International Journal of Pure and Applied Mathematics
Volume 72 No. 3 2011, 375-383

MATCHING NUMBER AND EDGE COVERING NUMBER ON KRONECKER PRODUCT OF C_n

Thanin Sitthiwirattham

Department of Mathematics
Faculty of Applied Science
King Mongkut's University of Technology
North Bangkok, Bangkok, 10800, THAILAND
Centre of Excellence in Mathematics, CHE
Sri Ayutthaya Road, Bangkok, 10400, THAILAND

Abstract: Let $\alpha'(G)$ and $\beta'(G)$ be the matching number and edge covering number, respectively. The Kronecker Product $G_1 \otimes G_2$ of graph G_1 and G_2 has vertex set $V(G_1 \otimes G_2) = V(G_1) \times V(G_2)$ and edge set $E(G_1 \otimes G_2) = \{(u_1v_1)(u_2v_2)|u_1u_2 \in E(G_1)$ and $v_1v_2 \in E(G_2)\}$. In this paper, let G is a simple graph with order m, we prove that

$$\alpha'(C_n \otimes G) = \max\{n\alpha'(G), m\lfloor \frac{n}{2} \rfloor\} \text{ and } \beta'(C_n \otimes G) = \min\{n\beta'(G), m\lceil \frac{n}{2} \rceil\}.$$

AMS Subject Classification: 05C69, 05C70, 05C76
Key Words: Kronecker product, matching number, edge covering number

1. Introduction

In this paper, graphs must be simple graphs which can be trivial graph. Let G_1 and G_2 be graphs. The Kronecker product of graph G_1 and G_2, denote by $G_1 \otimes G_2$, be the graph that $V(G_1 \otimes G_2) = V(G_1) \times V(G_2)$ and $E(G_1 \otimes G_2) = \{(u_1v_1)(u_2v_2)|u_1u_2 \in E(G_1)$ and $v_1v_2 \in E(G_2)\}$.

Next, we give the definitions about some graph parameters. A subset of the edge set E of G is said to be matching or an independent edge set of G, if no two distinct edges in M have a common vertex. A matching M is maximum matching in G if there is no matching M' of G with $|M'| > |M|$. The cardinality

Received: July 7, 2011

of maximum matching of G is called the matching number of G, denoted by $\alpha'(G)$.

An edge of graph G is said to cover the two vertices incident with it, and an edge cover of a graph G is a set of edges covering all the vertices of G. The minimum cardinality of an edge cover of a graph G is called the edge covering number of G, denoted by $\beta'(G)$.

By definitions of matching number , edge covering number, clearly that $\alpha'(C_n) = \lfloor \frac{n}{2} \rfloor$ and $\beta'(C_n) = \lceil \frac{n}{2} \rceil$.

In [1], there are some properties about Kronecker product of graph. We recall here.

Proposition 1. Let $H = G_1 \otimes G_2 = (V(H), E(H))$ then:

(i) $n(V(H)) = n(V(G_1)n(V(G_2)$;

(ii) $n(E(H)) = 2n(E(G_1)n(E(G_2)$;

(iii) for every $(u, v) \in V(H), d_H((u, v)) = d_{G_1}(u)d_{G_2}(v)$.

Note that for any graph G, we have $G_1 \otimes G_2 \cong G_2 \otimes G_1$

Theorem 2. *Let G_1 and G_2 be connected graphs, The graph $H = G_1 \otimes G_2$ is connected if and only if G_1 or G_2 contains an odd cycle.*

Theorem 3. *Let G_1 and G_2 be connected graphs with no odd cycle then $G_1 \otimes G_2$ has exactly two connected components.*

Next we get that general form of graph of Kronecker Product of C_n and a simple graph.

Proposition 4. Let G be connected graph order m, the graph of $C_n \otimes G$ *is*

$$(\bigcup_{i=1}^{n-1} H_i) \cup H_n$$

where $V(H_i)$=$W_i \cup W_{i+1}$ for $i = 1, 2, ..., n-1$; $W_i = \{(i, 1), (i, 2), ..., (i, m)\}$; $E(H_i) = \{(i, u)(i+1, v)/uv \in E(G)\}$ and $V(H_n) = W_n \cup W_{n+1}$; $E(H_n) = \{(n, u)/uv \in E(G)\}$ Moreover, if G has no odd cycle then for each H_1 and H_n has exactly two connected components isomorphic to G.

Example.

2. Matching Number of the Graph of $C_n \otimes G$

We begin this section by giving the definition and theorem for alternating path and augmenting path, Lemma 7 that show character of matching for each H_i.

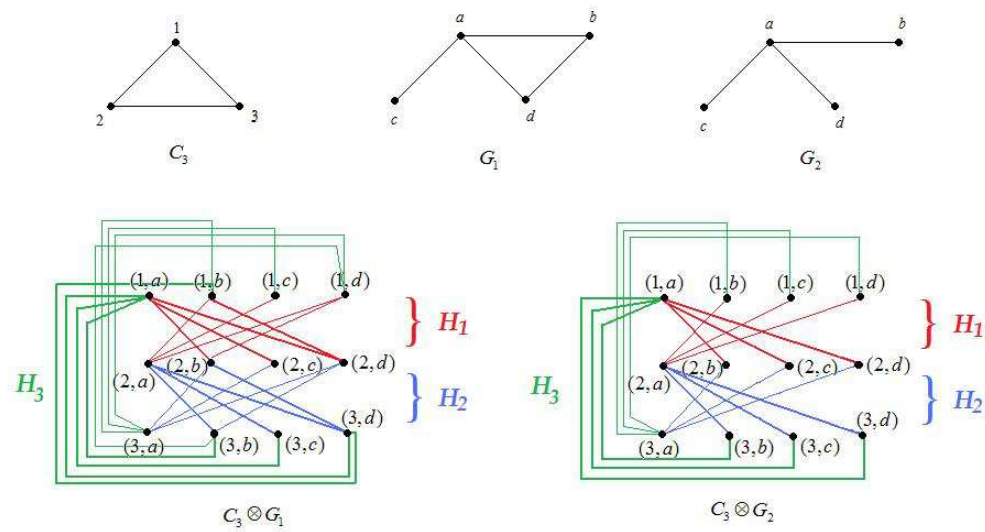

Figure 1: The graph of $C_3 \otimes G_1$ and $C_3 \otimes G_2$

Definition 5. Given a matching M, an M-alternating path is a path that alternates between edges in M and edges not in M. An M-alternating path whose endpoints are unsaturated by M is an M-augmenting path.

Theorem 6. *A matching M in a graph G is a maximum matching in G if and only if G has no M-augmenting path.*

Next, we giving Lemma 7 which show character of matching for each H_i.

Lemma 7. Let $C_n \otimes G = (\bigcup_{i=1}^{n-1} H_i) \cup H_n$. For each H_i and H_n, then $\alpha'(H_i) = \alpha'(H_n) = 2\alpha'(G)$.

Proof. Suppose G has no odd cycle, by proposition 1.4 we get H_i=$2G$. So $\alpha'(H_i) = 2\alpha'(G)$. If G has odd cycle, for each H_i , vertex $(u_i, v) \in W_i$ and $(u_{i+1}, v) \in W_{i+1}$ have $d_{H_i}((u_i, v)) = d_{H_i}(u_{i+1}, v)) = d_G(v)$. Let $\bigcup_{i=1}^{n-1} \overline{H_i} = C_n \otimes (G - \overline{e})$ when $\overline{e}$ is an edge in odd cycle, M be the maximum matching of

G . We get $\overline{H_i} = 2(G - \overline{e})$ then

$$\alpha'(\overline{H_i}) = 2\alpha'(G - \overline{e}) = \begin{cases} 2[\alpha'(G) - 1], & \text{if } \overline{e} \text{ is in } M, \\ 2\alpha'(G), & \text{otherwise.} \end{cases}$$

When we add $\overline{e}$ comeback, we get $\alpha'(H_i) = \alpha'(\overline{H_i}) + 1$. Hence $\alpha'(H_i) = 2\alpha' G$. Similarly, $\alpha'(H_n) = 2\alpha' G$. ☐

Next, we establish Theorem 8 for a matching number of $C_n \otimes G$

Theorem 8. *Let G be connected graph order m, then*

$$\alpha'(C_n \otimes G) = \max\{n\alpha'(G), m\lfloor\frac{n}{2}\rfloor\}.$$

Proof. Let $V(C_n) = \{u_i, i = 1, 2, ..., n\}$, $V(G) = \{v_j, j = 1, 2, ..., m\}$, $S_i = \{(u_i, v_j) \in V(C_n \otimes G)/j = 1, 2, ..., m\}, i = 1, 2, ..., n$ and since $\alpha'(C_n) = \lfloor\frac{n}{2}\rfloor$.

Let $\alpha'(G) = k$, assume that the maximum matching of C_n, G be

$$M_1 = \{u_1u_2, u_3u_4, ..., u_{2\lfloor\frac{n}{2}\rfloor-1}u_{2\lfloor\frac{n}{2}\rfloor}\},$$

$$M_2 = \{v_jv_{j+1}/j = 1, 3, ..., 2k - 1\},$$

respectively.

By Lemma 2.2 we have $\alpha'(H_i) = 2\alpha'(G)$. Since $C_n \otimes G$ is $(\bigcup_{i=1}^{n-1} H_i) \cup H_n$ which have matching in $H_1, H_3, ..., H_{2\lfloor\frac{n}{2}\rfloor-1}$, then $\alpha'(C_n \otimes G) \geq n\alpha'(G)$.

By definition of matching, we get another matching of $C_n \otimes G$ be set of edges such that incident with vertices in S_i and S_{i+1}, $i = 1, 3, ..., 2\lfloor\frac{n}{2}\rfloor - 1$. So $\alpha'(C_n \otimes G) \geq m\lfloor\frac{n}{2}\rfloor$.

Hence $\alpha'(C_n \otimes G) \geq \max\{n\alpha'(G), m\lfloor\frac{n}{2}\rfloor\}$.

If $n\alpha'(G) > m\lfloor\frac{n}{2}\rfloor$, suppose that $\alpha'(C_n \otimes G) > n\alpha'(G)$, then there exist a matching M is a augmenting path. That is not true because each vertices in $C_n \otimes G$ always incident with edges in

$$M = [\bigcup_{i=1,3,2\lfloor\frac{n}{2}\rfloor-1} \{(u_i, v_j)(u_{i+1}, v_{j+1})/j = 1, 3, ..., 2k - 1\}] \cup [\bigcup_{i=1,3,...,2\lfloor\frac{n}{2}\rfloor-1} \{(u_i, v_j)(u_{i+1}, v_{j-1})/j = 2, 4, ..., 2k\}]$$

and another edges which are not in M:

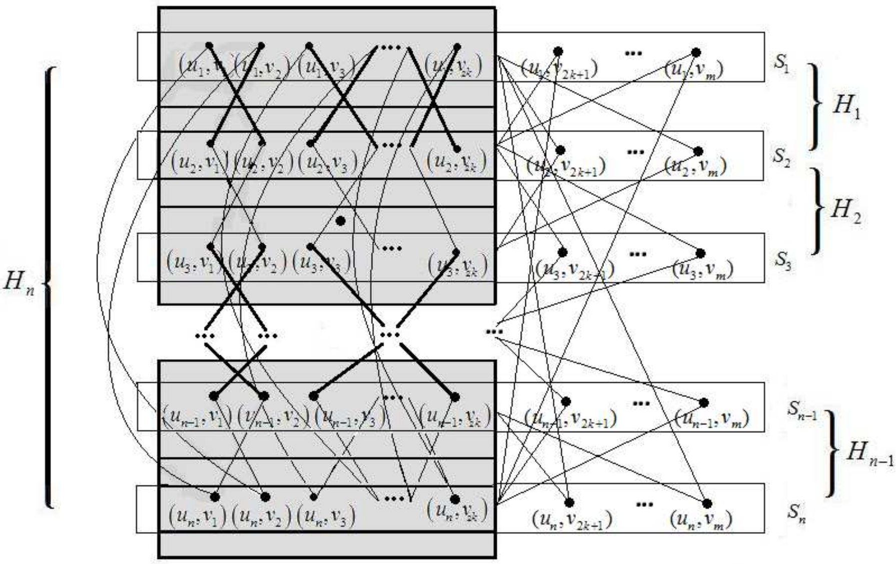

Figure 2: The Matching M when $n\alpha'(G) > m\lfloor \frac{n}{2} \rfloor$ and n is odd

$$N = [\bigcup_{i=2,4,2\lfloor \frac{n}{2} \rfloor} \{(u_i, v_j)(u_{i+1}, v_{j+1})/j = 1, 3, ..., 2k-1\}]$$

$$\cup [\bigcup_{i=2,4,2\lfloor \frac{n}{2} \rfloor} \{(u_i, v_j)(u_{i+1}, v_{j-1})/j = 2, 4, ..., 2k\}]$$

$$\cup \{(u_1, v_j)(u_n, v_{j+1})/j = 1, 3, ..., 2k-1\} \cup \{(u_1, v_j)(u_n, v_{j-1})/j = 2, 4, ..., 2k\},$$

so the endpoints of M are saturated by M.

If $n\alpha'(G) < m\lfloor \frac{n}{2} \rfloor$, suppose that $\alpha'(C_n \otimes G) > m\lfloor \frac{n}{2} \rfloor$, it is not true because every S_i have $|S_i| = m$.

Hence $\alpha'(C_n \otimes G) = \max\{n\alpha'(G), m\lfloor \frac{n}{2} \rfloor\}$. □

3. Edge Covering number of the graph of $C_n \otimes G$

We begin this section by giving Lemma 9 that shows a relation of matching number and edge covering number and Lemma 10 that show character of edge cover number for each H_i.

Lemma 9. Let G be a simple graph with order n. Then $\alpha'(G) + \beta'(G) = n$

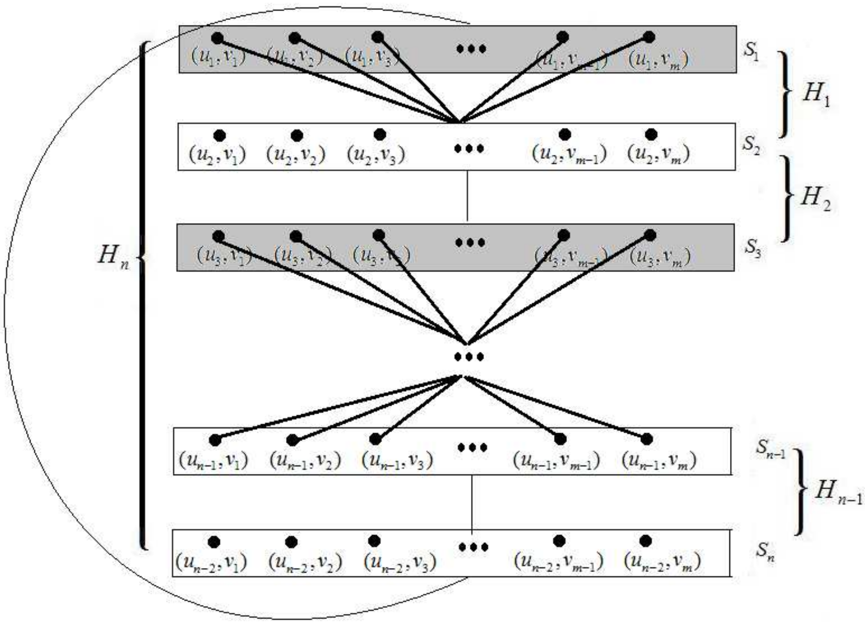

Figure 3: The Matching M when $n\alpha'(G) < m\lfloor \frac{n}{2} \rfloor$ and n is odd

Lemma 10. Let $C_n \otimes G = (\bigcup_{i=1}^{n-1} H_i) \cup H_n$. For each H_i and H_n then $\beta'(H_i) = \beta'(H_n) = 2\beta'(G)$

Proof. Suppose G has no odd cycle, by proposition 1.4, we get H_i=$2G$. So $\beta'(H_i) = 2\beta'(G)$.

If G has odd cycle, for each $(u_{i+1}, v) \in W_i$, $(u_{i+1}, v) \in W_(i+1)$ in $V(H_i)$ and $(u_n, v) \in W_n$ in $V(H_n)$ have $d_{H_i}((u_i, v)) = d_{H_i}(u_{i+1}, v)) = d_G(v) = d_{H_n}((u_n, v)) = d_{H_n}(u_1, v))$. Let $\bigcup_{i=1}^{n-1} \overline{H_i} = C_n \otimes (G - \overline{e})$ when $\overline{e}$ is an edge in odd cycle, C be the minimum edge covering set of G . We get $\overline{H_i} = 2(G - \overline{e})$ then

$$\beta(\overline{H_i}) = \begin{cases} 2[\beta(G) + 2], & \text{if } \overline{e} = xy \in C \text{ with } d(x) > 1 \text{ and } d(y) > 1, \\ 2[\beta(G) - 1], & \text{if } \overline{e} = xy \in C \text{ with } d(x) \geq 1 \text{ or } d(y) \geq 1, \\ 2\beta(G), & \text{otherwise.} \end{cases}$$

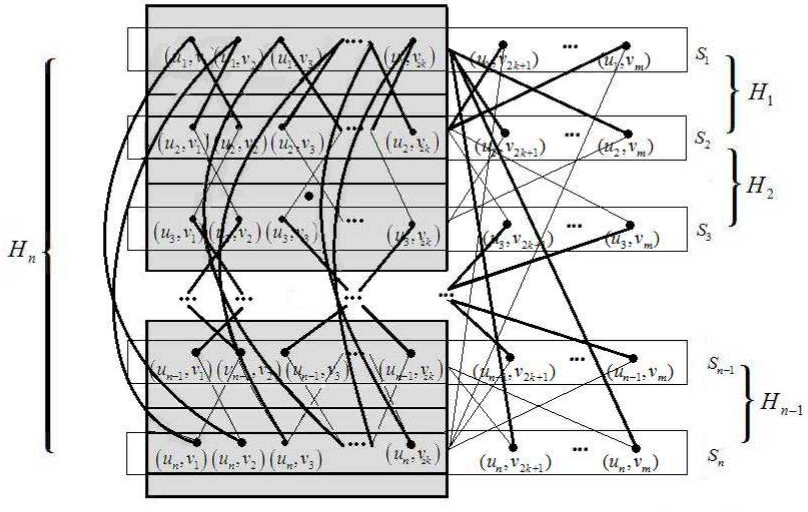

Figure 4: The edge cover when $n\beta'(G) < m\lceil \frac{n}{2} \rceil$ and n is odd

When we add $\overline{e}$ comeback, in the case $\beta'(G - \overline{e}) = \beta'(G) - 1$, we get $\beta'(H_i) = \beta'(\overline{H_i}) + 1$. And in the case $\beta'(G - \overline{e}) = \beta'(G) + 2$, we get $\bar{e} = xy \in C$ of G replace edges ux, yv (edge cover of $G - \bar{e}$), so $\beta'(G - \overline{e}) = \beta'(G) - 2$.

Hence $\beta'(H_i) = 2\beta'(G)$. Similarly, $\beta'(H_n) = 2\beta'(G)$. □

Next, we establish Theorem 11 for a minimum edge covering number of $C_n \otimes G$.

Theorem 11. *Let G be connected graph order m, then* $\beta'(C_n \otimes G) = \min\{n\beta'(G), m\lceil \frac{n}{2} \rceil\}$

Proof. Let $V(C_n) = \{u_i, i = 1, 2, ..., n\}$, $V(G) = \{v_j, j = 1, 2, ..., m\}$, $S_i = \{(u_i, v_j) \in V(C_n \otimes G)/j = 1, 2, ..., m\}$, $i = 1, 2, ..., n$ and since $\beta'(C_n) = \lceil \frac{n}{2} \rceil$. Let $\beta'(G) = k$, assume that the maximum matching of G be M_2, and minimum edge covering set of C_n, G be

$$C_1 = \begin{cases} \{u_1u_2, u_3u_4, ..., u_{n-1}u_n\} & \text{where } n \text{ is even,} \\ \{u_1u_2, u_3u_4, ..., u_{n-2}u_{n-1}, u_nu_1\} & \text{where } n \text{ is odd,} \end{cases}$$

$C_2 = M_2 \cup \{v_jv/j = 2k+1, 2k+2, ..., m \text{ and } v \text{ is endvertex of matching in } M_2\}$, respectively.

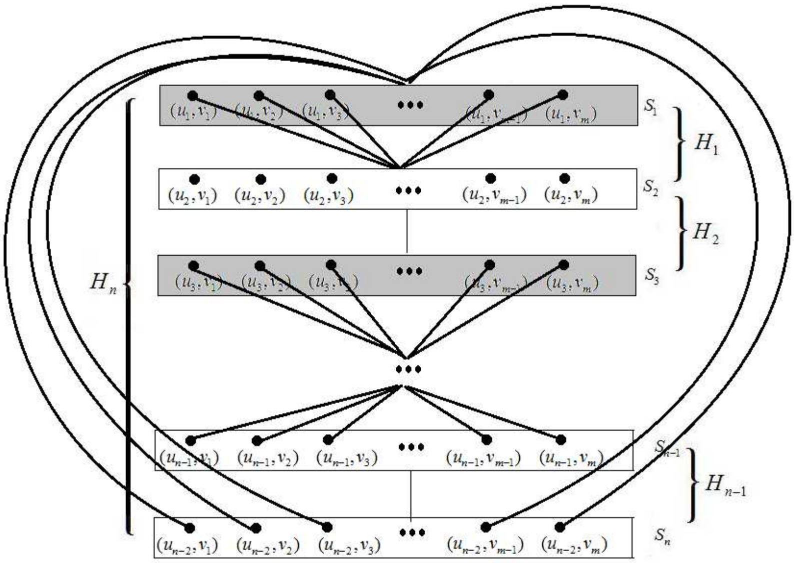

Figure 5: The edge cover when $n\beta'(G) > m\lceil \frac{n}{2} \rceil$ and n is odd

By Lemma 3.2 we have $\beta'(H_i) = 2\beta'(G)$. Since $C_n \otimes G$ is $(\bigcup_{i=1}^{n-1} H_i) \cup H_n$ which have edge cover in $H_1, H_3, ..., H_{2\lceil \frac{n}{2} \rceil - 1}$, then $\beta'(C_n \otimes G) \leq n\beta'(G)$.

Since definition of edge cover, we get another edge cover of $C_n \otimes G$ be set of edges, such that incident with vertices in S_i and S_{i+1}, $i = 1, 3, ..., 2\lceil \frac{n}{2} \rceil - 1$. So $\beta'(C_n \otimes G) \leq m\lceil \frac{n}{2} \rceil$.

Hence $\beta'(C_n \otimes G) \leq \min\{n\beta'(G), m\lceil \frac{n}{2} \rceil\}$.

If $n\beta'(G) < m\lceil \frac{n}{2} \rceil$, suppose that $\beta'(C_n \otimes G) < n\beta'(G)$, then there exist edges xy in edge covering of each $H_1, H_3, ..., H_{2\lceil \frac{n}{2} \rceil - 1}$, which is endvertex x and y incident with another edges in edge covering of each $H_1, H_3, ..., H_{2\lceil \frac{n}{2} \rceil - 1}$, it not impossible.

If $n\beta'(G) > m\lceil \frac{n}{2} \rceil$, suppose that $\beta'(C_n \otimes G) > m\lceil \frac{n}{2} \rceil$, that is not true because every S_i have $|S_i| = m$.

Hence $\beta'(C_n \otimes G) = \min\{n\beta'(G), m\lceil \frac{n}{2} \rceil\}$. □

By Theorem 2.3 and Lemma 3.1, we can also show that:

$$\alpha'(C_n \otimes G) + \beta'(C_n \otimes G) = mn,$$

$$\max\{n\alpha'(G), m\lfloor\frac{n}{2}\rfloor\} + \beta'(C_n \otimes G) = mn,$$

$$\begin{aligned}\beta'(C_n \otimes G) &= mn - \max\{n\alpha'(G), m\lfloor\frac{n}{2}\rfloor\}\\ &= mn + \min\{-n\alpha'(G), -m\lfloor\frac{n}{2}\rfloor\}\\ &= \min\{n(m - \alpha'(G)), m(n - \lfloor\frac{n}{2}\rfloor\}\\ &= \min\{n\beta'(G)), m\lceil\frac{n}{2}\rceil\}.\end{aligned}$$

Acknowledgments

This research is supported by the Centre of Excellence in Mathematics, Commission on Higher Education, Thailand.

References

[1] Z.A. Bottreou, Y. Metivier, Some remarks on the Kronecker product of graph, *Inform. Process. Lett*, **8** (1998), 279-286.

[2] B.W. Douglus, *Introduction to Graph Theory*, Prentice-Hall (2001).

[3] P.M. Weichsel, The Kronecker product of graphs, *Proc. Amer. Math. Soc.*, **8** (1962), 47-52.