\
&> \frac{1}{a^\varepsilon\,\varepsilon}(\theta(r)+\circ(1)) - \int_1^\infty \frac{u'(x)}{u(x)} \Big(\int_0^{\frac{b}{au(x)}} t^{\frac{1}{s}-1-\frac{\varepsilon}{q}}\,\mathrm{d}t\Big)\mathrm{d}x \\
&= \frac{1}{a^\varepsilon\,\varepsilon}(\theta(r)+\circ(1)) - b^{\frac{1}{s}-\frac{\varepsilon}{q}} \Big(\frac{1}{s}-\frac{\varepsilon}{q}\Big)^{-1} \int_1^\infty (a\,u(x))^{-\frac{1}{s}+\frac{\varepsilon}{q}-1}\, a\,u'(x)\,\mathrm{d}x \\
&= \frac{1}{a^\varepsilon\,\varepsilon}(\theta(r)+\circ(1)) + \Big(\frac{b}{a}\Big)^{\frac{1}{s}-\frac{\varepsilon}{q}} \Big(\frac{1}{s}-\frac{\varepsilon}{q}\Big)^{-2} \\
&= \frac{1}{a^\varepsilon\,\varepsilon}(\theta(r)+\circ(1)) + O(1). \qquad (2.14)
\end{aligned}
$$

Let $\varepsilon$ be sufficiently small, in view of (2.13) and (2.14), one has

$$\theta(r)+\circ(1)+\varepsilon a^\varepsilon\cdot O(1) < \varepsilon a^\varepsilon I < \Big(\frac{a}{b}\Big)^{\varepsilon/q} k.$$

Consequently, we have $\theta(r)\le k$ as $\varepsilon\to 0$, which contradicts the fact $k<\theta(r)$. Hence the constant factor $\theta(r)$ in (2.11) is the best possible.

Note that $u'(x)=u(x)\,h(x)$, by a simple computation, we get (2.5). Thus the proof is completed. □

**Remark 2.1.** (i) Let $a=b=\alpha=\beta=1$, $r=q$, $s=p$. (2.5) reduces to Theorem 2.1 in [7].

(ii) Let $a=b=\alpha=\beta=1$, $r=p$, $s=q$. (2.5) becomes the dual form of Theorem 2.1 in [7] as

$$
\begin{aligned}
\int_0^\infty\!\!\int_0^\infty \frac{f(x)g(y)}{x^{1+x}+y^{1+y}}\mathrm{d}x\mathrm{d}y &< \theta(p)\Big(\int_0^\infty x^{p-2-x}(1+x+x\ln x)^{1-p} f^p(x)\mathrm{d}x\Big)^{\frac{1}{p}} \\
&\quad\times\Big(\int_0^\infty y^{q-2-y}(1+y+y\ln y)^{1-q} g^q(y)\mathrm{d}y\Big)^{\frac{1}{q}}.
\end{aligned}
$$

(iii) Let $\alpha=\beta=1$, $r=q$, $s=p$. (2.5) implies Theorem 2.1 in [8].

(iv) Let $\alpha=\beta=1$, $r=p$, $s=q$. (2.5) follows the dual form of Theorem 2.1 in [8] as

$$\int_0^\infty\!\!\int_0^\infty \frac{f(x)g(y)}{a\,x^{1+x}+b\,y^{1+y}}\mathrm{d}x\mathrm{d}y$$

$$< a^{-\frac{1}{p}} b^{-\frac{1}{q}} \theta(p)\Big(\int_0^\infty x^{p-2-x}(1+x+x\ln x)^{1-p} f^p(x)\mathrm{d}x\Big)^{\frac{1}{p}}$$
$$\times \Big(\int_0^\infty y^{q-2-y}(1+y+y\ln y)^{1-q} g^q(y)\mathrm{d}y\Big)^{\frac{1}{q}}.$$

**Theorem 2.2.** *Assume (A2) holds. Suppose that there exist* $\alpha,\ \beta > 0$ *such that* $\alpha > e^{-2}\beta$. *Then for all* $a, b > 0$,

$$\int_0^\infty u'(y)u^{\frac{p}{s}-1}(y)\Big(\int_0^\infty \frac{f(x)}{a\,u(x)+b\,u(y)}\mathrm{d}x\Big)^p \mathrm{d}y < \Big(a^{-\frac{1}{r}} b^{-\frac{1}{s}} \theta(r)\Big)^p ||f||^p_{p,\,\omega}, \tag{2.15}$$

*where* $\theta(r)$ *is defined by (2.3), and the constant factor* $(a^{-\frac{1}{r}} b^{-\frac{1}{s}} \theta(r))^p$ *is the best possible. Moreover, (2.15) is equivalent to (2.5).*

*Proof.* Let

$$g(y) := u'(y)\,u^{\frac{p}{s}-1}(y)\Big(\int_0^\infty \frac{f(x)}{a\,u(x)+b\,u(y)}\mathrm{d}x\Big)^{p-1}, \quad y \in (0,\ \infty).$$

By (2.5),

$$\begin{aligned} 0 < ||g||^q_{q,\,\varpi} &= \int_0^\infty u^{\frac{q}{r}-1}(y)(u'(y))^{1-q} g^q(y)\mathrm{d}y \\ &= \int_0^\infty u'(y)\,u^{\frac{p}{s}-1}(y)\Big(\int_0^\infty \frac{f(x)}{a\,u(x)+b\,u(y)}\mathrm{d}x\Big)^p \mathrm{d}y \\ &= \int_0^\infty\int_0^\infty \frac{f(x)g(y)}{a\,u(x)+b\,u(y)}\mathrm{d}x\mathrm{d}y \\ &\le a^{-\frac{1}{r}} b^{-\frac{1}{s}} \theta(r)\,||f||_{p,\,\omega}||g||_{q,\,\varpi}\,, \end{aligned} \tag{2.16}$$

consequently,

$$0 < ||g||^q_{q,\,\varpi} \le \big(a^{-\frac{1}{r}} b^{-\frac{1}{s}} \theta(r)\big)^p ||f||^p_{p,\,\omega}\,. \tag{2.17}$$

Hence, we have $0 < ||g||_{q,\,\varpi} < \infty$. By (2.5), both (2.16) and (2.17) take the form of strict inequalities. Hence (2.15) holds.

On the other hand, if (2.15) holds, by the weighted Hölder's inequality Kuang[13], we have

$$I = \int_0^\infty \Big((u'(y))^{\frac{1}{p}} u^{\frac{1}{s}-\frac{1}{p}}(y)\int_0^\infty \frac{f(x)}{a\,u(x)+b\,u(y)}\mathrm{d}x\Big)\big(u^{\frac{1}{p}-\frac{1}{s}}(y)(u'(y))^{-\frac{1}{p}}\,g(y)\big)\mathrm{d}y$$

$$\begin{aligned}
&\leq \Big(\int_0^{\infty} u'(y)\, u^{\frac{p}{s}-1}(y) \Big(\int_0^{\infty} \frac{f(x)}{a\,u(x)+b\,u(y)} \mathrm{d}x\Big)^{p} \mathrm{d}y\Big)^{\frac{1}{p}} \\
&\quad \times \Big(\int_0^{\infty} u^{\frac{q}{r}-1}(y)(u'(y))^{1-q}\, g^{q}(y)\mathrm{d}y\Big)^{\frac{1}{q}} \\
&< a^{-\frac{1}{r}}\, b^{-\frac{1}{s}}\, \theta(r)\, ||f||_{p,\,\omega}||g||_{q,\,\overline{\omega}}\,. \qquad (2.18)
\end{aligned}$$

Hence inequalities (2.5) and (2.15) are equivalent.

If the constant factor $(a^{-1/r}\, b^{-1/s}\, \theta(r))^p$ in (2.15) is not the best possible, we can obtain a contradiction that the constant factor in (2.5) is not the best possible by (2.18). Hence the constant factor in (2.15) is still the best possible. This completes the proof. □

**Theorem 2.3.** *Assume (A3) holds. Suppose that there exist $\alpha,\ \beta > 0$ such that $\alpha > e^{-2}\beta$. Then for all $a, b > 0$,*

$$\int_0^{\infty} u'(x) u^{\frac{q}{r}-1}(x) \Big(\int_0^{\infty} \frac{g(y)}{a\,u(x)+b\,u(y)} \mathrm{d}y\Big)^{q} \mathrm{d}x < \Big(a^{-\frac{1}{r}}\, b^{-\frac{1}{s}}\, \theta(r)\Big)^{q} ||g||^{q}_{q,\,\overline{\omega}}, \qquad (2.19)$$

*where $\theta(r)$ is defined by (2.3), and the constant factor $(a^{-\frac{1}{r}}\, b^{-\frac{1}{s}}\, \theta(r))^q$ is the best possible. Moreover, (2.19) is equivalent to (2.5).*

*Proof.* Put

$$f(x) := u'(x)\, u^{\frac{q}{r}-1}(x) \Big(\int_0^{\infty} \frac{g(y)}{a\,u(x)+b\,u(y)} \mathrm{d}y\Big)^{q-1}, \quad x \in (0,\ \infty).$$

By (2.5),

$$\begin{aligned}
0 < ||f||^{p}_{p,\,\omega} &= \int_0^{\infty} u^{\frac{p}{s}-1}(x)(u'(x))^{1-p}\, f^{p}(x)\mathrm{d}x \\
&= \int_0^{\infty} u'(x)\, u^{\frac{q}{r}-1}(x) \Big(\int_0^{\infty} \frac{g(y)}{a\,u(x)+b\,u(y)} \mathrm{d}y\Big)^{q} \mathrm{d}x \\
&= \int_0^{\infty}\int_0^{\infty} \frac{f(x)g(y)}{a\,u(x)+b\,u(y)} \mathrm{d}x\mathrm{d}y \\
&\leq a^{-\frac{1}{r}}\, b^{-\frac{1}{s}}\, \theta(r)\, ||f||_{p,\,\omega}||g||_{q,\,\overline{\omega}}\,, \qquad (2.20)
\end{aligned}$$

consequently,

$$0 < ||f||^{p}_{p,\,\omega} \leq \Big(a^{-\frac{1}{r}}\, b^{-\frac{1}{s}}\, \theta(r)\Big)^{q} ||g||^{q}_{q,\,\overline{\omega}}\,. \qquad (2.21)$$

Hence, $0 < ||f||_{p,\,\omega} < \infty$. By (2.5), both (2.20) and (2.21) take the form of strict inequalities. Hence (2.19) holds.

On the other hand, if (2.19) holds, by the weighted Hölder's inequality Kuang [13], we have

$$\begin{aligned} I =& \int_0^\infty \Big(u^{\frac{1}{q}-\frac{1}{r}}(x)(u'(x))^{-\frac{1}{q}}f(x)\Big)\Big((u'(x))^{\frac{1}{q}}u^{\frac{1}{r}-\frac{1}{q}}(x)\int_0^\infty \frac{g(y)}{a\,u(x)+b\,u(y)}\mathrm{d}y\Big)\mathrm{d}x \\ \le& \Big(\int_0^\infty u^{\frac{p}{s}-1}(x)(u'(x))^{1-p}f^p(x)\mathrm{d}x\Big)^{\frac{1}{p}} \\ &\times \Big(\int_0^\infty u'(x)u^{\frac{q}{r}-1}(x)\Big(\int_0^\infty \frac{g(y)}{a\,u(x)+b\,u(y)}\mathrm{d}y\Big)^q\mathrm{d}x\Big)^{\frac{1}{q}} \\ <& a^{-\frac{1}{r}}b^{-\frac{1}{s}}\theta(r)\,||f||_{p,\,\omega}||g||_{q,\,\varpi}\,. \end{aligned} \tag{2.22}$$

Hence inequalities (2.5) and (2.19) are equivalent.

If the constant factor $(a^{-1/r}\,b^{-1/s}\,\theta(r))^q$ in (2.19) is not the best possible, we can obtain a contradiction that the constant factor in (2.5) is not the best possible by (2.22). Hence the constant factor in (2.19) is still the best possible. Thus we complete the proof. □

## 3. Applications

As applications of Theorem 2.1, in this section we should give some extensions of Hardy-Littlewood's integral inequality (see Hardy [1]).

Let $\varphi(x) \in L^2(0,1)$ and $\varphi(x) \neq 0$ for $x \in (0,1)$. If

$$a_n = \int_0^1 x^n\,\varphi(x)\,dx, \quad n = 0,\ 1,\ 2, \cdots,$$

then the Hardy-Littlewood's inequality [1, Theorem 324],

$$\sum_{n=0}^{\infty} a_n^2 < \pi \int_0^1 \varphi^2(x)dx, \tag{3.1}$$

holds, where the constant factor $\pi$ is the best possible.

Gao[10] extended (3.1) to the following:

$$\int_0^\infty f^2(x)dx < \pi \int_0^1 \varphi^2(t)dt, \tag{3.2}$$

where $f(x) = \int_0^1 t^x \varphi(t)dt$ for $x \in [0, +\infty)$.

Furthermore, Gao[11] got the refined form as following:

$$\int_0^{\infty} f^2(x)dx \le \pi \int_0^1 t\,\varphi^2(t)dt. \tag{3.3}$$

We will further extend the inequality (3.3) and establish more refined inequality.

**Theorem 3.1.** *Let $\varphi(t) \in L^2(0,1)$, $\varphi(t) \neq 0$ for $t \in (0,1)$ and (A1) holds. Let*

$$f(x) = \int_0^1 t^{u(x)} |\varphi(t)| dt, \quad x \in (0, +\infty),$$

*then*

$$\Big(\int_0^{\infty} f^2(x)dx\Big)^2 < \theta(r)\, ||f||_{p,\,\omega} ||f||_{q,\,\varpi} \int_0^1 t\,\varphi^2(t)dt, \tag{3.4}$$

*where $u(x)$ is defined by (2.2), $\theta(r)$ is defined by (2.3), and the constant factor $\theta(r)$ is the best possible.*

*Proof.* Let

$$f^2(x) = \int_0^1 f(x)\, t^{u(x)}\, |\varphi(t)| dt.$$

By Schwarz's inequality and Theorem 2.1, we get

$$\begin{aligned}
\Big(\int_0^{\infty} f^2(x)dx\Big)^2 &= \Big(\int_0^{\infty} \Big(\int_0^1 f(x)t^{u(x)}|\varphi(t)|dt\Big)dx\Big)^2 \\
&= \Big(\int_0^1 \Big(\int_0^{\infty} f(x)t^{(u(x)-\frac{1}{2})}dx\Big)t^{\frac{1}{2}}|\varphi(t)|dt\Big)^2 \\
&\le \int_0^1 \Big(\int_0^{\infty} f(x)t^{(u(x)-\frac{1}{2})}dx\Big)^2 dt \int_0^1 t\varphi^2(t)dt \\
&= \int_0^1 \Big(\int_0^{\infty} f(x)t^{(u(x)-\frac{1}{2})}dx\Big)\Big(\int_0^{\infty} f(y)t^{(u(y)-\frac{1}{2})}dy\Big)dt \\
&\quad \times \int_0^1 t\varphi^2(t)dt \\
&= \int_0^1 \Big(\int_0^{\infty}\int_0^{\infty} f(x)f(y)t^{u(x)+u(y)-1}dxdy\Big)dt \\
&\quad \times \int_0^1 t\varphi^2(t)dt
\end{aligned}$$

$$= \Big( \int_0^{\infty} \int_0^{\infty} \frac{f(x)f(y)}{u(x)+u(y)} dxdy \Big) \int_0^1 t\varphi^2(t)dt$$
$$\leq \theta(r)\,||f||_{p,\,\omega}||f||_{q,\,\varpi} \int_0^1 t\,\varphi^2(t)dt \tag{3.5}$$

Note that $\varphi(t) \neq 0$, $f^2(x) \neq 0$, so the equality in (3.5) is impossible. Thus we complete the proof. □

**Remark 3.1.** (i) Let $\alpha = \beta = 1$, $r = q$, $s = p$. (3.4) reduces to Theorem 3.1 in [8].

(ii) Let $\alpha = \beta = 1$, $r = p$, $s = q$. (3.4) becomes the dual form of Theorem 3.1 in [8] as following

$$\Big( \int_0^{\infty} f^2(x)dx \Big)^2 < \theta(p) \Big( \int_0^{\infty} x^{p-2-x}(1+x+x\ln x)^{1-p}\, f^p(x)dx \Big)^{\frac{1}{p}}$$
$$\times \Big( \int_0^{\infty} y^{q-2-y}(1+y+y\ln y)^{1-q}\, f^q(y)\,dy \Big)^{\frac{1}{q}} \int_0^1 t\,\varphi^2(t)dt.$$

Let $r = s = p = q = 2$, by Theorem 3.1, we have

**Corollary 3.1.** *Let the functions $\varphi(t)$, $f(x)$ and $u(x)$ satisfy the conditions in Theorem 3.1. Assume further that*

$$0 < \int_0^{\infty} (u(x)\ h(x))^{-1}\, f^2(x)\mathrm{d}x < +\infty.$$

*Then*

$$\Big( \int_0^{\infty} f^2(x)dx \Big)^2 < \pi \Big( \int_0^{\infty} (u(x)\,h(x))^{-1}\, f^2(x)\mathrm{d}x \Big) \int_0^1 t\,\varphi^2(t)dt. \tag{3.6}$$

*where the constant factor $\pi$ is the best possible.*

Obviously, the inequalities (3.4) and (3.6) are extensions of (3.3).

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