**International Journal of Pure and Applied Mathematics**
**Volume 72 No. 3 2011, 343-364**

# MODELING USING DIFFERENTIAL EQUATIONS WITH VARIABLE STRUCTURE AND IMPULSES

R.B. Chukleva

Department of Computer Systems and Technologies
Technical University Sofia, Branch Plovdiv
25, Tzanko Dyustabanov Str., Plovdiv, 4000, BULGARIA

**Abstract:** The modeling of shutter dynamics of a blow back valve is described with the systems of differential equations with variable structure and impulses in the present paper. The changes of the right-hand side of the system and impulsive perturbations take place at the moments when the so called "shifting functions" cancel (become zero). The structure of the modeling system changes depending on the states "closed" and "open" of the shutter valve. The impulses reflect the instantaneous change of the shutter valve speed during the switching from open to close position and intermittent shifting of the shutter valve. The main result in this paper is finding the suitable conditions which ensure the continuous dependence of the modeling system regarding the changes of the initial condition and switching functions.

**AMS Subject Classification:** 34A37, 70K20
**Key Words:** variable structure, variable impulsive moments, switching functions, continuous dependence, valve

## 1. Introduction

The qualitative theory of differential equations with variable structure and the same with impulses develops relatively intensively due to its numerous applications. The authors discuss the problem in which the modeling system of differential equations has not only a variable structure but also the impulsive perturbations.

The applications of differential equations with a variable structure are mainly

Received: May 2, 2011

in mechanical, electrical and automation control theory: [7], [8], [9], [14], [22], [23], [24], [29], [30] and [35]. The differential equations with impulses are used to describe the processes of electromechanics, pharmacokinetics etc. The dynamics of species, subjected to the instantaneous influences are investigated mainly by means of the impulsive systems: [1]-[5], [16]-[21], [26], [27], [28], [31]-[34] and [36].

The systems considered could be split into different classes, depending on the way of determining the variation of the structure and impulsive perturbations (further these moments are called shifting). We will point the following classes:

- The shifting moments are predefined [6], [10], [25] and [32];
- The shifting moments coincide with the moments at which the integral curve (the trajectory) cancels the predefined functions, defined in the extended phace space (or phace space) of the differential equation system [3], [8], [20], [25] and [33]. These functions are named shifting;
- The shifting moments coincide with the moments at which the trajectory of the system meets the predefined sets, situated in the expanded phace space (generally hypersurfaces) [11]-[15];
- The shifting moments coincide with the moments at which the solution minimizes the given functional [2]-[5] and [27];
- The shifting moments are occasional [37], [38] and [39] etc.

The shifting moments of the second type are considered in the present paper.

## 2. Preliminary Results

The following initial problem is investigated:

$$\frac{dx}{dt} = f_i\left(t, x\right), \qquad \phi_i\left(x\left(t\right)\right) \neq 0,\ t_{i-1} < t < t_i, \tag{1}$$

$$\phi_i\left(x\left(t_i\right)\right) = 0, \qquad i = 1, 2, \ldots, \tag{2}$$

$$x\left(t_i + 0\right) = x\left(t_i\right) + I_i\left(x\left(t_i\right)\right), \tag{3}$$

$$x\left(t_0\right) = x_0, \tag{4}$$

where:

- $f_i : \mathrm{R}^+ \times D \to \mathrm{R}^n$, $f_i = \left(f_i^1, f_i^2, \ldots, f_i^n\right)$, where the phase space $D$ is a non empty domain in $\mathrm{R}^n$;

- $\phi_i : D \to \mathrm{R}$;

- $\Phi_i = \{x \in D; \phi_i(x) = 0\}$, $i = 1, 2, \ldots,$ are shifting hypersurfaces of the problem considered;

- The functions $I_i : D \to \mathrm{R}^+$ and $(Id + I_i) : \Phi_i \to D$, where $Id$ is the identity in $\mathrm{R}^n$. Further, for convenience we assume that $I_0(x) = 0$ at $x \in \Phi_i$. The equality
$$(Id + I_0)(x) = x$$
is valid for each $x \in \Phi_i$;

- The initial point $(t_0, x_0) \in \mathrm{R}^+ \times D$.

The solution $x(t) = x(t; t_0, x_0, \phi_1, \phi_2, \ldots)$ of the initial problem is a piece-wise continuous function on the left at the moments $t_1, t_2, \ldots$

Consecutively for each $i = 1, 2, \ldots,$ it is satisfied:

1. For $t_{i-1} < t \leq t_i$ the solution of the problem (1), (2), (3), (4) coincides with the solution of the initial problem with a fixed structure and without impulses
$$\frac{dx}{dt} = f_i(t, x), \; x(t_{i-1} + 0) = (Id + I_{i-1})(x(t_{i-1}));$$

2. For any $t, t_{i-1} < t \leq t_i$, the following inequality is valid
$$\phi_i(x(t)) \neq 0;$$

3. Let $t_i$ is the first moment after $t_{i-1}$, for which the following equality is true
$$\phi_i(x_i(t_i)) = 0;$$

4. At the moment $t_i$ the system structure changes and an impulsive perturbation takes place. It is valid
$$x(t_i + 0) = x(t_i) + I_i(x(t_i)) = (Id + I_i)(x(t_i)).$$

In the common case, the functions $I_i(x) \neq 0,\ i = 1, 2, \ldots.$ The solution of the problem studied is a discontinuous function to the right at the points $t_1, t_2, \ldots$ There is a finite jump continuity at these points. We will recall that $t_1, t_2, \ldots$ are shifting moments; $I_1, I_2, \ldots$ are named impulsive functions and $\phi_1, \phi_2, \ldots$ are shifting functions.

Along with the fundamental problem we examine the so called perturbed initial problem:

$$\begin{aligned}
&\frac{dx^*}{dt} = f_i(t, x^*), && \phi_i^*(x^*(t)) \neq 0, \quad t_{i-1}^* < t < t_i^*,\\
&\phi_i^*(x^*(t_i^*)) = 0, && i = 1, 2, \ldots,\\
&x^*(t_i^* + 0) = x^*(t_i^*) + I_i(x^*(t_i^*)), &&\\
&x^*(t_0^*) = x_0^*, &&
\end{aligned}$$

where:

- $\phi_i^* : D \to \mathrm{R}$;
- $\Phi_i^* = \{x \in D; \phi_i^*(x) = 0\},\ i = 1, 2, \ldots$ are shifting hypersurfaces of the perturbed problem;
- $(t_0^*, x_0^*) \in \mathrm{R}^+ \times D$.

Let $x^*(t) = x^*(t; t_0^*, x_0^*, \phi_1^*, \phi_2^*, \ldots)$ denotes the solution of the upper problem.

**Definition 1.** (see [8]) We say that the solution of problem (1), (2), (3), (4) depends continuously on the initial condition and shifting functions if:

$$\begin{aligned}
&(\forall \varepsilon > 0)(\forall \eta > 0)(\forall T > t_0)(\exists \delta = \delta(\varepsilon, \eta, T) > 0):\\
&\left(\forall t_0^* \in \mathrm{R}^+,\ |t_0^* - t_0| < \delta\right)(\forall x_0^* \in D,\ \|x_0^* - x_0\| < \delta)\\
&(\forall \phi_i^* \in C[D, \mathrm{R}],\ \|\phi_i^*(x) - \phi_i(x)\| < \delta \text{ for } (x) \in D,\ i = 1, 2, \ldots)\\
&\Rightarrow \Big( \|x(t; t_0^*, x_0^*, \phi_1^*, \phi_2^*, \ldots) - x(t; t_0, x_0, \phi_1, \phi_2, \ldots)\| < \varepsilon\\
&\qquad\qquad \text{for } t_0^{\max} \leq t \leq T \text{ and } |t - t_i| > \eta,\ i = 1, 2, \ldots \Big),
\end{aligned}$$

where $t_0^{\max} = \max\{t_0^*, t_0\}$.

We note that, any "closeness" between both solutions (of the fundamental problem and the corresponding perturbed problem) is not required in the prefixed surroundings $(t_i - \eta, t_i + \eta),\ i = 1, 2, \ldots$ at the shifting moments of the problem.

Further, let $\|.\|$ and $\langle .,. \rangle$ denote the Euclidean norm and the dot product in $\mathrm{R}^n$, respectively.

The following conditions are introduced:

**H1.** $f_i \in C\left[\mathrm{R}^+ \times D, \mathrm{R}^n\right]$ and there exsists a positive constant $C_{Id+I}$, such that for each point $(t,x) \in \mathrm{R}^+ \times D$ and $i = 1, 2, \ldots$ the inequalities

$$\|f_i\,(t,x)\| \leq C_f$$

hold true;

**H2.** $\phi_i \in C^1\left[D, \mathrm{R}\right]$ and there exists a positive constant $C_{\mathrm{grad}\,\phi}$ such that for each point $x \in D$ and any $i = 1, 2, \ldots,$ the following inequalities are satisfied:

$$\|\mathrm{grad}\,\phi_i\,(x)\| \leq C_{\mathrm{grad}\,\phi};$$

**H3.** $I_i \in C\left[D, \mathrm{R}^n\right], (Id + I_i) : \Phi_i \to D$ and there exists a positive constant $C_{\phi(Id+I)}$ such that for each point $x \in \Phi_i$ and any $i = 1, 2, \ldots$ the following inequalities are fulfilled:

$$\left|\phi_{i+1}\left((Id + I_i)\,(x)\right)\right| = \left|\phi_{i+1}\left(x + I_i\,(x)\right)\right| \geq C_{\phi(Id+I)};$$

**H4.** For each point $(t,x) \in \mathrm{R}^+ \times D$ and any $i = 1, 2, \ldots$ the inequalities

$$\phi_{i+1}\left((Id + I_i)\,(x)\right)\left\langle \mathrm{grad}\,\phi_{i+1}\,(x)\,, f_{i+1}\,(t,x)\right\rangle < 0$$

hold true;

**H5.** For each point $(t_0, x_0) \in \mathrm{R}^+ \times D$ and for any $i = 1, 2, \ldots$ there exists a unique solution of the initial problem

$$\frac{dx}{dt} = f_i\,(t,x)\,, \quad x\,(t_0) = x_0,$$

where $t \geq t_0$.

The main result, used in the subsequent research, is present in following Theorem 1.

**Theorem 1.** (see [8]) *Let the conditions H1-H5 be satisfied.*

*Then the solution of problem (1), (2), (3), (4) depends continuously on the initial condition and the shifting functions.*

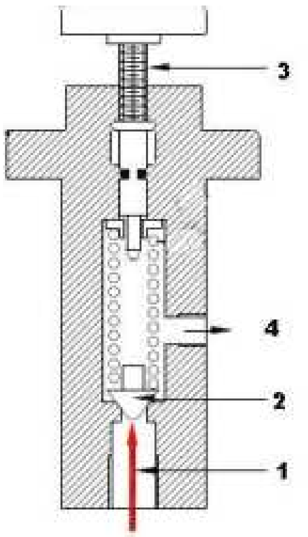


Figure 1

## 3. Mathematical Model of One Mechanical System of Hydrodynamics

We investigate the dynamic mathematical model of hydraulic blow down back-pressure valve. The model is borrowed from [14].

A model which optimizes some parameters and continuous dependence of the modeling system solution with respect to these parameters are studied in the article. The main elements of the valve (see Figure 1) are:

1. Intake mainfold of volum $V$;
2. Mitre shutter valve with an angle measure $2\alpha$ of the shutter crest and mass $M_c$;
3. Screw-shaped (spiral) spring with a mass $M_s$ and specific constant of elasticity $C_s$;
4. Exhaust collector.

The following notations are introduced:

- $\Delta = \Delta(t) \geq 0$ is a distance from a shutter valve (in vertical direction) to its bed;

- $P_1 = P_1(t) \geq 0$ and $P_2 = P_2(t) \geq 0$ are the pressure of the intake and exhaust collector respectively;

- $Q_1 = Q_1(t) \geq 0$ and $Q_2 = Q_2(t) \geq 0$ are the incoming and outgoing flow of the fluid respectively;

- $d$ is the diameter of the shutter bed;

In order to simplify the model, we will assume that the quantities $Q_1$ and $P_2$ do not change in the time.

The shutter valve opens at the moment, when the difference of the pressure in the intake and exhaust collector becomes greater than the power of the initial strength of the intensity spring $C_0$. Otherwise, the valve is closed.

From the dynamic point of view the valve can be regarded as a mechanical system that changes ("saltatory" in some moments) in the time. There are different options for determining the dominant factors in the dynamics of the shutter valve depending on the goals. One of these opportunities is to describe the changes in the mechanical system using the values of the functions: $\Delta = \Delta(t)$, $y = y(t)$ and $P_1 = P_1(t)$, i.e. by the magnitude of the port-hole between the shutter valve and its bed, the speed of the shutter and the fluid pressure magnitude in the intake collector.

The shutter valve has two states: "open" and "closed".

*Position One.* The valve is open, i.e. $\Delta(t) > 0$. Then the shutter valve moving is described by the fundamental equation of dynamics

$$m\frac{d^2}{dt^2}\Delta(t) = P(t),$$

where: the mass $m = M_c + \frac{1}{3}M_s$, power $P = P(t)$ is a vector sum of the initial intensity of the spring $C_0$, the intensity of the compression spring $C_s.\Delta(t)$ and the difference between pressures $P_1$ and $P_2$. More precisely, the following equality is valid

$$P = P(t) = \frac{\pi d^2}{4}(P_1 - P_2) - C_0 - C_s\Delta(t).$$

Thus we obtain that the function $\Delta = \Delta(t)$ satisfies the differential equation second order

$$\frac{d^2}{dt^2}\Delta(t) = \frac{1}{m}P(t) = \frac{1}{m}\left(\frac{\pi d^2}{4}(P_1 - P_2) - C_0 - C_s\Delta(t)\right) \text{ for } \Delta(t) > 0.$$

It is known (from the laws of hydrodynamics) that the pressure variations are defined by the differential equation first order

$$\frac{d}{dt}P_1(t) = \frac{E}{V}(Q_1 - Q_2),$$

where $E$ is a specific coefficient of the working fluid. The flow rate (output) $Q_2$, passing through the valve, is defined by the equality

$$Q_2 = \nu\pi D\Delta(t)\sin\alpha\sqrt{\frac{2}{\omega}(P_1 - P_2)}.$$

In the equation above $\nu$ is a specific coefficient of the flow rate $Q_2$, $\omega$ is the relative weight of the fluid and $\pi d.\Delta(t)\sin\alpha$ is the approximate surface of the hole (jaw opening) formed when the shutter valve is shifted away $\Delta = \Delta(t) > 0$ from his bed. The shutter speed we denote by $y$, i.e.

$$y = y(t) = \frac{d}{dt}\Delta(t).$$

The system, simulating the development of mechanical system in its first state has the form:

$$\frac{d}{dt}\Delta = y, \tag{5}$$

$$\frac{d}{dt}y = \frac{1}{m}P(t) = \frac{1}{m}\left(\frac{\pi d^2}{4}(P_1 - P_2) - C_0 - C_s\Delta\right), \tag{6}$$

$$\frac{d}{dt}P_1 = \frac{E}{V}(Q_1 - Q_2) = \frac{E}{V}\left(Q_1 - \nu\pi d\Delta\sin\alpha\sqrt{\frac{2}{\omega}(P_1 - P_2)}\right). \tag{7}$$

*Position two.* The valve is closed, i.e. $\Delta = \Delta(t) = 0$. In this case the shutter speed $y$ and the flow-rate $Q_2$ are zero. Only intake collector pressure $P_1$ changes, as its variation is proportional to the constantly intake flow. The changes of mechanical system in its second position are described by means of the system of differential equations:

$$\frac{d}{dt}\Delta = 0, \tag{8}$$

$$\frac{d}{dt}y = 0, \tag{9}$$

$$\frac{d}{dt}P_1 = \frac{E}{V}Q_1. \tag{10}$$

The valve transition from the first to second condition takes place at the moments $t_1, t_3, \ldots, t_{2i-1}, \ldots$, when the shutter valve reaches the bottom, i.e.

$$\Delta\left(t_{2i-1}\right) = 0, \quad i = 1, 2, \ldots$$

If we introduce the function $\phi^{\downarrow} = \phi^{\downarrow}(\Delta) = \Delta$, then we rewrite the equalities above like this:

$$\phi^{\downarrow}\left(\Delta\left(t_{2i-1}\right)\right) = 0, \quad i = 1, 2, \ldots$$

We denote by $\phi^{\uparrow} = \phi^{\uparrow}(P_1)$ the following function:

$$\phi^{\uparrow}(P_1) = \frac{1}{4}\pi d^2 \left(P_1 - P_2\right) - C_0 = P\left(\Delta, P_1, \mu\right) \text{ for } \Delta = 0.$$

If the function $\phi^{\uparrow}$ is negative, i.e.:

$$\frac{1}{4}\pi d^2 \left(P_1 - P_2\right) - C_0 < 0, \tag{11}$$

the valve is in its second position. The shifting of mechanical system from the second to first position takes place at the moments $t_2, t_4, \ldots, t_{2i}, \ldots$, when the function $\phi^{\uparrow}$ becomes zero, i.e. the following equations are satisfied:

$$\phi^{\uparrow}\left(P_1\left(t_{2i}\right)\right) = 0, \quad i = 1, 2, \ldots .$$

Obviously, at the moments $t_1, t_2, \ldots$ the nonlinear system of differential equations, which models the variations of the shutter valve, changes its structure, i.e. the right side of the system changes. More precisely, at the moments when an odd index of the system (5), (6), (7) is replaced by the system (8), (9), (10). At the moments with an even index the replacement of the system right hand sides is in the reverse order. Thus we conclude that the model of hydrodynamic mechanical system is a nonlinear system of differential equations with variable structure.

We will pay attention to another fact. The shutter valve speed becomes zero (regardless of the speed magnitude) with the shifting of right side. It means that the change (nullification) of the second component (speed) of the solution of model system takes place instantaneously in the form of impulses at the shifting moments with odd indices. Practically, the meaning is that the shutter valve "hits" its bed if it reaches the lowest position. We express this by the equations:

$$y\left(t_{2i-1} + 0\right) = 0, \quad i = 1, 2, \ldots .$$

The other modeling functions of the mechanical system ($\Delta$ and $P_1$) do not change intermittently at the shifting moments from open to close position. The following impulsive equalities are satisfied:

$$\begin{aligned}(\Delta(t_{2i-1}+0), &y(t_{2i-1}+0), P_1(t_{2i-1}+0)) \\ &= (\Delta(t_{2i-1}), y(t_{2i-1}), P_1(t_{2i-1})) + (0, -y(t_{2i-1}), 0) \\ &= \left(Id + I^{\downarrow}\right)(\Delta(t_{2i-1}), y(t_{2i-1}), P(t_{2i-1})), \quad i = 1, 2, \ldots,\end{aligned}$$

where the impulsive function is

$$I^{\downarrow}(\Delta, y, P_1) = (0, -y, 0).$$

The solution of modeling system is discontinuous at the shifting moments with even index: $t_2, t_4, \ldots, t_{2i}, \ldots$, i.e. the system of differential equations is subjected again to the impulsive effects. Indeed, at those moments the shutter valve moves from the downward position "instantaneously", compared to the total duration of the process

$$\Delta = \Delta_{\min} = 0$$

in upwardmost position

$$\Delta = \Delta_{\max} = \frac{1}{C_s}\left(\frac{\pi d^2}{4}(P_1 - P_2) - C_0\right).$$

We obtain the value of $\Delta_{\max}$ after nullifying the right side of (6), i.e. we assume that immediately after the model system shifting from close to open position, the shutter valve speed is 0. In other words, the speed of shutter valve nullifying in its upwardmost position, i.e. in the moments $t_2 + 0, t_4 + 0, \ldots$ The following equations are valid:

$$\Delta(t_{2i} - 0) = \Delta(t_{2i}) = \Delta_{\min}, \quad \Delta(t_{2i} + 0) = \Delta_{\max}, \quad i = 1, 2, \ldots.$$

The other phase functions of the model system ($y$ and $P_1$) do not change intermittently at the shifting moments from close to open position. The following impulsive equations are satisfied:

$$\begin{aligned}(\Delta(t_{2i}+0), y(t_{2i}+0), P_1(t_{2i}+0)) &= (\Delta(t_{2i}), y(t_{2i}), P_1(t_{2i})) + (\Delta_{\max}, 0, 0) \\ &= \left(Id + I^{\uparrow}\right)(\Delta(t_{2i}), y(t_{2i}), P(t_{2i})),\end{aligned}$$

where $i = 1, 2, \ldots$ and the impulsive function has the form

$$I^{\uparrow}(\Delta, y, P_1) = (\Delta_{\max}, 0, 0).$$

Thus, we find that the solution of model system is discontinuous (with respect to some phase coordinates) in the shifting moments $t_1, t_2, t_3, \ldots$ The solutions have a finite jump at these points. Without loss of generality, we consider that the solution of model system is continuous on the left in its domain.

The most general form of the modeling system with a variable structure is:

$$\frac{dx}{dt} = f_i(x), \quad \phi_i(x(t)) \neq 0, \; t_{i-1} < t < t_i, \; i = 1, 2, \ldots, \tag{12}$$

$$\phi_i(x(t_i)) = 0, \tag{13}$$

$$x(t_i + 0) = x(t_i) + I_i(x(t_i)), \tag{14}$$

$$x(t_0) = x_0, \tag{15}$$

where:

- The dimension is three: $n = 3$;
- The unknown function is $x = x(t) = (\Delta(t), y(t), P_1(t))$;
- The right hand sides of the system are determined by means of the equations:

$$\begin{aligned} f_i(x) =& f_i(\Delta, y, P_1) \\ =& \begin{cases} \left( y, \dfrac{1}{m} P(\Delta, P_1, \mu), \dfrac{E}{V}(Q_1 - Q_2(\Delta, P_1, \mu)) \right), & i = 2j-1, \\ \left( 0, 0, \dfrac{E}{V} Q_1 \right), & i = 2j, \end{cases} \\ =& \begin{cases} \Bigg( y, \dfrac{1}{m}\left( \dfrac{\pi d^2}{4}(P_1 - P_2) - C_0 - C_s \Delta \right), & \\ \quad \dfrac{E}{V}\left( Q_1 - \nu \pi d \Delta \sin\alpha \sqrt{\dfrac{2}{\omega}(P_1 - P_2)} \right) \Bigg), & i = 2j-1, \\ \left( 0, 0, \dfrac{E}{V} Q_1 \right), & i = 2j, \end{cases} \end{aligned}$$

  where $j = 1, 2, \ldots$;

- The shifting functions have the form:

$$\phi_i(x) = \phi_i(\Delta, y, P_1)$$

$$= \begin{cases} \phi^{\downarrow}(\Delta) = \Delta, & i = 2j-1, \\ \phi^{\uparrow}(P_1) = \frac{1}{4}\pi d^2 (P_1 - P_2) - C_0, & i = 2j, \ j = 1, 2, \dots; \end{cases}$$

- The impulsive functions are given by the equalities:

$$\begin{aligned} I_i(x) = & I_i(\Delta, y, P_1) \\ & = \begin{cases} I^{\downarrow}(\Delta, y, P) = (0, -y, 0), & i = 2j-1, \\ I^{\uparrow}(\Delta, y, P) = (\Delta_{\max}, 0, 0), & i = 2j, \ j = 1, 2, \dots; \end{cases} \end{aligned}$$

- We assume that the initial moment is $t_0 = 0$;
- We assume that the initial point $x_0 = (\Delta_0, y_0, P_{10})$ satisfies the following restrictions:
  - $0 = \Delta_{\min} \leq \Delta_0 \leq \Delta_{\max}$, where $\Delta_{\max}$ is a distance between the shutter valve and its bed at the moments $t_2 + 0, t_4 + 0, \dots$, in which the shutter valve is in its upwardmost position;
  - $y_{\min} \leq y_0 \leq y_{\max} = 0$, where $y_{\min}$ is the minimum speed of the shutter valve;
  - $P_{1\min} \leq P_{10} \leq P_{1\max}$, where

$$P_{1\min} = \frac{E}{V} \min Q_2 \quad \text{and} \quad P_{1\max} = \frac{E}{V} \max Q_2$$

    are the minimum and maximum value of the collector pressure, respectively;
  - The domain $D$ is defined by

$$\begin{aligned} D &= [\Delta_{\min}, \Delta_{\max}] \times [y_{\min}, y_{\max}] \times [P_{1\min}, P_{1\max}] \\ &= [0, \Delta_{\max}] \times [y_{\min}, 0] \times [P_{1\min}, P_{1\max}] \subset \mathrm{R}^3. \end{aligned}$$

## 4. Continuous Dependence of the Hydrodynamics Model Solution on the Initial Condition and the Shifting Functions

We will check the validity of the conditions of Section 2.

**Condition H1.** This condition is true by reason of the following facts:

- The functions $f_1$ and $f_2$ are continuous at the set $D$;
- The set $D$ is bounded and closed.

Then a positive constant $C_f$ exists, so that

$$\|f_1(\Delta, y, P_1)\| \le C_f, \|f_2(\Delta, y, P_1)\| \le C_f \text{ for } (\Delta, y, P_1) \in D.$$

Since

$$f_1 = f_3 = \cdots = f_{2i-1} = \ldots \text{ and } f_2 = f_4 = \cdots = f_{2i} = \ldots,$$

therefore, the inequalities above are valid for each right side of the modeling system of differential equations with variable structure and impulses, i.e.

$$\|f_i(\Delta, y, P_1)\| \le C_f \text{ for } (\Delta, y, P_1) \in D, \; i = 1, 2, \ldots .$$

**Condition H2.** For each $x = (\Delta, y, P_1) \in D$ and $i = 1, 2, \ldots$ it is fulfilled

$$\begin{aligned} \operatorname{grad}\phi_i(x) =& \operatorname{grad}\phi_i(\Delta, y, P_1) \\ =& \begin{cases} (1,0,0), & i = 2j-1, \\ \left(0,0,\frac{1}{4}\pi d^2\right), & i = 2j, \; j = 1,2,\ldots, \end{cases} \end{aligned}$$

hence it follows that the next inequalities are valid:

$$\|\operatorname{grad}\phi_i(x)\| \le \max\left\{1, \frac{1}{4}\pi d^2\right\} = C_{\operatorname{grad}\phi}, \quad i = 1, 2, \ldots .$$

**Condition H3.** The subsequent facts are valid:

- The functions $I_i \in C[D, \mathrm{R}^n]$, as they are linear;
- For each point $x \in \Phi_i$, i.e.

$$(\Delta, y, P_1) = \begin{cases} (0, y, P_1), & i = 2j-1, \\ (0,0,P_1), & i = 2j, \; j = 1,2,\ldots, \end{cases}$$

where $y_{\min} \le y \le 0$ and $P_{1\min} \le P_1 \le P_{1\max}$, the next equation is true:

$$\begin{aligned} &(Id + I_i)(x) = (Id + I_i)(\Delta, y, P_1) \\ &= \begin{cases} (0, y, P_1) + (0, -y, 0) = (0, 0, P_1), & i = 2j-1; \\ (0,0,P_1) + (\Delta_{\max}, 0, 0) = (\Delta_{\max}, 0, P_1), & i = 2j, \; j = 1, 2, \ldots . \end{cases} \end{aligned}$$

Since $(0, 0, P_1) \in D$ and $(\Delta_{\max}, 0, P_1) \in D$, we deduce that $(Id + I_i)(x) \in D$ and therefore $(Id + I_i) : \Phi_i \to D$, $i = 1, 2, \ldots$

- For each point $x \in \Phi_i$ there is:

$$
\begin{aligned}
\left|\phi_{i+1}\left(x+I_i\left(x\right)\right)\right| =& \begin{cases} \left|\phi_{i+1}\left(\left(0,0,P_1\right)\right)\right|, & i=2j-1, \\ \left|\phi_{i+1}\left(\left(\Delta_{\max},0,P_1\right)\right)\right|, & i=2j, \end{cases} \\
=& \begin{cases} \left|\frac{1}{4}\pi d^2\left(P_1-P_2\right)-C_0\right|, & i=2j-1, \\ \left|\Delta_{\max}\right| = \frac{1}{C_s}\left|\frac{\pi d^2}{4}\left(P_1-P_2\right)-C_0\right|, & i=2j, \end{cases} \\
\geq& \left|\frac{1}{4}\pi d^2\left(P_1-P_2\right)-C_0\right| \min\left\{1,\frac{1}{C_s}\right\} \\
\geq& \left|\frac{1}{4}\pi d^2\left(P_{1\min}-P_2\right)-C_0\right| \min\left\{1,\frac{1}{C_s}\right\} \\
=& C_{\phi(Id+I)} \\
>& 0.
\end{aligned}
$$

**Condition H4.** We will discuss two cases:

*Case 1.* $i=2j-1$. In accordance with (11) it is sattisfied

$$
\begin{aligned}
&\phi_{i+1}\left(\left(Id+I_i\right)\left(x\right)\right)\left\langle \operatorname{grad}\phi_{i+1}\left(x\right), f_{i+1}\left(x\right)\right\rangle \\
&\quad= \phi_{2j}\left(\left(Id+I_{2j-1}\right)\left(\Delta,y,P_1\right)\right)\left\langle \operatorname{grad}\phi_{2j}\left(\Delta,y,P_1\right), f_{2j}\left(\Delta,y,P_1\right)\right\rangle \\
&\quad= \left(\frac{1}{4}\pi d^2\left(P_1-P_2\right)-C_0\right)\left\langle\left(0,0,\frac{1}{4}\pi d^2\right),\left(0,0,\frac{E}{V}Q_1\right)\right\rangle \\
&\quad= \left(\frac{1}{4}\pi d^2\left(P_1-P_2\right)-C_0\right)\frac{\pi d^2 E Q_1}{4V} \\
&\quad< 0.
\end{aligned}
$$

*Case 2.* $i=2j$. It is fulfilled

$$
\begin{aligned}
&\phi_{i+1}\left(\left(Id+I_i\right)\left(x\right)\right)\left\langle \operatorname{grad}\phi_{i+1}\left(x\right), f_{i+1}\left(x\right)\right\rangle \\
&\quad= \Delta\left\langle\left(1,0,0\right),\left(y,\frac{1}{m}P\left(\Delta,P_1,\mu\right),\frac{E}{V}\left(Q_1-Q_2\left(\Delta,P_1,\mu\right)\right)\right)\right\rangle \\
&\quad= \Delta y \\
&\quad< 0.
\end{aligned}
$$

The last inequality is valid because in this case $\Delta > 0$ is true (the shutter valve is in the open position and hence the distance to his bed is positive). On the other hand, the function $\Delta = \Delta\left(t\right)$ decreases in the time. Concequently

$$
y = y\left(t\right) = \frac{d}{dt}\Delta\left(t\right) < 0.
$$

Thus condition H4 is valid.

**Condition H5.** We go to the initial problem of the differential equation system:

$$\frac{dx}{dt} = f_i(x), \quad x(0) = x_0. \tag{16}$$

We consider again two cases:

*Case 1.* If $i$ is odd, then the system above degenerates into system (8), (9), (10), which has permanent right hand side and therefore it has a singular solution for $t \geq 0$.

*Case 2.* The system above degenerates into system (5), (6), (7) if $i$ is even. The right sides of the system considered are continuous functions in the domain$D$. We prove that these functions are uniformly and the equal power Lipschitz-continuous functions in$D$. Recall that the following relation is valid in the case

$$\begin{aligned} f_i(x) =& f_i(\Delta, y, P_1) \\ =& \left(f_i^1(\Delta, y, P_1), f_i^2(\Delta, y, P_1), f_i^3(\Delta, y, P_1)\right) \\ =& \left(y, \frac{1}{m}\left(\frac{\pi d^2}{4}(P_1 - P_2) - C_0 - C_s\Delta\right),\right. \\ & \left.\frac{E}{V}\left(Q_1 - \nu\pi d\Delta \sin\alpha\sqrt{\frac{2}{\omega}(P_1 - P_2)}\right)\right). \end{aligned}$$

One by one, for each of the function's $f$ coordinates, we obtain:

- For function $f_i^1$ it is satisfied:

$$\begin{aligned} \left|f_i^1(x') - f_i^1(x'')\right| =& \left|f_i^1(\Delta', y', P_1') - f_i^1(\Delta'', y'', P_1'')\right| \\ =& \left|y' - y''\right| \leq \left\|x' - x''\right\|. \end{aligned}$$

- For function $f_i^2$ it is valid:

$$\begin{aligned} \left|f_i^2(x') - f_i^2(x'')\right| =& \left|f_i^2(\Delta', y', P_1') - f_i^2(\Delta'', y'', P_1'')\right| \\ =& \left|\frac{1}{m}\left(\frac{\pi d^2}{4}(P_1' - P_2) - C_0 - C_s\Delta'\right)\right. \\ & \left. - \frac{1}{m}\left(\frac{\pi d^2}{4}(P_1'' - P_2) - C_0 - C_s\Delta''\right)\right| \\ \leq& \frac{\pi d^2}{4m}\left|P_1' - P_1''\right| + \frac{C_s}{m}\left|\Delta' - \Delta''\right| \end{aligned}$$

$$\leq \left( \frac{\pi d^2}{4m} + \frac{C_s}{m} \right) \left\| x' - x'' \right\| .$$

- For function $f_i^3$ it is true:

$$\begin{aligned}
\left| f_i^2 \left( x' \right) - f_i^2 \left( x'' \right) \right| &= \left| f_i^3 \left( \Delta', y', P_1' \right) - f_i^3 \left( \Delta'', y'', P_1'' \right) \right| \\
&= \left| \frac{E}{V} \left( Q_1 - \nu \pi d \Delta' \sin \alpha \sqrt{\frac{2}{\omega} \left( P_1' - P_2 \right)} \right) \right. \\
&\quad \left. - \frac{E}{V} \left( Q_1 - \nu \pi d \Delta'' \sin \alpha \sqrt{\frac{2}{\omega} \left( P_1'' - P_2 \right)} \right) \right| \\
&= \frac{E}{V} \nu \pi d \sin \alpha \left| \Delta' \sqrt{\frac{2}{\omega} \left( P_1' - P_2 \right)} - \Delta'' \sqrt{\frac{2}{\omega} \left( P_1'' - P_2 \right)} \right| \\
&= \frac{E}{V} \nu \pi d \sin \alpha \left| F \left( \Delta', P_1' \right) - F \left( \Delta'', P_1'' \right) \right| ,
\end{aligned}$$

where

$$F \left( \Delta, P_1 \right) = \Delta \sqrt{\frac{2}{\omega} \left( P_1 - P_2 \right)}.$$

We apply the Intermediate Value Theorem for the difference $F \left( \Delta', P_1' \right) - F \left( \Delta'', P_1'' \right)$ and it follows that

$$\begin{aligned}
&\left| f_i^3 \left( \Delta', y', P_1' \right) - f_i^3 \left( \Delta'', y'', P_1'' \right) \right| \\
\leq &\frac{E}{V} \nu \pi d \sin \alpha \left( \max \left\{ \frac{\partial}{\partial \Delta} F \left( \Delta, P_1 \right) ; \left( \Delta, P_1 \right) \in [0, \Delta_{\max}] \times [P_{1\min}, P_{1\max}] \right\} \right. \\
&\times \left| \Delta' - \Delta'' \right| \\
&+ \max \left\{ \frac{\partial}{\partial P_1} F \left( \Delta, P_1 \right) ; \left( \Delta, P_1 \right) \in [0, \Delta_{\max}] \times [P_{1\min}, P_{1\max}] \right\} \\
&\left. \times \left| P_1' - P_1'' \right| \right) \\
\leq &\frac{E}{V} \nu \pi d \sin \alpha \left( \max \left\{ \sqrt{\frac{2}{\omega} \left( P_1 - P_2 \right)} ; \left( \Delta, P_1 \right) \in [0, \Delta_{\max}] \times [P_{1\min}, P_{1\max}] \right\} \right. \\
&\times \left| \Delta' - \Delta'' \right| \\
&+ \max \left\{ \frac{\Delta}{\sqrt{2\omega \left( P_1 - P_2 \right)}} ; \left( \Delta, P_1 \right) \in [0, \Delta_{\max}] \times [P_{1\min}, P_{1\max}] \right\}
\end{aligned}$$

$$
\begin{aligned}
&\times \left|P_1' - P_1''\right|\Bigg) \\
&\le \frac{E}{V}\nu\pi d\sin\alpha\left(\sqrt{\frac{2}{\omega}\left(P_{1\max} - P_2\right)}\left|\Delta' - \Delta''\right| + \frac{\Delta}{\sqrt{2\omega\left(P_{1\min} - P_2\right)}}\right. \\
&\left.\times \left|P_1' - P_1''\right|\right) \\
&\le \frac{E}{V}\nu\pi d\sin\alpha\left(\sqrt{\frac{2}{\omega}\left(P_{1\max} - P_2\right)} + \frac{\Delta}{\sqrt{2\omega\left(P_{1\min} - P_2\right)}}\right)\left\|x' - x''\right\|.
\end{aligned}
$$

Finally, the following estimate is obtained:

$$
\begin{aligned}
\left\|f_i\left(x'\right) - f_i\left(x''\right)\right\| \le &\left|f_i^1\left(x'\right) - f_i^1\left(x''\right)\right| + \left|f_i^2\left(x'\right) - f_i^2\left(x''\right)\right| \\
&+ \left|f_i^3\left(x'\right) - f_i^3\left(x''\right)\right| \\
\le &\left\|x' - x''\right\| + \left(\frac{\pi d^2}{4m} + \frac{C_s}{m}\right)\left\|x' - x''\right\| \\
&+ \frac{E}{V}\nu\pi d\sin\alpha\left(\sqrt{\frac{2}{\omega}\left(P_{1\max} - P_2\right)}\right. \\
&\left.+ \frac{\Delta}{\sqrt{2\omega\left(P_{1\min} - P_2\right)}}\right)\left\|x' - x''\right\| \\
= &\left(1 + \frac{\pi d^2}{4m} + \frac{C_s}{m} + \frac{E}{V}\nu\pi d\sin\alpha\left(\sqrt{\frac{2}{\omega}\left(P_{1\max} - P_2\right)}\right.\right. \\
&\left.\left.+ \frac{\Delta}{\sqrt{2\omega\left(P_{1\min} - P_2\right)}}\right)\right)\left\|x' - x''\right\| \\
= &C_{fLip}\left\|x' - x''\right\|,
\end{aligned}
$$

where the constant $C_{fLip}$ is expressed by the parameters of the mechanical system as follows:

$$
\begin{aligned}
C_{fLip} = &1 + \frac{\pi d^2}{4m} + \frac{C_s}{m} \\
&+ \frac{E}{V}\nu\pi d\sin\alpha\left(\sqrt{\frac{2}{\omega}\left(P_{1\max} - P_2\right)} + \frac{\Delta}{\sqrt{2\omega\left(P_{1\min} - P_2\right)}}\right).
\end{aligned}
$$

The functions $f_i$ are continuous, uniformly and the equal power Lipschitz-continuous functions. Therefore a unique solution of the initial problem (16) exists. The solution is global by reason of its specific character. Thus the validity of condition H5 is shown.

The conditions of Theorem 1 are valid according to the previous considerations. Therefore the solution of the modeling system of autonomous nonlinear differential equations with variable structure and impulses depends continuously on the initial condition and the shifting functions.

## 5. Comments

We suggest the following interpretation of the assertion, mentioned above. Let us investigate the dynamics of the shutter valve, subjected to the same external and internal conditions (parameters) of the system. We perform two experiments. The first one is called fundamental (basic), while the second is perturbed. The permissible initial variations in the realization of both experiments are:

- Different initial moments;
- Different initial conditions of the mechanical system. More precisely it is permissible:
  - difference in the offsets of shutter on its bed,
  - difference in the initial shutter speed,
  - difference in the initial pressure in the intake collector;
- Various shifting functions, such as changes in the height of the bed (appearing after prolonged use), i.e. the valve bed wear, the parameters changes, determining the spring force, etc;
- Both experiments, although with different starting points are observed to the same final point.

Then, if the initial differences, described above are "close enough" to each other, then the three states of dynamic mechanical system:

- The magnitude of the shifting (offset) of the shutter valve;
- The shutter valve speed;
- The pressure in the intake collector;

for both experiments can be "arbitrarily close" to each other during the total observation. The proximity between these dynamic states of the mechanical system is possible to be affected only in the symmetrical surroundings of shifting moments of the system in the basic fundamental experiment. These surroundings may have arbitrarily small radiuses.

## References

[1] S. Ahmad, G. Stamov, Almost periodic solutions of $N$-dimensional impulsive competitive systems, *Nonlinear Analysis: Real World Applications*, **10**, No. 3 (2009), 1846-1853.

[2] J. Angelova, A. Dishliev, Optimization problems in population dynamics, *Applicable Analysis*, **69** (1999), 207-221.

[3] J. Angelova, A. Dishliev, S. Nenov, I-optimal curve for impulsive Lotka-Volterra predator-prey model, *Computers & Mathematics with Applications*, **43**, No-s: 10-11 (2002), 1203-1218.

[4] J. Angelova, A. Dishliev, S. Nenov, Optimization problems for impulsive Lotka-Volterra predator-prey model, *J. International of Differential Equations and Applications*, **3**, No. 4 (2000), 7-18.

[5] D. Bainov, A. Dishliev, I. Stamova, Lipschitz quasistability of impulsive differential-difference equations with variable inpulsive perturbations, *J. of Computational and Applied Mathematics*, **70**, No. 2 (1996), 267-277.

[6] D. Bainov, S. Nenov, Limit sets of impulsive dynamical systems, In: *Proceedings of the Fourth International Colloquium on Differential Equations* (1994), 31-34.

[7] M. Branicky, Multiple Lyapunov functions and other analysis tools for switched and hybrid systems, *IEEE Transactions on Automatic Control*, **43**, No. 4 (1998), 475-482.

[8] R. Chukleva, A. Dishliev, K. Dishlieva, Continuous dependence of the solutions of the differential equations with variable structure and impulses in respect of switching functions, *International Journal of Applied Science and Technology*, **1**, No. 5 (2011), 46-59.

[9] DeR. Carlo, S. Zak, G. Matthews, Variable structure control of nonlinear multivariable systems: A tutorial, *Proceedings on the IEEE*, **76**, No. 3 (1988), 212-232.

[10] H. Dimov, S. Nenov, Some features of uncontinuable solutions of impulsive dynamical systems, *Extracta Mathematicae*, **11**, No. 3 (1996), 443-456.

[11] A. Dishliev, D. Bainov, Conditions for the absence of the phenomenon "beating" for systems of impulse differential equations, *Bulletin of the Institute of Mathematics Academia Sinica*, **13**, No. 3 (1985), 237-256.

[12] A. Dishliev, D. Bainov, Continuous dependence of the solution of a system of differential equations with impulses on the impulse hypersurfaces, *J. of Math. Analysis and Applications*, **135**, No. 2 (1988), 369-382.

[13] A. Dishliev, D. Bainov, Continuous dependence on initial condition and a parameter of a class of differential equations with variable structure and impulses, *J. International Systems Science*, **22**, No. 4 (1991), 641-658.

[14] A. Dishliev, D. Bainov, Dependence upon initial conditions and parameters of solutions of impulsive differential equations with variable structure, *J. International of Theoretical Physics*, **29**, No. 6 (1990), 655-675.

[15] A. Dishliev, D. Bainov, Sufficient conditions for absence of "beating"in systems of differential equations with impulses, *Applicable Analysis*, **18**, No-s: 1, 2 (1984), 67-73.

[16] A. Dishliev, K. Dishlieva, Continuous dependence and stability of solutions of impulsive differential equations on the initial conditions and impulsive moments, *International Journal of Pure and Applied Mathematics*, **70**, No. 1 (2011), 39-64.

[17] A. Dishliev, K. Dishlieva, Continuous dependence of the solutions of differential equations under "short" perturbations on the right hand side, *Communications in Applied Analysis*, **2** (2006), 1-13.

[18] A. Dishliev, K. Dishlieva, Orbital Hausdorff continuous dependence of the solutions of impulsive differential equations with respect to impulsive perturbations, *International Journal of Pure and Applied Mathematics*, **70**, No. 2 (2011), 167-187.

[19] A. Dishliev, K. Dishlieva, S. Nenov, *Specific Asymptotic Properties of the Solutions of Impulsive Differential Equations. Methods and Applications*, Academic Publications (2011), To Appear.

[20] K. Dishlieva, Continuous dependence of the solutions of impulsive differential equations on the initial conditions and barrier curves, *Acta Mathematica Scientia* (2011), To Appear.

[21] K. Dishlieva, Differentiability of solutions of impulsive differential equations with respect to the impulsive perturbations, *Nonlinear Analysis Series B: Real World Applications*, **12**, No. 6 (2011), 3541-3551.

[22] A. Filippov, *Differential Equations with Discontinuous Right-Hand Sides*, Kluwer Academic Publishers, Dordrecht (1988).

[23] W. Gao, J. Hung, Variable structure control of nonlinear systems: a new approach, *Industrial Electronics, IEEE Transactions*, **40**, No. 1 (1993), 45-55.

[24] J. Hung, W. Gao, J. Hung, Variable structure control: A survey, *Industrial Electronics, IEEE Transactions*, **40**, No. 1 (1993), 2-22.

[25] V. Lakshmikantham, D. Bainov, P. Simeonov, *Theory of Impulsive Differential Equations*, World Scientific, Singapore-New Jersey-London (1989).

[26] H.-Y. Lu, K. Wang, A class of autonomous single-species models and their optimal harvesting policies, *Journal of Systems Sciences and Mathematical Sciences*, **30**, No. 1 (2010), 33-42.

[27] S. Nenov, Impulsive controllability and optimizations problems in population dynamics, *Nonlinear Analysis*, **36**, No. 7 (1999), 881-890.

[28] L. Nie, Z. Teng, L. Hu, J. Peng, The dynamics of a Lotka-Volterra predator-prey model with state dependent impulsive harvest for predator, *BioSystems*, **98**, No. 2 (2009), 67-72.

[29] B. Paden, S. Sastry, A calculus for computing Filippov's differential inclusion with application to the variable structure control of robot manipulators, *Circuits and Systems, IEEE Transactions*, **34**, No. 1 (1987), 73-82.

[30] D. Shevitz, B. Paden, Lyapunov stability theory of nonsmooth systems, *Automatic Control, IEEE Transactions*, **39**, No. 9 (1994), 1910-1914.

[31] Z. Shuai, L. Bai, K. Wang, Optimization problems for general simple population with n-impulsive harvest, *J. of Mathematical Analysis and Applications*, **329**, No. 1 (2007), 634-646.

[32] G. Stamov, I. Stamova, Almost periodic solutions for impulsive neural networks with delay, *Applied Math. Modeling*, **31**, No. 7 (2007), 1263-1270.

[33] I. Stamova, *Stability Analysis of Impulsive Functional Differential Equations*, Walter de Gruyter, Berlin-New York (2009).

[34] S. Tang, R. Cheke, Y. Xiao, Optimal impulsive harvesting on non-autonomous Beverton-Holt difference equations, *Nonlinear Analysis*, **65**, No. 12 (2006), 2311-2341.

[35] V. Utkin, Variable structure systems with sliding modes, *Automatic Control, IEEE Transactions*, **22**, No. 2 (1977), 212-222.

[36] H. Wang, E. Feng, Z. Xiu, Optimality control of the nonlinear impulsive system in fed-batch fermentation, *Nonlinear Analysis: Theory, Methods & Applications*, **68**, No. 1 (2008), 12-23.

[37] S. Wu, X. Meng, Boundedness of nonlinear differential systems with impulsive effect on random moments, *Acta Mathematicae Applicatae Sinica*, **20**, No. 1 (2004), 147-154.

[38] S. Wu, The Euler scheme for random impulsive differential equations, *Applied Mathematics and Computation*, **191**, No. 1 (2007), 164-175.

[39] L. Zhao, Q. Zhang, Q. Yang, Chaos control of a ratio-dependent food chain model, *International Journal of Information and Systems Sciences*, **5**, No-s: 3, 4 (2009), 400-411.