International Journal of Pure and Applied Mathematics
Volume 72 No. 3 2011, 319-329

A COMPARATIVE ANALYSIS OF SOME ROBUST BOUNDARY INTEGRAL EQUATION METHODS FOR ACOUSTIC SCATTERING

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Abstract: A review of integral equation formulation of the exterior acoustic problem with the hard boundary condition is presented. For comparison, three particular methods for the numerical determination of boundary values are considered, viz. Burton and Miller method [1], Schenck method [8] and the Modified Layer Formulation Method [5] (MLF). The Burton and Miller basically incorporates the results of Terai [9] and combines the surface Helmholtz equation with its normal derivative form. Both Burton and Miller, and Schenck fail for structure surfaces that are plate-like, nevertheless, the Burton and Miller approach is suitable for thinner plates. Schenck approach faces problems with the selection of interior collocation points at higher frequencies, however, Burton and Miller faces no such problems provided the surface mesh is sufficiently fine. Unlike the other two, the MLF method utilizes layer potentials. This method circumvents the hyper-singularity problem and the non-uniqueness problem at a much reduced cost.

AMS Subject Classification: 35J05, 45B05
Key Words: Helmholtz equation, collocation method, boundary integral equation

Received: March 12, 2011

1. Introduction

It has been well established that analytical solutions for exterior acoustic problem can only be found for structure surfaces of simple shapes. Also, the analytical solutions are quite often given in an infinite series form with slow convergence and the problem becomes compounded when the size of the structure surface is comparable to the wavelength of the incident field. Hence, the only recourse is to use numerical methods. However, the boundary integral equation representation of the formulation may present weaknesses, despite the advantage over FEM or FDM. The main difficulty is the apparent conflict between the interior and exterior solution, particularly the non-uniqueness of the exterior solution for wave- numbers which coincide with the eigen-frequencies of the corresponding interior problem. Nevertheless, several formulations have been proposed (see [1], [5], [8]) to remedy this difficulty.

Schenck [8] combines the surface Helmholtz equation with the interior Helmholtz relation which is collocated at a finite number of well selected and well suited interior points. However, Jones [7] points out that the job of selection of these interior points is a cumbersome one, as there is no criteria for how and where best to choose these.

Burton and Miller [1] superposes the surface Helmholtz equation with its normal derivative form to obtain a linear coupling and the best suited value of the coupling parameter is determined. This ensures uniqueness of solution for all wave-numbers. Nevertheless, the boundary integral equation gives rise to a hyper-singular kernel. However, a regularization technique gets round this problem costing a huge computer time. Also, a result, due to Terai [9], enables the numerical evaluation of a singular integral equation of that type.

Modified Layer Formulation method [5] utilizes layer potentials, by a hybrid coupling of a simple-layer and a double-layer potential. Here too the optimum value of the coupling parameter is determined. The resulting boundary integral equation does not present any hyper-singularity as the dipoles are located in the interior on a fictitious surface. Moreover, the method ensures the uniqueness of solution for all wave-numbers.

The present work is motivated by a similar line of work done by Reut [18]. Recently, Wu She-Wu and Wu She-Wen [14], Krutitskii [15], Qian et al [16] have presented some boundary integral formulation for acoustic problems. Krutitskii utilizes an additional boundary with an appropriate boundary condition inside the scatterer. The approach is similar to [5]. A. Mohsen and M. Hesham [17] have presented an efficient method for solving the uniqueness problem in acoustic scattering. They have utilized basically the Schenck [8] approach which is termed as CHIEF (Combined Helmholtz Integral Equation Formulation).

2. Formulation via Integral Method

Consider the time harmonic propagation of small amplitude acoustic waves in an ideal three-dimensional homogeneous medium exterior to a smooth (Liapunov), bounded, closed surface ∂B. Let $\varpi_{in}(\widetilde{x})$ denote the pressure due to a given incident acoustic wave, and $\varpi_{sc}(\widetilde{x})$ the pressure due to the scattered acoustic wave produced by ∂B. The time harmonic acoustic propagation is given by $e^{i\omega t}$. The total acoustic pressure is given by

$$\varpi(\widetilde{x}) = \varpi_{in}(\widetilde{x}) + \varpi_{sc}(\widetilde{x}) \tag{2.1}$$

which satisfies the Helmholtz integral formula and its differentiated form (see [3], [9]). We assume a hard boundary condition, i.e.

$$\frac{\partial \varpi}{\partial(\widetilde{n})}(\widetilde{x}) = 0, \ \widetilde{x} \in \partial B, \tag{2.2}$$

where $\widetilde{n}$ is the outward normal unit vector. Under this assumption, Helmholtz integral formula and its differentiated forms are as given below respectively by eqns. (2.3) and (2.4), are

$$\varpi_{in}(\widetilde{x}) + \int_{\partial B} \varpi(\widetilde{y}) \frac{\partial G}{\partial n_y}(\widetilde{x}, \widetilde{y}) dy = \varepsilon(\widetilde{x}) \varpi(\widetilde{x}), \tag{2.3}$$

where

$$\varepsilon(\widetilde{x}) = \begin{cases} 4\pi, & \text{if } \widetilde{x} \in B^+ \\ 2\pi, & \text{if } \widetilde{x} \in \partial B \\ 0, & \text{if } \widetilde{x} \in B^- \end{cases}$$

and

$$\frac{\partial \varpi_{in}}{\partial(\widetilde{n})}(\widetilde{x}) + \int_{\partial B} \varpi(\widetilde{y}) \frac{\partial^2 G}{\partial \tilde{n}\, \partial \tilde{n}_y}(\widetilde{x}, \widetilde{y}) dy = \hat{\varepsilon}(\widetilde{x}) \frac{\partial \varpi}{\partial \widetilde{n}}(\widetilde{x}), \tag{2.4}$$

where

$$\hat{\varepsilon}(\widetilde{x}) = \begin{cases} 4\pi, & \text{if } \widetilde{x} \in B^+ \\ 0, & \text{if } \widetilde{x} \in \partial B \cup B^- \end{cases}.$$

Here, B^+ represents the exterior, and B^- the interior of the surface ∂B. The unit normal $\widetilde{n}_y$ at $\widetilde{y}$ on ∂B points outward, i.e. from ∂B into B^+.

Utilizing layer potentials (indirect method), the solution is formulated as follows,

$$\begin{aligned} \varpi(\widetilde{x}) &= \int_{\partial B} G_k(\widetilde{x},\widetilde{y})v(\widetilde{y})dy + \mu\int_{\partial B}\frac{\partial G_k}{\partial \widetilde{n}_{y^*}}(\widetilde{x},\widetilde{y}^*)dy; \qquad (2.5)\\ &\widetilde{x}\in B^+\cup\partial B,\ \ \widetilde{y}\in\partial B,\ \widetilde{y}^*\in\partial B^*. \end{aligned}$$

The coupling parameter $\mu = +i$, as suggested by the uniqueness of the solution for all k [12]. The boundary equation takes the following form

$$\begin{aligned} \frac{\partial\varpi}{\partial\widetilde{n}_x} &= -2\pi\mu(\widetilde{x}) + \int_{\partial B}\frac{\partial G_k}{\partial\widetilde{n}_x(\widetilde{x},\widetilde{y})} + \mu\frac{\partial^2 G_k}{\partial\widetilde{n}_x\partial\widetilde{n}_{y^*}}(\widetilde{x},\widetilde{y}^*)v(\widetilde{y})dy; \qquad (2.6)\\ &\widetilde{x}\in\partial B, \end{aligned}$$

where $v(\widetilde{x})$ is the unknown source density function defined on the surface and the free space Green's function is

$$G(\widetilde{x},\widetilde{y}) = \frac{e^{-ik|\widetilde{x}-\widetilde{y}|}}{|\widetilde{x}-\widetilde{y}|}, \qquad (2.7)$$

where k is the wave-number $2\pi\lambda^{-1}$ and λ is the wavelength.

We now introduce a mesh subdividing the surface ∂B into m (surface) boundary elements ∂B_i $(1 \le i \le m)$. The boundary elements are all of regular geometrical shapes. We assume the pressure $\varpi(\widetilde{x})$ is constant and equal to ϖ_s on the boundary element ∂B_s $(1 \le s \le m)$. Hence the total acoustic pressure may be approximated by

$$\varpi(\widetilde{x}) = \sum_{s=1}^{m}\varpi_s\zeta_s(\widetilde{x}),\ \widetilde{x}\in\partial B, \qquad (2.8)$$

where

$$\zeta_s = \begin{cases} 1, & \widetilde{x}\in\partial B_s \\ 0, & \widetilde{x}\notin\partial B_s \end{cases}. \qquad (2.9)$$

3. Determination of Boundary Values

We assume that $\varpi_{in}(\widetilde{x})$ is a given function of position $\widetilde{x}$. Provided $\widetilde{x}$ and $\frac{\partial\varpi}{\partial n}(\widetilde{x})$ are known on the boundary ∂B, any exterior problem can be solved by using the Helmholtz formula. In the case of the hard boundary condition, it is sufficient to know $\varpi(\widetilde{x})$ on ∂B only. Nevertheless, determining boundary values is generally a difficult task firstly due to the non-uniqueness of solutions and secondly due to the singularity of kernels. Henceforth, the boundary equation in eqn. (2.3) will be termed as the Surface Helmholtz Equation (SHE), that in eqn. (2.4) will be termed as the Normal Derivative form of Surface Helmholtz equation (NDSHE) and the interior equation in (2.3) will be termed as the Interior Helmholtz Relation (IHR). It is already known (see Burton [3]) that SHE has no unique solution for wave-numbers coinciding with the characteristic wave-numbers (eigenfrequencies) of the interior Dirichlet problem, and that NDSHE has no unique solution for wave-numbers coinciding with the eigenfrequencies of the interior Neumann problem. Note that the kernel of eqn. (2.3) is weakly singular, whereas that of the eqn. (2.4) is highly singular for $\widetilde{x} \in \partial B$. Assuming a hard boundary condition, we present here three numerical schemes for determining the boundary values.

Scheme A. (see [8]) We use the centroids of m boundary elements to collocate SHE at these m boundary points and collocate IHR at M interior points. We thus obtain a system of linear equations,

$$\sum_{j=1}^{m} H_{ij}\varpi_j = \langle \varpi_{in} \rangle_i, \tag{3.1}$$

where $i = 1, 2, 3, \ldots, m + M$, and the matrix elements are given by

$$H_{ij} = 2\pi\delta_{ij} - \int_{\partial B_j} \frac{\partial G_k}{\partial \widetilde{n}_y}(\widetilde{x}_i, \widetilde{y})dy \tag{3.2}$$

for

$$\widetilde{x} \in \partial B_i \ (i = 1, 2, 3, \ldots, m), \widetilde{x}_i \in B^- \ (i = m+1, m+2, m+3, \ldots, m+M),$$

$$\delta_{ij} = 1 \text{ for all } i = 1, 2, 3, \ldots, m \text{ and } \delta_{ij} = 0 \text{ if i} \neq \text{j}.$$

The system (3.1) is an over determined system of linear equations. Nevertheless, we solve the system by the least squares method yielding the total acoustic pressure at the collocation points. However, the problem occurs whenever the collocation point $\widetilde{x}$ lies within the region of integration ∂B_j (i.e. for $i = j$), a singular integral emerges. Nevertheless, these are weak singularities and can be dealt with numerically using standard approach, see [2]. It has been shown by Jones [7] that for a flat rectangular boundary element ∂B_j, the principal value of the jth singular integral vanishes, i.e.

$$\int_{\partial B_j} \frac{\partial G_k}{\partial \widetilde{n}_y}(\widetilde{x}_i, \widetilde{y})dy = 0 \tag{3.3}$$

and the corresponding diagonal element $H_{ij} = 2\pi$.

Scheme B. For any given problem, the integral equations in (2.3) and (2.4) (viz. SHE and NDSHE) have only one common solution even when k coincides with the eigenfrequencies [1] of the corresponding Dirichlet and Neumann problems. We form the composite Helmholtz equation by coupling eqns. (2.3) and (2.4)

$$2\pi\varpi(\widetilde{x}) - \int_{\partial B} \varpi(\widetilde{y}) \left[\frac{\partial G_k}{\partial \widetilde{n}_y}(\widetilde{x},\widetilde{y}) + \alpha(k)\frac{\partial^2 G_k}{\partial \widetilde{n}\partial \widetilde{n}_y}(\widetilde{x},\widetilde{y})\right] dy$$
$$= \varpi_{in}(\widetilde{x}) + \alpha(k)\frac{\partial \varpi_{in}}{\partial \widetilde{n}}(\widetilde{x}), \quad (3.4)$$

where $\alpha(k)$, the coupling parameter, is a complex coefficient which may depend on the wave-number k only. Burton (see [2], [3]) has proved that the solution of (3.4) is unique when $Imaginary(\alpha(k)) \neq 0$ for real k. The collocation of eqn. (3.4) at m boundary points (taken at the centroids of m boundary elements) leads to a system of linear equations

$$\sum_{j=1}^{m} H_{ij}\varpi_j = \left\langle \varpi_{in} + \alpha(k)\frac{\partial \varpi_{in}}{\partial \widetilde{n}}\right\rangle_i ; \; i = 1, 2, 3, \ldots, m. \quad (3.5)$$

The matrix elements are given by

$$H_{ij} = 2\pi\delta_{ij} - \int_{\partial B_j} \left(\frac{\partial G_k}{\partial \widetilde{n}_y}(\widetilde{x},\widetilde{y}) + \alpha(k)\frac{\partial^2 G_k}{\partial n_i \partial \widetilde{n}_y}(\widetilde{x},\widetilde{y})\right) dy \quad (3.6)$$

for $\widetilde{x}_i \in \partial B_i,\ i, j = 1, 2, 3, \ldots, m$. The linear system (3.5) is solved algebraically giving the total acoustic pressure at boundary collocation points. However, the problem occurs whenever the collocation point $\widetilde{x}_i$ lies within the region of integration ∂B_j. In this case a highly singular integral emerges, which may be circumvented by the method of regularization. However, in three-dimensional cases, numerical procedures become very tedious. It has been found (see [2], [9]) that for a flat rectangular boundary element ∂B_i,

$$\int_{\partial B_i} \frac{\partial^2 G_k}{\partial \widetilde{n}_i \partial \widetilde{n}_y}(\widetilde{x}_i,\widetilde{y})dy = -2\pi ik - \int_0^{2\pi} \frac{e^{ik\rho_i(\theta_i)}}{\rho_i(\theta_i)} d\theta_i, \quad (3.7)$$

where $\rho_i = \rho_i(\theta_i)$ is the polar equation of the contour Γ_i of the boundary element ∂B_i with respect to its centroid as an origin. Using (3.3), (3.6) and (3.7), the diagonal elements of the matrix take the form,

$$H_{ii} = 2\pi + \alpha(k)\left(2\pi ik + \int_0^{2\pi} \frac{e^{ik\rho_i(\theta_i)}}{\rho_i(\theta_i)} d\theta_i\right). \quad (3.8)$$

Denoting $\overline{\rho}$ as the mean value of ρ_i (averaged over the polar angle θ_i and then over the subscript i), the quantity $k\overline{\rho}$ can be used as a measure of the boundary mesh fineness. It is found that the best value of the coupling coefficient is $\alpha(k) = (-i)\gamma(k)$, where $\gamma(k) = o(k^{-1})$.

Scheme C. This utilizes the Ansatz or the indirect approach [5] using the modified layer- potentials. The boundary equation takes the following form:

$$\frac{\partial \varpi}{\partial \widetilde{x}_x} = -2\pi\sigma(\widetilde{x}) + \int_{\partial B} \left(\frac{\partial G_k}{\partial \widetilde{n}_x}(\widetilde{x}, \widetilde{y}) + \eta \frac{\partial^2 K_k}{\partial \widetilde{n}_x \partial \widetilde{n}_{y^*}}(\widetilde{x}, \widetilde{y}^*) \right) \sigma(\widetilde{y}) dy, \quad \widetilde{x} \in \partial B, \quad (3.9)$$

where $q^* \in B^*$, a fictitious surface similar and similarly situated as ∂B, such that $q^* = \xi q$, where $0 < \xi \leq 1$. Clearly, $\xi = 1$ corresponds to the given surface ∂B and as $\xi \to 0$, the similar surface ∂B^* collapses into a point i.e. the origin of the co-ordinate system and consequently, has no relevance to the present discussion. It has been found [13] that the optimal value of the parameter $\xi \approx 0.5$ produced the best results for the test problems. The coupling parameter $\eta = i$.

A mesh is introduced which divides the spherical surface into m axisymmetric boundary elements ∂B $(1 \leq j \leq m)$. The collocation scheme used in this paper is based on a piecewise constant approximation to σ, with $\zeta_j(\widetilde{x}) : \widetilde{x} \in \partial B$ as the shape functions, as defined in [9]. The boundary function $f(\widetilde{x})$ and the source density function $\sigma(\widetilde{x})$ are both approximated as follows:

$$\sigma(\widetilde{x}) = \sum_{j=1}^{m} \sigma_j \zeta_j(\widetilde{x}); \ \widetilde{x} \in \partial B,$$

$$f(\widetilde{x}) = \frac{\partial \varpi}{\partial \widetilde{n}}(\widetilde{x}) \sum_{j=1}^{m} \left(\frac{\partial \varpi_j}{\partial \widetilde{n}} \zeta_j(\widetilde{x}) \right); \ \widetilde{x} \in \partial B,$$

where

$$\frac{\partial \varpi_j}{\partial \widetilde{n}} := \frac{\partial \varpi}{\partial \widetilde{n}}(\widetilde{x}_j); \ 1 \leq j \leq m.$$

Collocation of boundary equation (3.9) at $\widetilde{x}_j \in \partial B_j$ $(1 \leq j \leq m)$, gives a system of m linear equations

$$\sum_{j=1}^{m} A_{ij}\sigma_j = \left\langle \frac{\partial \varpi_i}{\partial \widetilde{n}} \right\rangle; \ 1 \leq i \leq m, \quad (3.10)$$

where

$$A_{ij} = -2\pi\delta_{ij} + \int_{\partial B_{\mathrm{j}}} \left(\frac{\partial G_k}{\partial \widetilde{n}_j}(\widetilde{x}, \widetilde{y}) + \eta \frac{\partial^2 G_k}{\partial \widetilde{n}_i \partial \widetilde{n}_{y^*}}(\widetilde{x}, \widetilde{y}^*) \right) dy, \quad 1 \leq i, j \leq m.$$

The system (3.10) is solved by Gaussian Elimination process.

For the treatment of singularity at $i = j$, the method of Jaswon & Symm [6] is used. For spherical surface

$$\int_{\partial B_{\mathrm{j}}} \frac{\partial G_k}{\partial \widetilde{n}_x}(\widetilde{x}_j, \widetilde{y}) dy = \int_{\partial B} \frac{\partial G_k}{\partial \widetilde{n}_j}(\widetilde{x}_j, \widetilde{y}) dy - \sum_{\substack{i=1 \\ \mathrm{i} \neq \mathrm{j}}}^{m} \int_{\partial B_{\mathrm{i}}} \frac{\partial G_k}{\partial \widetilde{n}_j}(\widetilde{x}_j, \widetilde{y}) dy, \qquad (3.11)$$

where the integrations on the right hand side of (3.11) can be performed analytically.

4. Analysis of Field Points

According to Helmholtz Integral formula, at any point in B^+ (exterior to the surface ∂B), the scattered acoustic pressure is determined by the total pressure and its normal derivative on ∂B. In the case of the hard boundary condition (2.2), only the total pressure on ∂B need be known and then use B^+ part of (2.3). At a near-field point the integral has to be evaluated directly. As for far-field point, the usual asymptotic analysis can be used as long as the coordinate system has its origin within the surface ∂B.

Using the same convention as before, the near-field and far-field approximations for scattered pressure become [18]

$$\varpi_{sc}(\widetilde{x}) = \sum_{i=1}^{m} \varpi_i \int_{\partial B_{\mathrm{i}}} \frac{\partial G_k}{\partial \widetilde{n}_y}(\widetilde{x}, \widetilde{y}) dy$$

$$\varpi_{sc}(\widetilde{x}) = \frac{e^{-ik|\widetilde{x}|}}{|\widetilde{x}|} \sum_{i=1}^{m} \varpi_i \int_{\partial B} ik\widetilde{n}_y \cdot \frac{\widetilde{x} - \widetilde{y}}{|\widetilde{x} - \widetilde{y}|} \exp\left(\frac{ik\widetilde{x} \cdot \widetilde{y}}{|\widetilde{x}|} \right) dy$$

respectively, where ϖ_i ($1 \leq i \leq m$) is the total pressure at the ith surface collocation point. Hence, the total pressure at $\widetilde{x}$ can be found by using the known scattered pressure at the same position and by equation (2.1).

5. Conclusion

The three numerical schemes have been implemented, tested and applied to some scattering problems (see [4], [11]). As expected [18], the singular integrals appearing in eq. (3.2) have very small values compared with the non-singular integrals, and could very well be neglected for all practical purposes. This is also confirmed by the eq. (3.3). The Scheme B was implemented and tested [10] along the same lines as that for Scheme A. Numerical tests have shown that for a given surface mesh the best accuracies are obtained with $\alpha(k) = -ik^{-1}$, as mentioned at the end of the discussion above on Scheme B. Reut [18] found that to achieve good accuracy, $k\overline{\rho} \leq 1$ needs to be satisfied, where $k\overline{\rho}$ is used as a measure of the boundary mesh fineness, and $\overline{\rho}$ is as defined in eq. (3.7). Both Schemes A and B fail for structures that are plate like which are very thin compared to the wavelength, however, Scheme B faired better in the sense that it can be applied to thinner plates than that used in the Scheme A. As also concluded by Reut and others (see [4], [10], [11], [18]). On the whole, The Schemes A and B needed $ka \geq 0.51$ and $ka \geq 0.15$ respectively, where a is half the thickness of the plate. Also, the Scheme A faced difficulties regarding selection of interior collocation points for wave-numbers greater than the lowest eigen-frequencies of the interior Dirichlet problem. The Scheme B did not suffer such problem as long as the surface mesh condition $k\overline{\rho} \leq 1$ is satisfied. The Scheme C has shown superiority over both A and B, in the sense that it circumvents the singularity and hyper-singularity problems as faced by A and B respectively. It is found that $\xi \approx 0.5$ yields the optimal accuracy, i.e. have shown tolerance near the known eigen- frequencies (of the interior Dirichlet problem).

Acknowledgments

Thanks to Professor M. A. Jaswon, City University, London, UK for his valuable suggestions and help during the preparation of this work. And I am grateful to Sultan Qaboos University, Oman for their financial help and support during this project.

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