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ELEMENTAL SOLUTIONS OF THE OPERATOR L^K AND THE DISTRIBUTIONAL PRODUCT BETEWEEN $pf\left\{P^{-j}\right\}$ AND $\nabla\left(L^k P_+^{k-\frac{n}{2}}\right)$

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Abstract: The object of this paper is obtain elemental solutions of the operator L^k and give a sense to distribution products between $pf\left\{P^{-j}\right\}$ and $\nabla\left(L^k P_+^{k-\frac{n}{2}}\right)$where ∇ is the operator defined by(25) and L^k is the $n-$dimensional ultrahyperbolic operator iterated k-times defined by the formula (22). As consequence our formulae(83) is a genralization of the formula

$$\Delta^k\left\{r^{2k-n}(A_{k,n}\log r+B_{k,n})\right\}=\delta$$

which appear in (see [10], p. 47) where $A_{k,n}$ is defined by(60)and $B_{k,n}$ by (43). Our product

$$pf\left\{P^{-j}\right\}.\nabla\left(L^k P_+^{k-\frac{n}{2}}\right)$$

generalizes of the product $r^{-2k}.\nabla(\Delta r^{2-m})$ given by Li Chen Kuan in (see [2], p. 346).

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1. Introduction

Let $x=(x_1,...x_{p,},x_{p+q,}...x_n)$be a point of the n-dimensional Euclidean space R^n,where $p+q=n$.

Consider a quadratic form in n variables defined by

$$P=P(x)=x_1^2+...x_p^2-x_{p+1}^2...-x_{p+q}^2. \tag{1}$$

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The hypersurface $P=0$ is an hypercone with a singular point (the vertex) at the origin.

We call $\varphi(x)$ the C^{∞}functions with compact support defined from R^n to R.

From [1]), p. 253, formula (2) the distribution P_+^{λ} is defined by

$$\left(P_+^{\lambda},\varphi\right)=\int_{P>0}(P(x))^{\lambda}\varphi(x)dx \tag{2}$$

and

$$\left(P_-^{\lambda},\varphi\right)=\int_{-P>0}(-P(x))^{\lambda}\varphi(x)dx \tag{3}$$

where $x=(x_1,...x_p,,x_{p+q,}...x_n)$, λ is a complex number and $dx=dx_1...dx_pdx_{p+1}$ $...dx_{p+q}$.

For Real(λ)$\geq$ 0, this integrals converges and are analytic function of λ. Analytic continuation to Real(λ) <0 can be used to extend the definitions of $\left(P_{\pm}^{\lambda},\varphi\right)$.

From see [1], p. 225, $\left(P_{\pm}^{\lambda},\varphi\right)$ have two sets of singularities, nameley

$$\lambda=-1,-2,-3,\ldots,-k,\ldots \tag{4}$$

and

$$\lambda=-\frac{n}{2},-\frac{n}{2}-1,\ldots,-\frac{n}{2}-s,\quad s=0,1,2,... \tag{5}$$

Let be the expansion in Laurent series of $P_{\pm}^{\lambda}$ in a neighborhood of point $\lambda=-s$, $s=1,2,...$ where n (dimension of the space) is odd,

$$P_{\pm}^{\lambda}=\frac{A_{-1}}{\lambda+k}+A_o+\sum_{\nu\geq 1}A_{\nu}(\lambda+k)^{\nu}. \tag{6}$$

The distribution A_o is by definition,the finite part of $P_{\pm}^{\lambda}$ at $\lambda=-s$, $s=1,2,...$,

$$A_o=pf_{\lambda=-s}\left\{P_{\pm}^{\lambda}\right\}=\lim_{\lambda\to -s}\left\{\frac{d}{d\lambda}(\lambda+s)P_{\pm}^{\lambda}\right\}=P_{\pm}^{-s} \tag{7}$$

(see [1], p. 86).

In follow $P_{\pm}^{-s}$ we mean finite part in the sense of definition (7).

On the other hand, from [1], pp. 260-269, the following formulae are valid

$$\operatorname{Re}s_{\lambda=-\frac{n}{2}-j}P_+^{\lambda}=a_{n,j}L^j\left\{\delta(x)\right\} \tag{8}$$

if p is odd and q is even (n odd),

$$\operatorname{Re}s_{\lambda=-\frac{n}{2}-j} P_+^\lambda = \frac{(-1)^{\frac{n}{2}+j-1}}{\Gamma(\frac{n}{2}+j)} \delta_1^{(\frac{n}{2}+j-1)}(P) + a_{n,j} L^j \{\delta(x)\} \tag{9}$$

if p and q are even and

$$\operatorname{Re}s_{\lambda=-\frac{n}{2}-j} P_+^\lambda = \frac{(-1)^{\frac{n}{2}+j-1}}{\Gamma(\frac{n}{2}+j)} \delta_1^{(\frac{n}{2}+j-1)}(P) + b_{n,j}\left[\psi(\frac{p}{2}) - \psi(\frac{n}{2})\right] L^j \{\delta(x)\} \tag{10}$$

if p and q are odd. Where

$$a_{n,j} = \frac{(-1)^{\frac{q}{2}} \pi^{\frac{n}{2}}}{2^{2j} j! \Gamma(\frac{n}{2}+j)}, \tag{11}$$

$$b_{n,j} = \frac{(-1)^{\frac{q+1}{2}} \pi^{\frac{n}{2}-1}}{2^{2j} j! \Gamma(\frac{n}{2}+j)}, \tag{12}$$

$$\psi(\alpha) = \frac{\Gamma'(\alpha)}{\Gamma(\alpha)}, \tag{13}$$

$\Gamma(\alpha)$ is the function gamma defined by

$$\Gamma(\alpha) = \int_o^\infty e^{-x} x^{\alpha-1} dx \tag{14}$$

and for integer and half integer value of the argument, $\psi(\alpha)$ is given by

$$\psi(k) = -\xi + 1 + \frac{1}{2} + ... + \frac{1}{k-1}, \tag{15}$$

$$\psi(k+\frac{1}{2}) = -\xi - 2\ln 2 + 2(1 + \frac{1}{3} + ... + \frac{1}{2k-1}) \tag{16}$$

and ξ is Euler's constant.

Here $\delta^{(k-1)}(P)$ is defined by

$$\delta^{(k-1)}(P) = (-1)^{k-1} \int_{P=0} w_{k-1}(\varphi), \tag{17}$$

$$w_{k-1}(\varphi) = \frac{\partial^k}{\partial u_1^k} \left\{ \varphi_1(u_1, u_2, ...u_n) D\binom{x}{u} du_2...du_n \right\} \tag{18}$$

and $D\binom{x}{u}$ is the jacobian of the transformation and

$$\varphi_1(u_1, u_2, ...u_n) = \varphi(x_1, x_2, ...x_n). \tag{19}$$

In (9) and (10), $\delta_1^{(k-1)}(P)$ we mean $\delta^{(k-1)}(P)$ if $k-1 < \frac{n}{2} - 1$, whereas if $k - 1 \geq \frac{n}{2} - 1$, it is to be understood in the sense of its regularization (see [1], Section 2.1, pp. 249-250), i.e.

$$\delta_1^{(k-1)}(P) = \delta^{(k-1)}(P) \tag{20}$$

if $k - 1 < \frac{n}{2} - 1$ and

$$\delta_1^{(k-1)}(P) = \text{ regularization of } \delta^{(k-1)}(P) \tag{21}$$

if $k - 1 \geq \frac{n}{2} - 1$.

On the other hand in(8),(9)and(10), L^j is the n-dimensional ultrahyperbolic operator iterated j times,defined by

$$L^j = \left\{ \frac{\partial^2}{\partial x_1^2} + ... + \frac{\partial^2}{\partial x_p^2} - \frac{\partial^2}{\partial x_{p+1}^2} - ... - \frac{\partial^2}{\partial x_{p+q}^2} \right\}^j . \tag{22}$$

From (see [2], p. 336),we know that the Neutrix Product of $r^{-k} \circ \nabla\delta$ exists and

$$r^{-2k} \circ \nabla\delta = \frac{1}{2^{k+1}(k+1)!(m+2)...(m+k)} . \sum_{i=1}^{m} (x_i \bigtriangleup^{k+1} \delta) \tag{23}$$

and

$$r^{1-2k} \circ \nabla\delta = 0, \tag{24}$$

where ∇ is the operator defined by

$$\nabla = \frac{\partial}{\partial x_1} + ... \frac{\partial}{\partial x_n}, \tag{25}$$

$m = n$ dimension of the space and

$$\Delta^j \delta = \left\{ \sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2} \right\}^j \delta \tag{26}$$

and

$$\Delta^o \delta = \delta. \tag{27}$$

The object of this paper is obtain elemental solutions of the operator L^k and give a sense to distribution products between $pf\left\{P^{-j}\right\}$ and $\nabla\left(L^k P_+^{k-\frac{n}{2}}\right)$where ∇ is the operator defined by(25) and L^k is the $n-$dimensional ultrahyperbolic operator iterated k-times defined by the formula(22). As consequence our formulae(83) is a genralization of the formula

$$\Delta^k\left\{r^{2k-n}(A_{k,n}\log r+B_{k,n})\right\}=\delta$$

which appear in (see [10], p. 47) where $A_{k,n}$ is defined by(60),$B_{k,n}$ by (43) and our product

$$pf\left\{P^{-j}\right\}.\nabla\left(L^k P_+^{k-\frac{n}{2}}\right)$$

generalizes the product $r^{-2k}.\nabla(\Delta r^{2-m})$ given by Li Chen Kuan in (see [2], p. 346).

2. Elemental Solution of the Operator L^k

We know that T distribution $\in D'(R^n)$ is called a elemental solution for the diferential operator $P(D)$ whith constant coefficients if

$$P(D)E=\delta. \tag{28}$$

On the other hand from (see [1], p. 258) the following formula is valid:

$$LP_+^{\lambda+1}=2(\lambda+1)(2\lambda+n)P_+^{\lambda}. \tag{29}$$

Iterated it k times, we arrive at,

$$L^kP_+^{\lambda+k}=2^{2k}(\lambda+1)...(\lambda+k)(\lambda+\frac{n}{2})...(\lambda+\frac{n}{2}+k-1)P_+^{\lambda}. \tag{30}$$

Using the formulae

$$\frac{\Gamma(z)}{\Gamma(z-m)}=(z-1)...(z-m) \tag{31}$$

(see [8], p. 4) and

$$\Gamma(z+m)=z(z+1)...(z+m-1)\Gamma(z) \tag{32}$$

the formula(30) can be written

$$L^kP_+^{\lambda+k}=\frac{(-1)^k2^{2k}\Gamma(-\lambda)\Gamma(\lambda+\frac{n}{2}+k)}{\Gamma(-\lambda-k)\Gamma(\lambda+\frac{n}{2})}P_+^{\lambda}\ . \tag{33}$$

Lemma 1. *Let n be odd dimension of the space and k nonnegative integers then exists a constant $c_{k,n,q}$ such that*

$$L^k\left\{c_{n,p,q,k}P_+^{k-\frac{n}{2}}\right\}=\delta(x_1,x_2,...x_n) \tag{34}$$

where L^k is the $n-$dimensional ultrahyperbolic operator iterated k-times defined by the formula(22), P_+^λ is defined by (21) and

$$c_{n,p,q,k}=\frac{\Gamma(\frac{n}{2}-k)}{(-1)^k2^{2k}(k-1)!(-1)^{\frac{q}{2}}\pi^{\frac{n}{2}}}. \tag{35}$$

Proof. Taking into account(4) and (5) P_+^λ is regular at $\lambda=k-\frac{n}{2}$ for n odd, therefore using the formula (33) we have,

$$L^k\left\{P_+^{k-\frac{n}{2}}\right\}=\lim_{\lambda\to-\frac{n}{2}}L^k\left\{P_+^{\lambda+k}\right\}=$$

$$=\lim_{\lambda\to-\frac{n}{2}}\tfrac{(-1)^k2^{2k}\Gamma(-\lambda)}{\Gamma(-\lambda-k)}\left((\lambda+\tfrac{n}{2}+1)...(\lambda+\tfrac{n}{2}+k-1)\right)(\lambda+\tfrac{n}{2})P_+^\lambda$$

$$=\tfrac{(-1)^k2^{2k}(k-1)!}{\Gamma(\frac{n}{2}-k)}(1.2....(k-1))\lim_{\lambda\to-\frac{n}{2}}(\lambda+\tfrac{n}{2})P_+^\lambda$$

$$=\tfrac{(-1)^k2^{2k}(k-1)!}{\Gamma(\frac{n}{2}-k)}(k-1)!\operatorname{Re}s_{\lambda=-\frac{n}{2}}P_+^\lambda. \tag{36}$$

Now using the formula

$$\operatorname{Re}s_{\lambda=-\frac{n}{2}}P_+^\lambda=\frac{(-1)^{\frac{q}{2}}\pi^{\frac{n}{2}}}{\Gamma(\frac{n}{2})}\delta(x) \tag{37}$$

if n is odd (see [1], p. 258, formula 23). From(36) we obtain the formula(34). □

We observe that the formula(34) is a genralization of the formula

$$\triangle\left\{\frac{1}{r^{n-2}}\right\}=(n-2)\Omega_n\delta(x) \tag{38}$$

(see [1], p. 29) and (see [2], p. 6), where Ω_n is the hypersurface area of the unit spehere in n-space and $\triangle$ is the Laplacian

$$\triangle=\sum_{i=1}^{n}\frac{\partial^2}{\partial x_i^2}. \tag{39}$$

In fact, putting $q = 0$ and $k = 1$ in (34) and (35) and using(22) we have

$$\triangle\left\{(x_1^2 + x_2^2 + ... + x_n^2)^{1-\frac{n}{2}}\right\} = \triangle\left\{r^{2-n}\right\} = \frac{(-1)2^2\pi^{\frac{n}{2}}}{\Gamma(\frac{n}{2}-1)}\delta = \frac{(-1)4^2\pi^{\frac{n}{2}}(\frac{n}{2}-1)}{\Gamma(\frac{n}{2})}\delta$$
$$-\frac{(n-.2)}{2}.\frac{\pi^{\frac{n}{2}}}{\Gamma(\frac{n}{2})}\delta = -(n-2)\Omega_n\delta(x). \tag{40}$$

where

$$\Omega_n = \frac{2\pi^{\frac{n}{2}}}{\Gamma(\frac{n}{2})}. \tag{41}$$

On the other hand, our formula(34) is a generalization of the formula

$$\Delta^k\left\{B_{k,n}r^{2k-n}\right\} = \delta \tag{42}$$

if $2k - n < 0$, as well as $2k - n \geq 0$ and n odd, which appear in [10], p. 47, formula (II,3,16), where

$$B_{k,n} = \left[(2k-n)(2k-2-n)...(4-n)(2-n)\frac{2^{k-1}(k-1)!2\pi^{\frac{n}{2}}}{\Gamma(\frac{n}{2})}\right]^{-1} \tag{43}$$

and δ is the measure of Dirac. In fact, using the formula

$$\frac{\Gamma(z)}{\Gamma(z-l)} = (z-1)...(z-l) \tag{44}$$

(see [8], p. 4), $l = 1, 2, ...$ $B_{k,n}$ can be written

$$B_{k,n} = \Big[2^k(-1)^k(\tfrac{n}{2}-1).....(\tfrac{n}{2}-(k-1))((\tfrac{n}{2}-k))$$
$$.\frac{2^{k-1}(k-1)!2\pi^{\frac{n}{2}}}{\Gamma(\frac{n}{2})}\Big]^{-1} = \left[\frac{2^{2k}(-1)^k(k-1)!\pi^{\frac{n}{2}}}{\Gamma(\frac{n}{2}-k)}\right]^{-1} \tag{45}$$

and putting $q = 0$ in (35) we obtain the formula(43).

Now we will study $L^k\left\{P_+^{k-\frac{n}{2}}\right\}$ when n is even.Taking into account(4) and (5) $P_+^{k-\frac{n}{2}}$ is singular if $k < \frac{n}{2}$ and is regular if $k \geq \frac{n}{2}$.

Now first study the case $k \geq \frac{n}{2}$.

Lemma 2. *Let n be even dimension of the space and k positive integers then exists constants $D_{n,p,q,k}$ and $\bar{D}_{n,p,q,k}$ such that the following formulae are valid*

$$L^k\left\{D_{n,p,q,k}P_+^{k-\frac{n}{2}}\ln P_+\right\} = \delta(x) \tag{46}$$

if $k \geq \frac{n}{2}$ and p, q are both even and

$$L^k \left\{ \bar{D}_{n,p,q,k} P_+^{k-\frac{n}{2}} \ln P_+ \right\} = \delta(x) \tag{47}$$

if $k \geq \frac{n}{2}$ and p, q are both odd. Where

$$D_{n,p,q,k} = \left[2^{2k} (-1)^{\frac{n}{2}} (-1)^{\frac{q}{2}} (k - \frac{n}{2})!(k-1)!\pi^{\frac{n}{2}} \right]^{-1} \tag{48}$$

and

$$\bar{D}_{n,p,q,k} = \left[2^{2k} (-1)^{\frac{n}{2}} (-1)^{\frac{q+1}{2}} (k - \frac{n}{2})!(k-1)!\pi^{\frac{n}{2}-1} \left(\psi(\frac{p}{2}) - \psi(\frac{n}{2}) \right) \right]^{-1} \tag{49}$$

and $\psi(x)$ is defined by(13),(15) and(16).

Proof. The expansion of $P_+^{\lambda+k}$ in the Taylor series about $\lambda = -\frac{n}{2}$ is

$$\begin{aligned} P_+^{\lambda+k} &= P_+^{k-\frac{n}{2}} + (P_+^{k-\frac{n}{2}} \ln P_+)(\lambda + \frac{n}{2}) + ... \\ &= \sum_{\nu \geq o} P_+^{k-\frac{n}{2}} \ln^\nu P_+ \ (\lambda + \frac{n}{2})^\nu, \end{aligned} \tag{50}$$

for $k \geq \frac{n}{2}$.

We observed from(30)and (33), that $L^k \left\{ P_+^{k-\frac{n}{2}} \right\} = 0$ if $k \geq \frac{n}{2}$ and n even. Therefore $P_+^{k-\frac{n}{2}}$ is a solution of the homogeneous equation $L^k u = f(x)$, where L^k is the n-dimensional ultrahyperbolic operator iterated k-times defined by the formula(22). Now we insert(44) into(30), taking into account that

$$\begin{aligned} &\frac{(\lambda+1)...(\lambda+k)(\lambda+\frac{n}{2}+1)...(\lambda+\frac{n}{2}+k-1)}{(\lambda+\frac{n}{2})} \\ &= \frac{(\lambda+1)...(\lambda+k)(\lambda+\frac{n}{2}+1)...(\lambda+\frac{n}{2}+k-1)}{\Gamma(\lambda+\frac{n}{2}+1)} (\lambda+1)...(\lambda+\frac{n}{2}-1)\Gamma(\lambda+1) \\ &= \frac{\Gamma(\lambda+k+1)(\lambda+\frac{n}{2}+1)...(\lambda+\frac{n}{2}+k-1)}{\Gamma(\lambda+1)\Gamma(\lambda+\frac{n}{2}+1)} (\lambda+1)...(\lambda+\frac{n}{2}-1)\Gamma(\lambda+1) \\ &= \frac{\Gamma(\lambda+k+1)(\lambda+\frac{n}{2}+1)...(\lambda+\frac{n}{2}+k-1)(\lambda+1)...(\lambda+\frac{n}{2}-1)}{\Gamma(\lambda+\frac{n}{2}+1)} \end{aligned} \tag{51}$$

and comparing coefficients of $(\lambda + \frac{n}{2})$ on both sides of the equation, we obtain,

$$L^k \left\{ P_+^{k-\frac{n}{2}} \ln P_+ \right\} = 2^{2k} (-1)^{\frac{n}{2}-1} (\frac{n}{2} - 1)!(k - \frac{n}{2})! \operatorname{Re}s_{\lambda=-\frac{n}{2}} P_+^\lambda. \tag{52}$$

On the other hand, from (see [1], pp. 260-269), P_+^λ has simple poles at $\lambda = -\frac{n}{2} - s, s = 0, 1, 2, ...$ and the residue is give by

$$\operatorname{Re}s_{\lambda=-\frac{n}{2}-s} P_+^\lambda = \frac{(-1)^{\frac{n}{2}+s-1}}{\Gamma(\frac{n}{2}+s)}\delta_1^{(\frac{n}{2}+s-1)}(P) + + \frac{(-1)^{\frac{q}{2}}\pi^{\frac{n}{2}}}{2^{2s}s!\Gamma(\frac{n}{2}+s)}L^s\delta \tag{53}$$

if p and q are both even and P_+^λ has poles of order2 at $\lambda = -\frac{n}{2} - s, s = 0, 1, 2, ...$and

$$c_{-1}^{(s)} = \operatorname{Re}s_{\lambda=-\frac{n}{2}-s} P_+^\lambda = \frac{(-1)^{\frac{n}{2}+s-1}}{\Gamma(\frac{n}{2}+s)}\delta_1^{(\frac{n}{2}+s-1)}(P) + + \frac{(-1)^{\frac{q+1}{2}}\pi^{\frac{n}{2}-1}}{2^{2s}s!\Gamma(\frac{n}{2}+s)}\left[\psi(\tfrac{p}{2}) - \psi(\tfrac{n}{2})\right]L^s\delta \tag{54}$$

if p and q are both odd (see [1], p. 269).

On the other hand, from (see [5], p. 261) the following formula are valid,

$$\delta_1^{(s)}(P) = \frac{(-2)(-1)^s(-1)^{\frac{q}{2}}\pi^{\frac{n}{2}}}{4^{s-\frac{n}{2}+1}(s-\frac{n}{2}+1)!}L^{s-\frac{n}{2}+1}\delta \tag{55}$$

if p and q are both even (n even) and $s \geq \frac{n}{2} - 1$, and

$$\delta_1^{(s)}(P) = \frac{2(-1)^{s-1}(-1)^{\frac{q+1}{2}}\pi^{\frac{n}{2}-1}}{4^{s-\frac{n}{2}+1}(s-\frac{n}{2}+1)!}\left[\psi(\frac{p}{2}) - \psi(\frac{n}{2})\right]L^{s-\frac{n}{2}+1}\delta \tag{56}$$

if p and q are both odd and and $s \geq \frac{n}{2} - 1$.

From (52) and using (53) and (55) we have

$$L^k\left\{P_+^{k-\frac{n}{2}}\ln P_+\right\} = 2^{2k}(-1)^{\frac{n}{2}-1}(\tfrac{n}{2}-1)!(k-\tfrac{n}{2})!(k-1)!. \left[\frac{(-1)^{\frac{n}{2}-1}}{\Gamma(\frac{n}{2})}\delta_1^{(\frac{n}{2}-1)}(P) + \frac{(-1)^{\frac{q}{2}}\pi^{\frac{n}{2}}}{\Gamma(\frac{n}{2})}\delta\right] = = 2^{2k}(-1)^{\frac{n}{2}}(k-\tfrac{n}{2})!(k-1)!.(-1)^{\frac{q}{2}}\pi^{\frac{n}{2}}\delta. \tag{57}$$

if p and q are both even.

Similarly from (52) and using (54) and (56) we have,

$$L^k\left\{P_+^{k-\frac{n}{2}}\ln P_+\right\} = 2^{2k}(-1)^{\frac{n}{2}-1}(\tfrac{n}{2}-1)!(k-\tfrac{n}{2})!(k-1)!.(-1)^{\frac{q+1}{2}}$$

$$\left\{\frac{2(-1)^{\frac{n}{2}-1}(-1)^{\frac{n}{2}-2}\pi^{\frac{n}{2}-1}}{\Gamma(\frac{n}{2})}\left[\psi(\tfrac{p}{2})-\psi(\tfrac{n}{2})\right]\delta + \frac{\pi^{\frac{n}{2}-1}}{\Gamma(\frac{n}{2})}\left[\psi(\tfrac{p}{2})-\psi(\tfrac{n}{2})\right]\delta\right\} =$$

$$= -2^{2k}(k-1)!(-1)^{\frac{n}{2}-1}(-1)^{\frac{q+1}{2}}\pi^{\frac{n}{2}-1}(k-\tfrac{n}{2})!\left[\psi(\tfrac{p}{2})-\psi(\tfrac{n}{2})\right]\delta. \tag{58}$$

From(57) and (58) we obtain the formulae(46) and (48) respectively. □

We observe that our formula(46) is a generalization of the formula

$$\triangle^k\left\{A_{k,n}r^{2k-n}\log r\right\} = \delta \tag{59}$$

if $2k-n\geq 0, n$ even and

$$A_{k,n} = \left[(2k-n)(2k-2-n)...(4-n)(2-n)2^{k-1}(k-1)!2\frac{\pi^{\frac{n}{2}}}{\Gamma(\frac{n}{2})}\right]^{-1} \tag{60}$$

which appear in (see [10], p. 47, formula (II,3,18).

In fact, putting $q=0$ in (46) and(48) and taking into account the conditions of the constan $A_{k,n}$ defined in (see [10], p. 48, formula (II,3,18)) we have

$$D_{n,p,o,k} = A_{k,n}. \tag{61}$$

Now we will study

$$L^k\left\{P_+^{k-\frac{n}{2}}\ln P_+\right\}. \tag{62}$$

when n is even and $k<\frac{n}{2}$.In this order to do it, we will consider two cases:

$$\text{case 1: } p \text{ and } q \text{ are both even.} \tag{63}$$

In this case P_+^{-s} we mean finite part in the sense of definition(7).

On the other hand, from (see [1], pp. 260-269) we know that P_+^{λ} has simple poles at $\lambda=-\frac{n}{2}-s, s=0,1,2,...$ and

$$c_{-2}^{(s)} = \lim_{\lambda\to-\frac{n}{2}-s}(\lambda+\frac{n}{2}+s)P_+^{\lambda} = 0 \tag{64}$$

(see [1], p. 262),where $c_{-2}^{(s)}$ is the coefficients of expansion of P_+^{λ} in the Laurent series:

$$P_+^{\lambda} = \frac{c_{-2}^{(s)}}{(\lambda+\frac{n}{2}+s)^2} + \frac{c_{-1}^{(s)}}{(\lambda+\frac{n}{2}+s)} + ... \tag{65}$$

Lemma 3. *Let n be even dimension of the space and k positive integers such that $k<\frac{n}{2}$ then exists a constant $d_{n,p,q,k}$ such that the following formula is valid*

$$L^k\left\{d_{n,p,q,k}P_+^{k-\frac{n}{2}}\right\}=\delta \tag{66}$$

where $P_+^{k-\frac{n}{2}}$ we mean finite part in the sense of definition(7),

$$d_{n,p,q,k}=-c_{n,p,q,k} \tag{67}$$

and $c_{n,p,q,k}$ is defined by(35).

Proof. From(33) and using the definition(7) we have

$$L^k\left\{P_+^{k-\frac{n}{2}}\right\}=\underset{\lambda=\frac{n}{2}}{pf}L^k\left\{P_+^{\lambda+k}\right\}=\lim_{\lambda\to-\frac{n}{2}}\frac{d}{d\lambda}\left[(\lambda+\tfrac{n}{2})L^kP_+^{\lambda+k}\right]=$$

$$=2^{2k}(-1)^k\lim_{\lambda\to-\frac{n}{2}}\frac{d}{d\lambda}\left[(\lambda+\tfrac{n}{2})\frac{\Gamma(-\lambda)}{\Gamma(-\lambda-k)}.\frac{\Gamma(\lambda+\frac{n}{2}+k)}{\Gamma(\lambda+\frac{n}{2})}P_+^{\lambda}\right]=2^{2k}(-1)^k.$$

$$\lim_{\lambda\to-\frac{n}{2}}\frac{d}{d\lambda}\left[(\lambda+\tfrac{n}{2})\left[(\lambda+\tfrac{n}{2})P_+^{\lambda}\right](\lambda+\tfrac{n}{2}+1)...(\lambda+\tfrac{n}{2}+k-1)\frac{\Gamma(-\lambda)}{\Gamma(-\lambda-k)}\right], \tag{68}$$

form(68) we have

$$L^k\left\{P_+^{k-\frac{n}{2}}\right\}=2^{2k}(-1)^k.\left\{\lim_{\lambda\to-\frac{n}{2}}\left[(\lambda+\tfrac{n}{2})P_+^{\lambda}(\lambda+\tfrac{n}{2}+1)...(\lambda+\tfrac{n}{2}+k-1).\right.\right.$$

$$\left.\frac{\Gamma(-\lambda)}{\Gamma(-\lambda-k)}\right]+\lim_{\lambda\to-\frac{n}{2}}\left\{(\lambda+\tfrac{n}{2})\frac{d}{d\lambda}\left[(\lambda+\tfrac{n}{2})P_+^{\lambda}\right].(\lambda+\tfrac{n}{2}+1)...(\lambda+\tfrac{n}{2}+k-1).\right.$$

$$\left.\frac{\Gamma(-\lambda)}{\Gamma(-\lambda-k)}\right\}+.\lim_{\lambda\to-\frac{n}{2}}(\lambda+\tfrac{n}{2})\left[(\lambda+\tfrac{n}{2})P_+^{\lambda}\right].$$

$$\left.\lim_{\lambda\to-\frac{n}{2}}\frac{d}{d\lambda}\left[(\lambda+\tfrac{n}{2}+1)...(\lambda+\tfrac{n}{2}+k-1)\frac{\Gamma(-\lambda)}{\Gamma(-\lambda-k)}\right]\right\}. \tag{69}$$

From(69) using that

$$\lim_{\lambda\to-\frac{n}{2}}\left[(\lambda+\frac{n}{2})P_+^{\lambda}\right]=\underset{\lambda=-\frac{n}{2}}{\mathrm{Re}\,s}P_+^{\lambda} \tag{70}$$

and

$$\lim_{\lambda\to-\frac{n}{2}} \frac{d}{d\lambda}\left[(\lambda+\frac{n}{2})P_+^\lambda\right] = pf_{\lambda=-\frac{n}{2}}\left\{P_+^\lambda\right\} \tag{71}$$

we have

$$L^k\left\{P_+^{k-\frac{n}{2}}\right\} = \frac{(-1)^k 2^{2k}(k-1)!\Gamma(\frac{n}{2})}{\Gamma(\frac{n}{2}-k)} \operatorname{Re}s_{\lambda=-\frac{n}{2}} P_+^\lambda. \tag{72}$$

From(72) and using(53) and (55) we obtain

$$L^k\left\{P_+^{k-\frac{n}{2}}\right\} = \frac{(-1)^k 2^{2k}(k-1)!\Gamma(\frac{n}{2})}{\Gamma(\frac{n}{2}-k)}.$$

$$\left\{\frac{(-1)^{\frac{n}{2}-1}}{\Gamma(\frac{n}{2})}\delta_1^{(\frac{n}{2}-1)}(P) + \frac{(-1)^{\frac{q}{2}}\pi^{\frac{n}{2}}}{\Gamma(\frac{n}{2})}\delta\right\} =$$

$$= \frac{(-1)^k 2^{2k}(k-1)!\Gamma(\frac{n}{2})}{\Gamma(\frac{n}{2}-k)}.\left\{\frac{(-1)^{\frac{n}{2}-1}}{\Gamma(\frac{n}{2})}\left[(-2)(-1)^{\frac{n}{2}-1}(-1)^{\frac{q}{2}}\pi^{\frac{n}{2}}\delta\right] + \frac{(-1)^{\frac{q}{2}}\pi^{\frac{n}{2}}}{\Gamma(\frac{n}{2})}\delta\right\} =$$

$$= -\frac{(-1)^k 2^{2k}(k-1)!(-1)^{\frac{q}{2}}\pi^{\frac{n}{2}}}{\Gamma(\frac{n}{2}-k)}\delta. \tag{73}$$

From(73) we obtain the formula(66). □

$$p \text{ and } q \text{ are both odd} \tag{74}$$

In this case from (see [1], p. 263), the coefficients $c_{-2}^{(s)}$ is given by the formula

$$c_{-2}^{(s)} = \lim_{\lambda\to-\frac{n}{2}-s}\left[(\lambda+\frac{n}{2})^2 P_+^\lambda\right] = \frac{(-1)^{\frac{q-1}{2}}\pi^{\frac{n}{2}-1}}{2^{2s}s!\Gamma(\frac{n}{2}+s)} L^s\delta, s = 0, 1, 2, ... \tag{75}$$

Lemma 4. *Let n be even dimension of the space and k positive integers such that $k < \frac{n}{2}$,then exists a constants $e_{n,p,q,k}$ such that the following formula is valid*

$$L^k\left\{e_{n,p,q,k}P_+^{k-\frac{n}{2}}\right\} = \delta(x) \tag{76}$$

if p and q are both odd. Where $P_+^{k-\frac{n}{2}}$ we mean finite part in the sense of definition(7) and $e_{n,p,q,k}$ is defined by

$$e_{n,p,q,k} = \frac{2^{2k}(k-1)!(-1)^k(-1)^{\frac{q-1}{2}}\pi^{\frac{n}{2}-1}}{\Gamma(\frac{n}{2}-k)}$$

$$\times \left[\psi(k) + \psi(\frac{n}{2} - k) - \gamma - \psi(\frac{p}{2}) - 2\psi(\frac{n}{2})\right] \quad (77)$$

and γ is the Euler´s constant.

Proof. From (69) and considering (72) and (73) we have,

$$L^k \left\{P_+^{k-\frac{n}{2}}\right\} = \frac{(-1)^k 2^{2k}(k-1)!\Gamma(\frac{n}{2})}{\Gamma(\frac{n}{2}-k)} \operatorname{Re} s_{\lambda=-\frac{n}{2}} P_+^\lambda$$

$$+ 0.\frac{2^{2k}(k-1)!\Gamma(\frac{n}{2})}{\Gamma(\frac{n}{2}-k)} Pf_{\lambda=-\frac{n}{2}} P_+^\lambda$$

$$+ c_{-2}^{(0)}.2^{2k}(-1)^k \lim_{\lambda\to-\frac{n}{2}} \frac{d}{d\lambda}\left[(\lambda+\frac{n}{2}+1)...(\lambda+\frac{n}{2}+k-1)\frac{\Gamma(-\lambda)}{\Gamma(-\lambda-k)}\right]. \quad (78)$$

On the other hand,

$$\frac{d}{d\lambda}\left[(\lambda+\frac{n}{2}+1)...(\lambda+\frac{n}{2}+k-1)\frac{\Gamma(-\lambda)}{\Gamma(-\lambda-k)}\right]$$
$$= \frac{d}{d\lambda}\left[\frac{\Gamma(-\lambda)}{\Gamma(-\lambda-k)}\frac{\Gamma(\lambda+\frac{n}{2}+k)}{\Gamma(\lambda+\frac{n}{2}+1)}\right]$$
$$= \frac{\Gamma(\lambda+\frac{n}{2}+1)\Gamma(-\lambda-k)}{(\Gamma(\lambda+\frac{n}{2}+1))^2(\Gamma(-\lambda-k))^2}$$
$$\times\left[\Gamma(\lambda+\frac{n}{2}+k)\Gamma'(-\lambda)(-1)+\Gamma(-\lambda)\Gamma'(\lambda+\frac{n}{2}+k)\right]$$
$$-\frac{\Gamma(\lambda+\frac{n}{2}+k)\Gamma(-\lambda)}{(\Gamma(\lambda+\frac{n}{2}+1))^2(\Gamma(-\lambda-k))^2}$$
$$times\left[\Gamma(\lambda+\frac{n}{2}+1)\Gamma'(-\lambda-k)(-1)+\Gamma(-\lambda-k)\Gamma'(\lambda+\frac{n}{2}+1)\right]. \quad (79)$$

Therefore

$$\lim_{\lambda\to-\frac{n}{2}}\frac{d}{d\lambda}\left[(\lambda+\frac{n}{2}+1)...(\lambda+\frac{n}{2}+k-1)\frac{\Gamma(-\lambda)}{\Gamma(-\lambda-k)}\right]$$
$$= \frac{1}{\Gamma(1)(\Gamma(\frac{n}{2}-k)}\left[-\Gamma(k)\Gamma'(\frac{n}{2})+\Gamma(\frac{n}{2})\Gamma'(k)\right]$$
$$-\frac{\Gamma(k)\Gamma(\frac{n}{2})}{(\Gamma(1))^2(\Gamma(\frac{n}{2}-k))^2}\left[-\Gamma(1)\Gamma'(\frac{n}{2}-k)+\Gamma(\frac{n}{2}-k)\Gamma'(1)\right]. \quad (80)$$

From (78), using (71), (13) and (80) we have

$$L^k\left\{P_+^{k-\frac{n}{2}}\right\} = \frac{(-1)^k 2^{2k}(k-1)!\Gamma(\frac{n}{2})}{\Gamma(\frac{n}{2}-k)} \operatorname{Re}s_{\lambda=-\frac{n}{2}} P_+^{\lambda}$$

$$+\frac{(-1)^k(-1)^{\frac{q-1}{2}}2^{2k}\pi^{\frac{n}{2}-1}}{\Gamma(\frac{n}{2})}.\Gamma(k)\frac{\Gamma(\frac{n}{2})}{\Gamma(\frac{n}{2}-k)}$$

$$\times\left[\psi(k)-\psi(\frac{n}{2})+\psi(\frac{n}{2}-k)-\psi'(1)\right]\delta = \frac{(-1)^k 2^{2k}(k-1)!\Gamma(\frac{n}{2})}{\Gamma(\frac{n}{2}-k)}$$

$$\times\left\{\frac{(-1)^{\frac{n}{2}-1}}{\Gamma(\frac{n}{2})}\delta_1^{(\frac{n}{2}-1)}+\frac{(-1)^{\frac{q+1}{2}}\pi^{\frac{n}{2}-1}}{\Gamma(\frac{n}{2})}\left[\psi(\frac{p}{2})-\psi(\frac{n}{2})\right]\delta\right\}$$

$$\times\frac{(-1)^k 2^{2k}(-1)^{\frac{q-1}{2}}\pi^{\frac{n}{2}-1}\Gamma(k)}{\Gamma(\frac{n}{2}-k)}\left[\psi(k)-\psi(\frac{n}{2})+\psi(\frac{n}{2}-k)-\psi'(1)-2\psi(\frac{n}{2})\right]\delta. \tag{81}$$

Therefore from (81) and using (56) we have

$$L^k\left\{P_+^{k-\frac{n}{2}}\right\} = \frac{(-1)^k 2^{2k}(k-1)!\Gamma(\frac{n}{2})}{\Gamma(\frac{n}{2}-k)}$$

$$\left\{\frac{(-1)^{\frac{n}{2}-1}}{\Gamma(\frac{n}{2})}.2(-1)^{\frac{n}{2}}(-1)^{\frac{q+1}{2}}\pi^{\frac{n}{2}-1}\left[\psi(\frac{p}{2})-\psi(\frac{n}{2})\right]\delta\right.$$

$$+\frac{(-1)^{\frac{q+1}{2}}\pi^{\frac{n}{2}-1}}{\Gamma(\frac{n}{2})}\left[\psi(\frac{p}{2})-\psi(\frac{n}{2})\right]\delta$$

$$\left.+\frac{(-1)^{\frac{q-1}{2}}\pi^{\frac{n}{2}-1}}{\Gamma(\frac{n}{2})}\left[\psi(k)-\psi(\frac{n}{2})+\psi(\frac{n}{2}-k)-\psi'(1)\right]\delta\right\}$$

$$=\frac{2^{2k}(k-1)!\Gamma(\frac{n}{2})(-1)^{\frac{q-1}{2}}\pi^{\frac{n}{2}-1}}{\Gamma(\frac{n}{2}-k)}\left[\psi(k)+\psi(\frac{n}{2}-k)-\gamma-\psi(\frac{p}{2})-2\psi(\frac{n}{2})\right]\delta. \tag{82}$$

where $\psi(x)$ is defined by(13),(15) and (16) and $\Gamma'(1)=\gamma$ Euler´s constant (see [11], p. 361). From(82) we obtain the formulae(76) and (77). □

From(34),(46),(47),(66) and (76) we conclude for all n, p, q and k that there is constants $H_{k,n,p,q}$ and $T_{k,n,p,q}$ such that

$$L^k\left\{H_{k,n,p,q}P_+^{k-\frac{n}{2}}\log P_+ + T_{k,n,p,q}P_+^{k-\frac{n}{2}}\right\}=\delta(x) \tag{83}$$

where

$$H_{k,n,p,q}=0 \tag{84}$$

if n is odd,

$$T_{k,n,p,q} = 0 \tag{85}$$

if n is even and $k \geq \frac{n}{2}$,

$$H_{k,n,p,q} = 0 \tag{86}$$

if n is even and $k < \frac{n}{2}$,

$$H_{k,n,p,q} = \begin{cases} D_{n,p,q,k} \text{ if } p \text{ and } q \text{ are both even and } k \geq \frac{n}{2} \text{ and} \\ \\ \overset{-}{D}_{n,p,q,k} \text{ if } p \text{ and } q \text{ are both odd and } k \geq \frac{n}{2}, \end{cases} \tag{87}$$

$$T_{k,n,p,q} = \begin{cases} c_{n,p,q,k} \text{ if } n \text{ is odd,} \\ \\ d_{n,p,q,k} \text{ if } p \text{ and } q \text{ are both even and } k < \frac{n}{2} \text{ and} \\ \\ e_{n,p,q,k} \text{ if } p \text{ and } q \text{ are both odd and } k < \frac{n}{2}. \end{cases} \tag{88}$$

In (87) and (88) $D_{n,p,q,k}, \overset{-}{D}_{n,p,q,k}, c_{n,p,q,k}, d_{n,p,q,k}$ and $e_{n,p,q,k}$ are defined by(48), (49),(34),(67) and (75) respectively.

From(83) follows that

$$N = P_+^{k-\frac{n}{2}} \left\{ H_{k,n,p,q} \log P_+ + T_{k,n,p,q} \right\} \tag{89}$$

is an elemental solution of the n-dimensional ultrahyperbolic operator iterated k times defined by(22).

The formula(83) is a generalization of the formula

$$\Delta^k \left\{ r^{2k-n}(A_{k,n} \log r + B_{k,n}) \right\} = \delta \tag{90}$$

which appear in (see [10], p. 47, formula (II,3,19)) where $A_{k,n}$ is defined by (60) and $B_{k,n}$ by (43).

3. The Distributional Product between pfP^{-j} and $\nabla \left(L^k P_+^{k-\frac{n}{2}} \right)$

In this paragraph we will a sense to distributional product of $pf \left\{ P^{-j} \right\}$ and $\nabla \left(L^k P_+^{k-\frac{n}{2}} \right)$ .In order to do it we need the following results

$$pf \left\{ P^{-j} \right\} . \nabla L^s \left\{ \delta \right\} = A_{s,j,n} \left(\sum_{i=1}^{p} x_i - \sum_{i=p+1}^{p+q} x_i \right) L^{j+s+1} \left\{ \delta \right\} \tag{91}$$

for odd n, as well as for even n if $j < \frac{n}{2}$ (see [3]) and

$$pf\left\{P^{-j}\right\}.\nabla L^{s}\left\{\delta\right\} = A_{s,j,n}\left(\sum_{i=1}^{p} x_i - \sum_{i=p+1}^{p+q} x_i\right) L^{j+s+1}\left\{\delta\right\} \tag{92}$$

under condition $j \geq \frac{n}{2}$ for n even (see [3]). Where $pf\left\{P^{-j}\right\}$ we mean finite part of P_{+}^{λ} in he sense of definition(7) and

$$A_{s,j,n} = -\frac{1}{2}\frac{1}{2^k}\frac{1}{(s+1)...(s+j+1)(n+2(s+1))...(n+2(s+j))}. \tag{93}$$

Theorem 5. *Let k and j be nonnegative integers, n odd dimension of the space and ∇ operator defined by(25) then the following formula is valid*

$$pf\left\{P^{-j}\right\}.\nabla\left(L^{k}P_{+}^{k-\frac{n}{2}}\right) = E_{k,n,q}\left(\sum_{i=1}^{p} x_i - \sum_{i=p+1}^{p+q} x_i\right) L^{j+1}\left\{\delta\right\}. \tag{94}$$

Where L^k is the operator defined by(22) and

$$E_{k,n,p,q} = -\frac{2^{2k}(k-1)!(-1)^{\frac{q}{2}}\pi^{\frac{n}{2}}(-1)^{k}}{\Gamma(\frac{n}{2}-k)}.\frac{1}{2^{j+1}(j+1)!(n+2)...(n+2j)}. \tag{95}$$

Proof. From(34) and using(65) we have,

$$\begin{aligned} pf\left\{P^{-j}\right\}.\nabla\left(L^{k}P_{+}^{k-\frac{n}{2}}\right) &= \left(pf\left\{P^{-j}\right\}\right).\nabla\left(c_{n,p,q,k}\,\delta(x)\right) = \\ &= c_{n,p,q,k}\; pf\left\{P^{-j}\right\}.\nabla\delta(x) = \\ &= c_{n,p,q,k}\; A_{s,j,n}\left(\sum_{i=1}^{p} x_i - \sum_{i=p+1}^{p+q} x_i\right) L^{j+1}\left\{\delta\right\}. \end{aligned} \tag{96}$$

if n is odd. Where $c_{n,p,q,k}$ is defined by(35) and $A_{s,j,n}$ is given by(93). In particular putting $s = 0$ in(93) we have

$$A_{o,j,n} = .-\frac{1}{2}\frac{1}{2^j}\frac{1}{1.2...(j+1)(n+2)...(n+2j))}. \tag{97}$$

From(35) and (97) we obtain $E_{k,n,p,q}$. From (96) we obtain the formula (94). □

The formula(94) generalized the product

$$r^{-2k}.\nabla(\Delta r^{2-m}) = \frac{(m-2)\pi^{\frac{m}{2}}}{2^k(k+1)!(m+2)...(m+2k)\Gamma(\frac{m}{2})}\sum_{i=1}^{m}\left(x_i\Delta^{k+1}\delta\right) \tag{98}$$

given by C.K. Li (c.f. [2], p. 346, formula (9)), where m is the dimension of the space. In fact, letting $k=1$ and $q=0$ in (94) and (95) we have

$$\begin{aligned} r^{-2j}.\nabla(\Delta r^{2-n}) &= \frac{2^2\pi^{\frac{n}{2}}}{\Gamma(\frac{n}{2}-1)}\cdot\frac{1}{2^{j+1}(j+1)!(n+2)...(n+2j)}\left(\sum_{i=1}^{n}x_i\right)\Delta^{j+1}\delta = \\ &= \frac{(n-2)\pi^{\frac{n}{2}}}{\Gamma(\frac{n}{2})}\cdot\frac{1}{2^j(j+1)!(n+2)...(n+2j)}\left(\sum_{i=1}^{n}x_i\right)\Delta^{j+1}\delta. \end{aligned} \tag{99}$$

It is clear that the formula(99) is the same that the formula(98).

Theorem 6. *Let k and j be nonnegative integers, n even dimension of the space and ∇ operator defined by(25) then the following formula is valid*

$$pf\left\{P^{-j}\right\}.\nabla\left(L^k P_+^{k-\frac{n}{2}}\right) = -E_{k,n,q,j}\left(\sum_{i=1}^{p}x_i - \sum_{i=p+1}^{p+q}x_i\right)L^{j+1}\{\delta\}. \tag{100}$$

if p and q are both even and $j < \frac{n}{2}$.Where $p+q=n$ is the dimension of the space,L^k is defined by(22) and $E_{k,n,q}$ by(95).

Proof. The proof of formula(100) is consequence of the formulae(66) and(91). □

Theorem 7. *Let k and j be nonnegative integers, n even dimension of the space and ∇ operator defined by(25) then the following formula is valid*

$$pf\left\{P^{-j}\right\}.\nabla\left(L^k P_+^{k-\frac{n}{2}}\right) = F_{k,n,q}\left(\sum_{i=1}^{p}x_i - \sum_{i=p+1}^{p+q}x_i\right)L^{j+1}\{\delta\}. \tag{101}$$

if p and q are both odd and $j \geq \frac{n}{2}$.Where $p+q=n$ is the dimension of the space,L^k is defined by(22),

$$\begin{aligned} F_{k,n,q} &= \frac{(-1)^k(-1)^{\frac{q-1}{2}}\pi^{\frac{n}{2}-1}}{\Gamma(\frac{n}{2}-k)}\left[\psi(k)+\psi(\tfrac{n}{2}-k)-\gamma-\psi(\tfrac{p}{2})\right]. \\ &\cdot\left[-\frac{1}{2}\frac{1}{2^j}\frac{1}{(j+1!)(n+2)...(n+2j))}\right] \end{aligned} \tag{102}$$

and $\psi(x)$ is defined by(13),(15) and (16).

Proof. From(76) and using(92) we have

$$pf\left\{P^{-j}\right\}.\nabla\left(L^k P_+^{k-\frac{n}{2}}\right) = \left(pf\left\{P^{-j}\right\}\right).\nabla\left(e_{n,p,q,k}\ \delta(x)\right)$$
$$= e_{n,p,q,k}\ pf\left\{P^{-j}\right\}.\nabla\left(\delta(x)\right)$$
$$= e_{n,p,q,k}\ A_{o,j,n}.\left(\sum_{i=1}^{p} x_i - \sum_{i=p+1}^{p+q} x_i\right) L^{j+1}\left\{\delta\right\}. \quad (103)$$

From(77) and (97) we obtain the formula(102).

From(103) we obtain the formula(101). □

The formula(100) for the case n even, generalize the product(98), given by C.K. Li (c.f. [2], p. 346, formula (9)). In fact, letting $k = 1$ and $q = 0$ in(100) and (95) and considering(27) we have

$$r^{-2j}.\nabla(\Delta r^{2-n}) = -E_{1,n,o,j}\left(\sum_{i=1}^{n} x_i\right)\Delta^{j+1}\delta = E_{1,n,o,j}\left(\sum_{i=1}^{n} x_i\right)\Delta^{j+1}\delta$$
$$\frac{2^2\pi^{\frac{n}{2}}}{\Gamma(\frac{n}{2}-1)}.\frac{1}{2^{j+1}(j+1)!(n+2)...(n+2j)}\left(\sum_{i=1}^{n} x_i\right)\Delta^{j+1}\delta$$
$$= \frac{(n-2)\pi^{\frac{n}{2}}}{\Gamma(\frac{n}{2})}.\frac{1}{2^{j}(j+1)!(n+2)...(n+2j)}\left(\sum_{i=1}^{n} x_i\right)\Delta^{j+1}\delta, \quad (104)$$

if n is even, where $p + q = n$ is the dimension of the space.The formula(104) coincide with the formula (98).

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