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DIRICHLET PROBLEM IN INFINITE NETWORKS

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Abstract: Functions on infinite graphs are studied, keeping infinite electrical grids and Markov chains as models. The techniques used for this study are based on those from the classical and axiomatic potential theory and the analytic function theory on the complex plane and Riemann surfaces. This paper investigates Dirichlet problem in the context of an infinite network.

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1. Introduction

Graph theory generally deals with a finite collection of objects and their interconnections. Many problems in transport, telecommunication, data structures, methods of optimisation, business etc. can be formulated as problems in finite graph theory. However, infinite graphs cannot be ignored. For example, because of our inability to find the exact solutions of many differential equations, we try to obtain their approximate solutions by various methods. One such method is to solve partial differential equations by the finite difference approximation, which involves horizontal and vertical displacements. This in effect looks like an electrical grid which is a graph. However, for some differential equations, like wave equations, the domain of existence of the solution may be unbounded, suggesting a problem in a graph with infinite vertices and consequently a need to consider situations like infinite electrical grids.

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The first thing we notice about an infinite electrical network is that it is not just a graph and its geometrical properties, but on the basic structure of a graph there is a *voltage-current regime* subject to Ohm's law and Kirchhoff's voltage and current laws. Thus, one is interested here in functions defined on nodes and edges satisfying certain conditions. For the development of the current article, we shall treat an infinite electrical network as a basic example.

We mentioned above the connection between differential equations and infinite electrical grids (see Zemanian [12]). A similar connection exists between infinite electrical networks and Markov chains (see Woess [10]). A *Markov chain* consists of a countable number of vertices X provided with a *transition probability* $p(x, y)$ and the *Markov property* (which basically states that given the present, the past and the future are independent, and this property of independence, expressed mathematically in the context of an infinite network, characterizes a Markov chain which is a useful theory for the industrial production managers, the government planning commissioners etc.). We shall write $x \sim y$ if and only if $p(x, y) > 0$. Note the constraint that $\sum_{y \in X} p(x, y) = 1$ for any x in X, assuming $p(x, x) = 0$.

In the context of a finite electrical network, Bendito et al. [5] studies conductance, effective resistance, capacities, equilibrium measures etc. These notions have their obvious counterparts in the study of Newtonian potentials. There is also a parallel study of these notions with probabilistic interpretations. For example, the effective resistance in an electrical network has a close relation to the escape probability for a reversible Markov chain (Ponzio [8] and Tetali [9]) which is characterized by the transition probability from one state to another, given the similarity between the conductance and the transition probability.

2. Preliminaries

By an *infinite network* (see Yamasaki [11]), we mean a triple (X, Y, t) where X is a countable collection of vertices; Y is a countable number of edges, each edge $[x, y]$ joining two vertices x and y; and for any pair of vertices x and y is associated a number $t(x, y) = t(y, x) \geq 0$ such that $t(x, y) > 0$ if and only if there exists an edge $[x, y]$ joining x and y. We say that x and y are neighbours (write $x \sim y$) if there is an edge $[x, y]$ joining x and y. For convenience, we denote the infinite network simply by writing X.

By a path joining x and y, we mean a collection $\{x = x_0, x_1, x_2, \ldots, x_n = y\}$ of vertices where $x_i \sim x_{i+1}$ for $0 \leq i \leq n$. Then the length of the path is n. There may be many such paths joining a given pair of vertices x and y with different lengths, the minimum of which is called the distance $d(x, y)$ between x and y. We usually fix a vertex e, and for any vertex x write $|x| = d(e, x)$ which measures the distance of x from e. Write $B_n = \{x : |x| \leq n\}$ and $S_n = \{x : |x| = n\}$. We assume that the network is *connected* (that is, for any two vertices x and y there is at least one path joining them), *locally finite* (that is, any vertex has only finitely many neighbours), and *without self-loops* (that is, no edge of the form $[x, x]$). Consequently, if $t(x) = \sum t(x, y)$, then the summation is over only a finite number of terms, $y \sim x$, and $t(x) > 0$ for every vertex x. Hence, if we write $p(x, y) = \frac{t(x,y)}{t(x)}$ then $p(x, y) > 0$ if and only if $x \sim y$ and $\sum_{y \in X} p(x, y) = 1$ for any vertex x (taking $p(x, x) = 0$). An infinite network is said to be a *tree* if there is *no circuit* in X, that is there is no path in X of the form $\{a = x_0, x_1, \ldots, x_n = a\}$with $n \geq 3$. Cartier [7] studies in detail harmonic functions, potentials, the Green's function, Martin compactification etc. in a tree provided with a system of transient transition probabilities. This paper also is very relevant for the development of the present article.

Given a subset E of X, a vertex x is said to be an interior point of E if and only if x and all its neighbours are in E. Denote by $\overset{o}{E}$ the set of all the interior points of E; write $\partial E = E \backslash \overset{o}{E}$ called the boundary of E. Note that $E = \overset{o}{E}$ if and only if $E = X$. A set E is said to be *circled* if and only if every vertex in ∂E has at least one neighbour in $\overset{o}{E}$. Let us denote by $V(E)$ the set of all vertices z in X such that either z is in E or a neighbour of z is in E. In particular, $V(x)$ is the set consisting of x and all its neighbours. Note that E is contained in the interior of $V(E)$.

3. Harmonic and Superharmonic Functions on an Infinite Network

In a finite electrical network, suppose v is the electrostatic potential function due to the electrical charges; then the electrical field $\overrightarrow{E} = -\nabla v$ and the total electric charge at a node z is $\nabla . \overrightarrow{E}(z) = (-\Delta)v(z)$. Now in the case of a finite network, we can calculate the total electric charge at a node by using the Ohm's law which gives the total electric charge at z as $\sum_i C_i[v(z) - v(x_i)]$; here each x_i is a node connected to z by a branch $[x_i, z]$ whose conductance is C_i. Thus we write

$$\Delta v(z) = \sum_i C_i[v(x_i) - v(z)].$$

If the charge per unit volume is denoted by ρ then the differential form of the Gauss Flux Theorem is $\nabla.(\varepsilon_0 \overrightarrow{E}) = \rho$. That is, $\Delta v = \nabla^2 v = -\frac{\rho}{\varepsilon_0}$, which is known as the Poisson equation. If $\rho = 0$ then $\Delta v = 0$ which is known as the *Laplace equation* and in this case v is called a harmonic function. Note that the Poisson equation $\Delta v = -f$ can be represented as $\Delta v = f^{-} - f^{+}$. Suppose we have found the solutions v_1 and v_2 for the equations $\Delta v_1 = -f^{+}$ and $\Delta v_2 = f^{-}$, then $\Delta(v - v_1 - v_2) = 0$, so that $v = v_1 + v_2$ up to an additive harmonic function. Thus, it is of importance to know how to solve an inequality of the form $\Delta u \leq 0$. Considering the importance of the Laplace operator in a finite electrical network, it is of interest to introduce such an operator in the abstract setting of an infinite network.

Let u be a real-valued function on X. Write

$$\Delta u(x) = \sum_{x_i \in X} t(x, x_i)[u(x_i) - u(x)].$$

Note that this is a finite summation, since $t(x, x_i) = 0$ if x_i is not a neighbour of x. The introduction of the Laplace operator Δ in the framework of an infinite network is very fruitful. Recall that the Laplace operator plays a very important role in the study of (Newton) potential theory and also in the theory of analytic functions in the complex plane. Consequently, the development of the theory of functions in an infinite network will be guided not only by the problems in infinite electrical grids and in Markov chains, but also by those in the classical and the axiomatic potential theory and in Riemannian manifolds.

Definition 1. Let u be a real-valued function defined on a subset E of X. u is said to be harmonic (respectively superharmonic) on E if and only if $\Delta u = 0$ (respectively $\Delta u \leq 0$) at every vertex x in $\overset{o}{E}$; a real-valued function v is said to be subharmonic on E if and only if $-v$ is superharmonic on E.

Now we introduce some properties of superharmonic functions.

Lemma 2. *Let u_1 and u_2 be two superharmonic functions on a subset E. Then $\alpha_1 u_1 + \alpha_2 u_2$, where $\alpha_1 \geq 0$ and $\alpha_2 \geq 0$, is superharmonic on E; so is* $\inf(u_1, u_2)$.

Lemma 3. (Harnack Property) *Let E be a set such that $\overset{o}{E}$ is connected. Let a and b be two vertices in $\overset{o}{E}$. Then, for any superharmonic function $s > 0$ on E, $\alpha s(a) \leq s(b) \leq \beta s(a)$, where α and β are two positive numbers independent of the superharmonic function s.*

Proof. Let $\{a = a_0, a_1, \ldots, a_m = b\}$ be a path connecting a and b in $\overset{o}{E}$. Let

$s > 0$ be a superharmonic function on E. Since $\Delta s(a) \leq 0$ and $a \sim a_1$, we have

$$t(a)s(a) \geq \sum_{x_i} t(a, x_i)s(x_i) \geq t(a, a_1)s(a_1).$$

Since $\Delta s(a_1) \leq 0$, we obtain analogously $t(a_1)s(a_1) \geq t(a_1, a_2)s(a_2)$. Proceeding similarly, we arrive at the inequality

$$t(a)t(a_1)\ldots t(a_{m-1})s(a) \geq t(a, a_1)t(a_1, a_2)\ldots t(a_{m-1}, b)s(b),$$

which can be written as $\beta s(a) \geq s(b)$. The other inequality is obtained similarly. □

Consequence. *Let E be a circled set (that is, every $z \in \partial E$ has a neighbour in $\overset{o}{E}$) such that $\overset{o}{E}$ is connected. Let s_n be an increasing sequence of superharmonic (respectively harmonic) functions on E. Suppose* $\sup_n s_n(a) < \infty$ *for some $a \in \overset{o}{E}$ Then, $s(x) = \sup_n s_n(x)$, $x \in E$, is finite and superharmonic (respectively harmonic) on E.*

Proof. Let $b \in \overset{o}{E}$ be arbitrary. Then $s_n(b) \leq \beta s_n(a)$ for every n. Hence $s(b) \leq \beta s(a) < \infty$. Let $z \in \partial E$. Since E is circled, there is some $b \in \overset{o}{E}$ such that $z \sim b$. Since $t(b)s_n(b) \geq \sum_{x_i} t(b, x_i)s_n(x_i) \geq t(b, z)s_n(z)$, taking limits we conclude that $s(z) < \infty$. Thus s is finite at every vertex of E.

To show that s is superharmonic on E, note that for $b \in \overset{o}{E}$, $\Delta s_n(b) \leq 0$ which implies that $t(b)s_n(b) \geq \sum_x t(b, x)s_n(x)$. Hence, taking limits as $n \to \infty$, we conclude that $t(b)s(b) \geq \sum_x t(b, x)s(x)$. That is, $s(x)$ is superharmonic at b. The vertex b being arbitrary in $\overset{o}{E}$ we conclude that s is superharmonic on E.

If each s_n is harmonic on E, a similar argument shows that $s(x)$ is harmonic on E. □

Corollary 4. *Let E be a circled set such that $\overset{o}{E}$ is connected. Let $s \geq 0$ be superharmonic on E. Suppose $s(a) = 0$ for some $a \in \overset{o}{E}$. Then $s \equiv 0$ on E.*

Proof. Let $s_n = ns$, so that s_n is superharmonic on E and the sequence s_n is increasing and $\lim_n s_n(a) = 0$. Hence, $\lim_n ns$ is finite and superharmonic on E. This is possible only if $s \equiv 0$ on E. □

Lemma 5. (Minimum Principle) *Let u be a superharmonic function defined on a finite set F. Then, $\inf_{\partial F} u = \inf_F u$.*

Proof. Let $\alpha = \inf_{\partial F} u$ and $\beta = \inf_F u$. Then, $\beta \le \alpha$. Suppose $\beta < \alpha$. Then, for some $z \in \overset{o}{F}$, $u(z) = \beta$. Choose $y \notin F$. Since X is connected, there is a path $\{z = z_0, z_1, \ldots, z_n = y\}$ connecting z and y. Let i be the index such that $z_j \in \overset{o}{F}$ if $j < i$ and $z_i \notin \overset{o}{F}$. Then $1 \le i \le n$, and since $z_{i-1} \in \overset{o}{F}$ and $z_{i-1} \sim z_i$, we find that $z_i \in \partial F$, so that $u(z_i) \ge \alpha$.

Now $\Delta u(z) \le 0$ means that

$$0 \ge \Delta u(z) = \sum t(z,x)[u(x) - u(z)] = \left[\sum t(z,x)u(x)\right] - t(z)u(z).$$

Hence, $t(z)\beta = t(z)u(z) \ge \sum t(z,x)u(x) \ge \sum t(z,x)\beta = t(z)\beta$. This means that $u(x) = \beta$ for every $x \sim z$. In particular, $u(z_1) = \beta$. Proceeding similarly, we find that $u(z_k) = \beta$ for $k \le i$. In particular, $u(z_i) = \beta$; but this is a contradiction since $u(z_i) \ge \alpha > \beta$. Hence the assumption that $\beta < \alpha$ is false. Thus, the stated minimum principle is established. □

Lemma 6. (Poisson Modification) *For a real-valued function u on X, and a vertex $a \in X$, let us write $P_a u$, the discrete analogue of the Poisson integral, as $P_a u(x) = u(x)$ if $x \ne a$, and $P_a u(a) = \sum \frac{t(a,z)}{t(a)} u(z)$, and refer to $P_a u$ as the Poisson modification of u at the vertex a.*

Assume that u is superharmonic on a subset E of X. Let $a \in \overset{o}{E}$. Then $P_a u$ is superharmonic on E, harmonic at a and $P_a u(x) \le u(x)$ on E.

Proof. Since u is superharmonic at a, $P_a u(a) \le u(a)$. For $x \notin V(a)$, we have $P_a u = u$ on $E \cap V(x)$, so that $\Delta P_a u(x) = \Delta u(x) \le 0$ at each $x \in \overset{o}{E} \setminus V(a)$. In case $x \in \overset{o}{E} \cap V(a)$, we have for $x \ne a$,

$$\begin{aligned} \Delta P_a u(x) &= -t(x)P_a u(x) + \sum t(x,z)P_a u(z) \\ &\le -t(x)u(x) + \sum t(x,z)u(z) = \Delta u(x) \le 0. \end{aligned}$$

For $x = a$, we have $\Delta P_a u(a) = -t(a)P_a u(a) + \sum t(a,z)u(z) = 0$. □

Theorem 7. (Perron Family) *A non-empty subset $\mathcal{F}$ of superharmonic functions on E is said to be a Perron family if it satisfies the following conditions.*

1. *For any $v_1, v_2 \in \mathcal{F}$, there exists $v \in \mathcal{F}$ such that $v \le \min(v_1, v_2)$;*

2. *$P_a u \in \mathcal{F}$ for every $u \in \mathcal{F}$ and $a \in \overset{o}{E}$;*

3. *There exists a real-valued function u_0 on E such that $v \ge u_0$ for all $v \in \mathcal{F}$.*

If $\mathcal{F}$ is a Perron family, then $h(x) = \inf\{v(x) : v \in \mathcal{F}\}$ is harmonic on E.

Proof. By c), we have $h(x) \geq u_0(x)$ on E. Let $a \in \overset{o}{E}$ be fixed arbitrarily. Since $V(a)$ is a finite set, we can find a sequence $\{v_x^{(n)}\}$ in $\mathcal{F}$ for every $x \in V(a)$ such that $v_x^{(n)} \to h(x)$ for $n \to \infty$. By a), there exists $u_n \in \mathcal{F}$ such that $u_n \leq \min\{v_x^{(n)} : x \in V(a)\}$. Then $u_n(x) \to h(x)$ as $n \to \infty$ for every $x \in V(a)$. Let $u_n^\star = P_a u_n$. Then $u_n^\star \in \mathcal{F}$ and $u_n^\star \to h(x)$ as $n \to \infty$ for every $x \in V(a)$ and $u_n^\star$ is harmonic at a. Consequently, $\Delta h(a) = \lim_{n\to\infty} \Delta u_n^\star = 0$. Thus h is harmonic at a. □

Theorem 8. (The Greatest Harmonic Minorant (g.h.m.)) *Let u be superharmonic and v be subharmonic on a subset E of X, such that $v \leq u$ on E. Then there exists a harmonic function h on E such that $v \leq h \leq u$ on E, and if h' is any harmonic function on E such that $v \leq h' \leq u$ on E, then $h' \leq h$. (h is referred to as the greatest harmonic minorant of u on E.)*

Proof. Let $\mathcal{F}$ be the family of all subharmonic functions q on E such that $q \leq u$ on E. If $a \in \overset{o}{E}$ then $P_a q(a) = \sum t(a,z)q(z) \leq \sum t(a,z)u(z) \leq t(a)u(a)$, and if $a \neq x \in E$, then $P_a q(x) = q(x) \leq u(x)$. Also, $P_a q(x)$ is subharmonic on E and $P_a q(x) \geq q(x)$. Consequently, it is easy to see that $\mathcal{F}$ is a Perron family. Let $h = \sup_{\mathcal{F}} q$. Then, h is harmonic on E, and $v \leq h \leq u$ since $v \in \mathcal{F}$ If h' is another such harmonic function between v and u, then $h' \in \mathcal{F}$ so that $h' \leq h$ on E. □

4. Dirichlet Solution and Condenser Principle

The Dirichlet problem can be seen to play an important role in the study of functions on infinite networks. We can deduce some important results like balayage, equilibrium principle, domination principle etc. as solutions to some suitable Dirichlet problems (See [1, 2]). In this section, we obtain the solution to a generalized form of the Dirichlet problem and as an example deduce the condenser principle.

Theorem 9. *Let E be an arbitrary set in X and $F = V(E)$. Let f be a real-valued function on $F \setminus E$. Suppose u is superharmonic on F and v is subharmonic on F such that $v \leq u$ on F and $v \leq f \leq u$ on $F \setminus E$. Then there exists a function h on F such that $v \leq h \leq u$ on F, $h = f$ on $F \setminus E$ and $\Delta h = 0$ at every vertex of E.*

Proof. Let $\mathcal{F}$ be the family of all real-valued functions ϕ on F such that $\phi \leq u$ on F, $\Delta\phi \geq 0$ on E, and $\phi = f$ on $F \setminus E$.

Consider the function v_1 on F such that $v_1 = v$ on E and $v_1 = f \geq v$ on $F \setminus E$. Note that $v_1 \leq u$ on F. If $z \in E$, then

$$\begin{aligned} t(z)v_1(z) &= t(z)v(z) \leq \sum_y t(z,y)v(y) \text{ (since } \Delta v \geq 0 \text{ on } E) \\ &\leq \sum_y t(z,y)v_1(y) \text{ (since } v \leq v_1), \end{aligned}$$

and hence, $\Delta v_1(z) \geq 0$ if $z \in E$. Consequently, $v_1 \in \mathcal{F}$.

Let $h(x) = \sup_{\phi\in\mathcal{F}} \phi(x)$ for every $x \in F$, so that h is subharmonic at every vertex of E and $v \leq h \leq u$ on F. For any $z \in E$, and any $\phi \in \mathcal{F}$, denote by ϕ_1 the function $\phi_1(x) = \phi(x)$ if $x \neq z$, and $t(z)\phi_1(z) = \sum_y t(z,y)\phi(y)$. Then, ϕ_1 is subharmonic at every vertex of E, and $\phi_1(z) \leq u(z)$ so that $\phi_1 \leq u$ on F, ϕ_1 is harmonic at the vertex z, $\phi \leq \phi_1$ on F and $\phi_1 = f$ on $F \setminus E$. Consequently $\phi_1 \in \mathcal{F}$ so that $h(x) = \sup_{\phi\in\mathcal{F}} \phi(x) = \sup_{\phi_1\in\mathcal{F}} \phi_1(x)$. Thus $\mathcal{F}$ is a Perron family on E, and hence $h(x)$ is harmonic at every vertex of E. This proves the theorem. □

Corollary 10. *(Classical Dirichlet problem in a network) Let E be a finite set in X. Let f be a real-valued function on ∂E. Then there exists a unique harmonic function h on E such that $h = f$ on ∂E.*

Proof. Let $F = V(\overset{o}{E})$. If $z \in \overset{o}{E}$, then all the neighbours of z are in E by definition. Hence $F \subset E$. Consequently $\overset{o}{F} \subset \overset{o}{E}$, but $\overset{o}{E} \subset \overset{o}{F}$ so that $\overset{o}{F} = \overset{o}{E}$. Thus $\overset{o}{F} \cup \partial F = F \subset E = \overset{o}{E} \cup \partial E$, so that $\partial F \subset \partial E$, and $F \setminus \overset{o}{E} = F \setminus \overset{o}{F} = \partial F \subset \partial E$. Now, $F \setminus \overset{o}{E}$ being finite, f is bounded on $F \setminus \overset{o}{E}$. Hence by applying the above theorem, we obtain a real-valued function h on F such that $h = f$ on $F \setminus \overset{o}{E}$, and $\Delta h = 0$ at every vertex of $\overset{o}{E}$. Extend h by f on $\partial E \setminus \partial F$. Then h is a bounded function on E, $h = f$ on ∂E and $\Delta h = 0$ at every vertex of $\overset{o}{E}$.

The uniqueness of h follows from the minimum principle on a finite set. □

Condenser Principle: A perfect *conductor* is a medium in which the electric intensity is zero. Consequently, the potential is a constant on such a conductor. Suppose now there are two conductors in isolation (called a *condenser*). In this situation the relation between the potential and the charges on the conductors is significant. This is expressed as the *condenser principle* in the following theorem which is, in fact, a corollary to the above theorem.

Theorem 11. *Let A and B be two disjoint sets. Then there exists a function v on X such that*

1. $0 \leq v \leq 1$ *on* X,
2. $v = 1$ *on* A *and* $\Delta v \leq 0$ *on* A,
3. $v = 0$ *on* B *and* $\Delta v \geq 0$ *on* B, *and*
4. $\Delta v = 0$ *on* $X \setminus (A \cup B)$.

Proof. Let $F = V(E)$, where $E = (A \cup B)^c$. Let $f = 1$ on $(F \setminus E) \cap A$, and $f = 0$ on $(F \setminus E) \cap B$. Let ϕ be the function (the Dirichlet solution) such that $\phi = f$ on $F \setminus E$ and $\Delta \phi = 0$ at every vertex of E. Extend ϕ by 1 on A and by 0 on B. Let the thus extended function be denoted by v. Then v satisfies the conditions stated in the theorem. □

Balayage: An important problem in Newton potential theory (and a similar problem in electricity) is as follows: Start with a mass distribution on $\mathbf{R}^3$, to which is associated a certain potential function. Let E be a given subset of $\mathbf{R}^3$. How to redistribute the mass so that there is no mass on E but the potential outside E remains unchanged? The solution to this problem depends on the Poincaré's method of sweeping-out the mass from E onto its boundary. This process is called balayage. Here we show that the balayage in the discrete case can be presented as a Dirichlet solution.

Theorem 12. *Let E be an arbitrary set in a network X. Let $F = V(E)$. Let u be a superharmonic function on X, lower bounded on F. Then there exists a superharmonic function s on X such that $\Delta s = 0$ at every vertex of E and $s = u$ outside E.*

Proof. Suppose $u \geq \alpha$ on F. Take $f = u$ on $F \setminus E$. Then $\alpha \leq f \leq u$ on $F \setminus E$. Then by Theorem 9, there exists h on F such that $h = f$ on $F \setminus E$ and $\Delta h = 0$ on E. Extend h by u outside F. Thus extended function s is the balayage function of u. □

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