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OSCILLATION CRITERIA FOR THE SOLUTIONS OF A FIRST ORDER NEUTRAL NONCONSTANT DELAY IMPULSIVE DIFFERENTIAL EQUATIONS WITH VARIABLE COEFFICIENTS

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Abstract: In this paper we consider a first order neutral impulsive differential equation with variable coefficients and nonconstant delay. Sufficient conditions for oscillation of all solutions of such equations are found.

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1. Introduction

The Neutral Impulsive Differential Equations (NIDE) are part of the Impulsive Differential Equations with Deviating Arguments (IDEDA). Among the numerous publications concerning the oscillation properties of IDEDA - with delayed or advanced arguments, we choose to refer to [1], [2], [7], [8], [9], [13] and [14]. NIDE are characterized with neutral argument in which the highest-order derivative of the unknown function appears in the equation both with and without delay. Moreover, the unknown function in them, may have discontinuities of first kind at points, which we call jump points. Such equations can be used to model processes, that occur in the theory of optimal control, industrial robotics, biotechnologies, etc. Some results on the oscillation theory of this type of equations can be found in [3], [4] and [6].

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As concern the behavior of the solutions of the differential equations with deviating arguments, but without impulses, we choose to refer to [5], [10], [11] and [12].

The authors investigated neutral delay impulsive differential equations with constant coefficients and found there necessary and sufficient conditions for existence of eventually positive solutions in [3] and established oscillation criteria in [4], as well. In the present paper we study the asymptotic behavior of the eventually non-oscillatory solutions of (E_1) and obtain oscillation criteria when the coefficients are variable.

2. Preliminary Notes

The object of investigation in the present work is the first order impulsive differential equation with variable coefficients and nonconstant neutral delay argument of the form

$$\frac{d}{dt}\{y(t)-c(t)y(h(t))\}+p(t)y(\sigma(t))=0,\ t\neq\tau_k\ k\in N \qquad (E_1)$$
$$\Delta\{y(\tau_k)-c_{\tau_k}y(h(\tau_k))\}+p_{\tau_k}y(\sigma(\tau_k))=0,\ k\in N$$

as well as the corresponding to it inequalities

$$\frac{d}{dt}\{y(t)-c(t)y(h(t))\}+p(t)y(\sigma(t))\leq 0,\ t\neq\tau_k,\ k\in N \qquad (N_{1,\leq})$$
$$\Delta\{y(\tau_k)-c_{\tau_k}y(h(\tau_k))\}+p_{\tau_k}y(\sigma(\tau_k))\leq 0,\ k\in N$$

and

$$\frac{d}{dt}\{y(t)-c(t)y(h(t))\}+p(t)y(\sigma(t))\geq 0,\ t\neq\tau_k,\ k\in N \qquad (N_{1,\geq})$$
$$\Delta\{y(\tau_k)-c_{\tau_k}y(h(\tau_k))\}+p_{\tau_k}y(\sigma(\tau_k))\geq 0,\ k\in N.$$

The points $\tau_k\in(0,+\infty),\ k\in N$ are the moments of impulsive effect (let us call them jump points), where the unknown function reveals its discontinuities of first kind as jumps. In order to manifest these jumps of the unknown function $y(t)$, we use the notation

$$\Delta\{y(\tau_k)-c_{\tau_k}y(h(\tau_k))\}=\Delta y(\tau_k)-c_{\tau_k}\Delta y(h(\tau_k)),\ \ \Delta y(\tau_k)=y(\tau_k+0)-y(\tau_k-0).$$

Denote by $P_\tau C(R,R)$ the set of all functions $u\colon R\to R$, which satisfy the following conditions:

(i) u is piecewise continuous on $(\tau_k,\tau_{k+1}],\ k\in N$,

(ii) u is continuous from the left at the points τ_k, i.e.

$$u(\tau_k - 0) = \lim_{t \to \tau_k - 0} u(t) = u(\tau_k),$$

(iii) there exists a sequence of reals $\{u(\tau_k + 0)\}_{k=1}^{\infty}$, such that

$$u(\tau_k + 0) = \lim_{t \to \tau_k + 0} u(t),$$

(iv) u may have discontinuities of first kind at the jump points τ_k, $k \in N$, that we qualify as down-jumps when $\Delta u(\tau_k) < 0$, or as up-jumps when $\Delta u(\tau_k) > 0$, $k \in N$.

Introduce the following hypotheses, where $R^+ = (0, +\infty)$ and $R_0^+ = [0, +\infty)$:

(H1) $0 < \tau_1 < \tau_2 < \ldots < \tau_k < \ldots$, $\lim\limits_{k \to +\infty} \tau_k = +\infty$, $\max\{\tau_{k+1} - \tau_k\} < +\infty$, $k \in N$;

(H2) h, $\sigma \in C^1(R^+, R^+)$, $h'(t) > 0$, $\sigma'(t) > 0$, $\sigma(t) < t$, $h(t) < t$ and $\lim\limits_{t \to +\infty} h(t) = +\infty$, $\lim\limits_{t \to +\infty} \sigma(t) = +\infty$;

(H3) $c \in C(R, (0, c_0])$, $c_0 \in (0, 1)$, $c_{\tau_k} = c(\tau_k)$;

(H4) $p \in P_\tau C(R, R)$, $p(t)$ is not identically zero on any positive half-line, $\sum\limits_{k=1}^{+\infty} p_{\tau_k}^2 \neq 0$;

(H5) $p \in P_\tau C(R, R^+)$, $p_{\tau_k} \in R_0^+$, $k \in N$, $\int\limits_0^{+\infty} p(s)ds + \sum\limits_{k=1}^{+\infty} p_{\tau_k} = +\infty$.

Let $\rho(t) = \min\limits_{t \in R^+}\{\sigma(t), h(t)\}$. We say that a real valued function $y(t)$ is a *solution* of the equation (E_1), if there exists a number $T_0 \in R$ such that $y \in P_\tau C([\rho(T_0), +\infty), R)$, the function $z(t) = y(t) - c(t)y(h(t))$ is continuously differentiable for $t \geq T_0$, $t \neq \tau_k$, $k \in N$ and $y(t)$ satisfies (E_1) for all $t \geq T_0$.

Without further mentioning we will assume throughout this paper, that every solution $y(t)$ of equation (E_1) that is under consideration here, is continuable to the right and is nontrivial. That is, $y(t)$ is defined on some ray of the form $[T_y, +\infty)$ and for each $T \geq T_y$ it is fulfilled $\sup\{|y(t)| : t \geq T\} > 0$. Such a solution is called a *regular solution* of (E_1).

We say that a real valued function u defined on an interval $[a, +\infty)$ has some property *eventually,* if there is a number $b \geq a$ such that u has this property on the interval $[b, +\infty)$.

A regular solution $y(t)$ of equation (E_1) is said to be *nonoscillatory,* if there exists a number $t_0 \geq 0$ such that $y(t)$ is of constant sign for every $t \geq t_0$. Otherwise, it is called *oscillatory*. Also, note that a *nonoscillatory* solution is called *eventually positive* (*eventually negative*), if the constant sign that determines its *nonoscillation* is positive (negative). Equation (E_1) is called *oscillatory* , if all its solutions are oscillatory.

Moreover, in this article, when we write a functional relation (or inequality), we will mean that it holds for all sufficiently large values of the argument.

In order to assist our investigations on the oscillation of the equation (E_1), we shall consider in the next section the delay impulsive differential equation with variable coefficient of the form

$$z'(t) + q(t)z(s(t)) = 0,\ t \neq \tau_k \qquad (E_2)$$
$$\Delta z(\tau_k) + q_{\tau_k} z(s(\tau_k)) = 0,\ k \in N$$

and the corresponding to it inequalities

$$z'(t) + q(t)z(s(t)) \leq 0,\ t \neq \tau_k \qquad (N_{2,\leq})$$
$$\Delta z(\tau_k) + q_{\tau_k} z(s(\tau_k)) \leq 0,\ k \in N$$

and

$$z'(t) + q(t)z(s(t)) \geq 0,\ t \neq \tau_k \qquad (N_{2,\geq})$$
$$\Delta z(\tau_k) + q_{\tau_k} z(s(\tau_k)) \geq 0,\ k \in N,$$

under the hypotheses:

H2$*$ $\quad s \in C^1(R^+, R^+),\ s'(t) > 0,\ \lim\limits_{t\to+\infty} s(t) = +\infty,$ and $s(t) < t.$

H3$*$ $\quad q \in P_\tau C(R^+, R^+),\ q_{\tau_k} \in R_0^+,\ k \in N.$

3. Some Useful Lemmas

Consider $y(t)$ as a solution of equation (E_1) and set the auxiliary function

$$z(t) = y(t) - c(t)y(h(t)), \qquad \Delta z(\tau_k) = \Delta y(\tau_k) - c_{\tau_k}\Delta y(h(\tau_k)),\quad c_{\tau_k} = c(\tau_k),\ k \in N. \qquad (*)$$

We introduce two lemmas, which investigate the asymptotic behavior of the function $z(t)$, when $y(t)$ is a non-oscillatory solution of (E_1). The first one is formulated and proved for an eventually positive solution $y(t)$ of the equation (E_1).

Lemma 1. *Assume that the hypotheses* $(H_1) - (H_5)$ *are satisfied and* $y(t)$ *be an eventually positive solution of* (E_1)*. Then* $z(t)$*, defined by* $(*)$*, is a decreasing eventually positive function of* t *with not strict down-jumps and* $\lim_{t\to+\infty} z(t) = 0$ *with* $\lim_{\tau_k\to+\infty} |\Delta z(\tau_k)| = 0$.

Proof. Let $y(t)$ be an eventually positive solution of the equation (E_1), i.e. $y(t)$ is a solution of (E_1) and there exists a number $T_0 > 0$ such that $y(t) > 0$ for $t \geq \rho(T_0)$. Then, from (E_1) and utilizing $(*)$, we have

$$z'(t) = -p(t)y(\sigma(t)), \qquad t \neq \tau_k,\ k \in N,\ t \geq T_0, \tag{1}$$

$$\Delta z(\tau_k) = -p_{\tau_k} y(\sigma(\tau_k)), \qquad k \in N,\ \tau_k \geq T_0.$$

From (1), in view of (H_5), it follows that $z(t)$ is an eventually decreasing function of t $(z'(t) < 0)$ with not strict down-jumps $(\Delta z(\tau_k) \leq 0)$ for $t \in [T_0, +\infty)$.

Assume $z(t) < 0$ eventually. Then, for some $t_1 \geq T_0$ there exists $\delta_\nu > 0$ such that $z(t) \leq -\delta_\nu$, for every $t \geq t_1, t \neq \tau_k$, i.e. $y(t) - c(t)y(h(t)) \leq -\delta_\nu$, $t \neq \tau_k$, $t \geq t_1$. In the meantime, for the same $\delta_\nu > 0$, there will be such a position ν in the sequence of the impulsive moments $\{\tau_k\}$, whereafter $z(\tau_k) \leq -\delta_\nu$, for every $\tau_k \geq \tau_\nu$ where $k \geq \nu$, $k \in N, \nu \in N$. Hence, $y(\tau_k) - c_{\tau_k} y(h(\tau_k)) \leq -\delta_\nu$, $\tau_k \geq \tau_\nu$, $k \geq \nu$. Denote $t_\nu = \max\{t_1, \tau_\nu\}$. Using (H_3), we can combine the last two inequalities as

$$y(t) \leq -\delta_\nu + c(t)y(h(t)) \leq -\delta_\nu + c_0 y(h(t)), \qquad t \geq t_\nu.$$

By iterations, from the last inequality we get

$$y(t) \leq -\delta_\nu(1 + c_0 + c_0^2 + ... + c_0^{n-1}) + c_0^n y(h^n(t)), \qquad t \geq t_\nu, \tag{2}$$

In view of (H_3), the inequality (2) implies

$$y(t) \leq -\frac{\delta_\nu}{1-c}, \qquad t \geq t_\nu.$$

This is a contradiction. Hence, our assumption, that eventually $z(t) < 0$, is impossible.

Assume $z(t) \equiv 0$. Then, from (1), it follows

$$p(t)y(\sigma(t)) \equiv 0 \text{ and } p(\tau_k)y(\sigma(\tau_k)) = 0.$$

But $y(t)$ is an eventually positive function, so we should have $p(t) \equiv 0$, $p_{\tau_k} = 0$, $k \in N$, which contradicts (H_4).

Thus, $z(t) \geq 0$ eventually. Moreover, in view of (1) and (H_5), we conclude that there exists $\lim_{t\to+\infty} z(t)$ and it is a finite positive number or zero. Observe, that the last fact implies $\lim_{\tau_k\to+\infty} |\Delta z(\tau_k)| = 0$.

Assume $\lim_{t\to+\infty} z(t) = L,\ L > 0$. Then, if we integrate (E_1) from T_0 to t, we obtain

$$\int_{T_0}^{t} z'(s)ds + \int_{T_0}^{t} p(s)y(\sigma(s))ds = 0,$$

or

$$z(t) - z(T_0) - \sum_{T_0<\tau_k<t} \Delta z(\tau_k) + \int_{T_0}^{t} p(s)y(\sigma(s))ds = 0,$$

i.e.

$$z(t) = z(T_0) + \sum_{T_0<\tau_k<t} \Delta z(\tau_k) - \int_{T_0}^{t} p(s)y(\sigma(s))ds. \tag{3}$$

But $\Delta z(\tau_k) = -p_{\tau_k} y(\sigma(\tau_k))$ and from (3) we get

$$z(t) = z(T_0) - \sum_{T_0<\tau_k<t} p_{\tau_k} y(\sigma(\tau_k)) - \int_{T_0}^{t} p(s)y(\sigma(s))ds. \tag{4}$$

Note, that $L < z(t) < y(t)$, i.e. $y(t)$ is bounded from below. Then (4) reduces to

$$z(t) \leq z(T_0) - L\left(\sum_{T_0<\tau_k<t} p_{\tau_k} + \int_{T_0}^{t} p(s)ds\right),$$

which together with (H_5) implies $\lim_{t\to+\infty} z(t) = -\infty$ and contradicts our assumption. Therefore, $\lim_{t\to+\infty} z(t) = 0$. The proof is complete.

The second lemma is only formulated for an eventually negative solution $y(t)$ of the equation (E_1), but the proof is carried out respectively to the proof of Lemma 1.

Lemma 2. *Assume that the hypotheses $(H_1) - (H_5)$ are satisfied. Let $y(t)$ be an eventually negative solution of (E_1). Then $z(t)$, defined by $(*)$, is an increasing eventually negative function of t with not strict up-jumps and $\lim_{t\to+\infty} z(t) = 0$ with $\lim_{\tau_k\to+\infty} |\Delta z(\tau_k)| = 0$.*

Our aim into the next lemma is to establish appropriate sufficient condition under which the equation (E_2) is oscillatory. To this end, we introduce the following result.

Lemma 3. *Assume the hypotheses* $(H_1), (H_2*), (H_3*)$ *are satisfied. Suppose also that:*

$$\liminf_{t\to+\infty}\left(\int\limits_{s(t)}^{t} q(r)dr + \sum_{s(t)<\tau_k<t} q_{\tau_k}\right) > 1.$$

Then:

(a) the equation (E_2) *is oscillatory;*

(b) the inequality $(N_{2,\leq})$ *has no eventually positive solutions;*

(c) the inequality $(N_{2,\geq})$, *has no eventually negative solutions.*

Proof. Since the proofs of (a),(b) and (c) can be carried out by similar arguments, it suffices to prove only the case (a). To this end, we assume for the sake of contradiction, that equation (E_2) has a nonoscillatory solution. Since the negative of a solution of (E_2) is again a solution of (E_2), it suffices to prove the lemma considering this solution as an eventually positive function. So, suppose that $z(t)$ is a solution of (E_2) and there exists a number $t_0 > 0$, such that $z(t) > 0$, for every $t \geq s(t_0)$. Then, it follows from (E_2) that $z'(t) = -q(t)z(s(t)) < 0$ and $\Delta z(\tau_k) = -q_{\tau_k} z(s(\tau_k)) \leq 0$, for every t, $\tau_k \geq s(t_0)$, $k \in N$, i.e. $z(t)$ is a decreasing function with not strict down-jumps.

If we integrate (E_2) from $s(t)$ to t, we find

$$\int\limits_{s(t)}^{t} z'(r)dr + \int\limits_{s(t)}^{t} q(r)z(s(r))dr = 0,$$

i.e.

$$z(t) - z(s(t)) - \sum_{s(t)<\tau_k<t} \Delta z(\tau_k) + \int\limits_{s(t)}^{t} q(r)z(s(r))dr = 0. \tag{5}$$

By omitting the first term in (5) and replacing $\Delta z(\tau_k)$ with $-q_{\tau_k} z(s(\tau_k))$, we obtain

$$-z(s(t)) + \sum_{s(t)<\tau_k<t} q_{\tau_k} z(s(\tau_k)) + \int\limits_{s(t)}^{t} q(r)z(s(r))dr \leq 0. \tag{6}$$

Counting on the decreasing nature of the function $z(t)$ and the fact that $z(s(t)) < z(s(\tau_k))$, when $s(t) < \tau_k < t$, we get from (6)

$$-z(s(t)) + z(s(t)) \sum_{s(t)<\tau_k<t} q_{\tau_k} + z(s(t)) \int_{s(t)}^{t} q(r)dr \leq 0,$$

or more precisely

$$z(s(t)) \left(-1 + \int_{s(t)}^{t} q(r)dr + \sum_{s(t)<\tau_k<t} q_{\tau_k} \right) \leq 0.$$

Here, because of the eventually positive nature of the function $z(t)$, we conclude

$$\int_{s(t)}^{t} q(r)dr + \sum_{s(t)<\tau_k<t} q_{\tau_k} \leq 1.$$

The last inequality contradicts the condition of the lemma and completes the proof.

The last lemma in this section affords us new opportunity to investigate the oscillation of the equation (E_1). It indicates, that the function $z(t)$, which is constructed from an eventually none-oscillatory solution $y(t)$ of equation (E_1), is found to satisfy an auxiliary relation with useful properties. To this purpose, we introduce the following hypotheses, which describe the possible location of the points of impulse effect:

(**AH**) $\exists\, n \geq 1,\ n \in N\ :\ h(\tau_s) \in \{\tau_k\}_{k=1}^{\infty},\ \tau_s \in \{\tau_i\}_{i=n+1}^{\infty}$.

(**BH**) There exists a strictly increasing sequence $\{k_\nu\}_{\nu=1}^{\infty} \subseteq N$ (not obligatory consistent), for which $\{\tau_{k_\nu}\}_{\nu=1}^{\infty} \subseteq \{\tau_k\}_{k=1}^{\infty}$, but $h(\tau_{k_\nu}) \bar{\in} \{\tau_k\}_{k=1}^{\infty}$.

Lemma 4. *Let the hypotheses $(H_1) - (H_5)$ are satisfied and $y(t)$ be a positive solution of (E_1). Then the function $z(t)$, defined by $(*)$, satisfies the neutral impulsive differential relation*

$$z'(t) - r(t)z'(h(t)) + p(t)z(\sigma(t)) = 0,\ t \neq \tau_k,\ k \in N \qquad (E_1^*)$$

$$\Delta z(\tau_k) - r_{\tau_k}\Delta z(h(\tau_k)) + p_{\tau_k} z(\sigma(\tau_k)) \leq 0,\ k \in N,$$

where $r(t) = c(\sigma(t))\dfrac{p(t)}{p(h(t))},\ r_{\tau_k} = r(\tau_k) = c_{\sigma(\tau_k)}\dfrac{p_{\tau_k}}{p_{h(\tau_k)}}$.

Proof. A direct substitution shows that $z(t) = y(t) - c(t)y(t-h)$, $t \neq \tau_k$, $k \in N$ is a continuously differentiable solution of the differential part of (E_1^*) in $J = \bigcup_{k=0}^{+\infty}(\tau_k, \tau_{k+1})$.

Using the definition of the function $z(t)$ in $(*)$, we can consider the left side of the difference part of (E_1*), as follows:

$$\Delta\left\{y(\tau_k) - c_{\tau_k}y(h(\tau_k))\right\} - r_k\Delta\left\{y(h(\tau_k)) - c_{h(\tau_k)}y(h^2(\tau_k))\right\} \\ + p_{\tau_k}\left\{y(\sigma(\tau_k)) - c_{\sigma(\tau_k)}y(\sigma(h(\tau_k)))\right\}.$$

This, by means of the difference part of (E_1), can be reduced to

$$-r_{\tau_k}\left\{\Delta\left(y(h(\tau_k)) - c_{h(\tau_k)}y(h^2(\tau_k)\right) + p_{h(\tau_k)}y(\sigma(h(\tau_k)))\right\} = I_k.$$

Observe, that when the hypothesis (**AH**) is valid, then $h(\tau_k)$ does belong to the sequence $\{\tau_k\}_{k=n+1}^{\infty}$. Therefore, we have eventually $I_k = 0$.

Assume, the hypothesis (**BH**) is valid. Then, $I_k = 0$ for every $k < k_1$, while when $k = k_1$, we have $\Delta y(h(\tau_{k_1})) = 0$ and subsequently, we obtain

$$I_k = I_{k_1} = r_{\tau_{k_1}}c_{h(\tau_{k_1})}\Delta y(h^2(\tau_{k_1})) - r_{\tau_{k_1}}p_{h(\tau_{k_1})}y(\sigma(h(\tau_{k_1}))).$$

But, in view of (**BH**), $h^2(\tau_{k_1})$ is not a point of impulse effect, i.e. $h^2(\tau_{k_1})$ does not belong to the sequence $\{\tau_k\}_{k=n+1}^{\infty}$ and $\Delta y(h^2(\tau_{k_1})) = 0$. Indeed, let we suppose, that $h^2(\tau_{k_1}) \in \{\tau_k\}_{k=n+1}^{\infty}$, whereas $h(\tau_{k_1}) \bar{\in} \{\tau_k\}_{k=n+1}^{\infty}$. From the hypotheses (H_3) and (**BH**), it follows $h^{-1}(\tau_i) \in \{\tau_k\}_{k=1}^{k_1}$, $i = \overline{0, k_1 - 1}$. Thus, we should have $h^{-1}(h^2(\tau_{k_1})) \in \{\tau_k\}_{k=1}^{k_1}$, i.e. $h(\tau_{k_1}) \in \{\tau_k\}_{k=1}^{k_1}$, which is a contradiction. Hence, when $k = k_1$ we have

$$I_k = -r_{\tau_k}p_{h(\tau_k)}y(\sigma(h(\tau_k))) \leq 0. \tag{13}$$

Further, we have $I_k = 0$, $k_1 < k < k_2$, whereas $\Delta y(h(\tau_{k_2})) = 0$ and subsequently

$$I_k = I_{k_2} = r_{\tau_{k_2}}c_{h(\tau_{k_2})}\Delta y(h^2(\tau_{k_2})) - r_{\tau_{k_2}}p_{h(\tau_{k_2})}y(\sigma(h(\tau_{k_2}))).$$

Here, analogically as above, we see that $h^2(\tau_{k_2})$ is not a point of impulse effect, i.e. $h^2(\tau_{k_2})$ does not belong to the sequence $\{\tau_k\}_{k=n+1}^{\infty}$ i.e. $\Delta y(h^2(\tau_{k_2})) = 0$ and (13) is fulfilled, for $k = k_2$. This procedure can be repeated for every $k \in \{k_\nu\}_{\nu=1}^{\infty} \subseteq N$.

So, we conclude that under the conditions of the lemma, the difference part of (E_1*) is satisfied by $\Delta z(\tau_k) = \Delta y(\tau_k) - c_{\tau_k}\Delta y(h(\tau_k))$, as well. The proof of the lemma is complete.

4. Oscillation Criteria for the Solutions of (E_1)

In this section we study the oscillatory properties of the solutions of the equation (E_1). The next theorems will establish sufficient conditions for oscillation of (E_1).

Theorem 1. *Let the hypotheses $(H_1)-(H_5)$ are satisfied. Suppose also that*

$$(i) \quad \liminf_{t\to\infty}\left(\int\limits_{\sigma(t)}^{t} p(s)ds + \sum_{\sigma(t)<\tau_k<t} p_{\tau_k}\right) > 1.$$

Then, all solutions of equation (E_1) are oscillatory.

Proof. Assume, for the sake of contradiction, that (E_1) has a non-oscillatory solution. Since the negative of a solution of (E_1) is again a solution of (E_1), it suffices to prove the theorem considering an eventually positive solution of (E_1). So, let us suppose that $y(t)$ is a solution of (E_1) and there exists a number $T_0 > 0$, such that $y(t) > 0,$ for $t \geq \rho(T_0)$. Recall $(*)$. By Lemma 4, $z(t)$ satisfies (E_1^*). Moreover, from Lemma 1, it follows that $z(t)$ is a decreasing eventually positive function of t with not strict down-jumps, i.e.

$$z(t) > 0, \qquad z'(t) < 0, \qquad t \geq T_0, \qquad \Delta z(\tau_k) \leq 0, \qquad \tau_k \geq T_0,\ k \in N. \quad (8)$$

In view of (8), the relation (E_1^*) leads us to the conclusion, that $z(t)$ satisfies the impulsive differential inequality with retarded argument

$$z'(t) + p(t)z(\sigma(t)) \leq 0,\ t \neq \tau_k,\ t \geq T_0, \quad (9)$$

$$\Delta z(\tau_k) + p_{\tau_k} z(\sigma(\tau_k)) \leq 0,\ \tau_k \geq T_0,\ k \in N,$$

which has the form of $(N_{2,\leq})$. But, the conclusion obtained, under the condition (i) of the theorem and in view of Lemma 3(b), contradicts to (8) and completes the proof.

Theorem 2. *Let the hypotheses $(H_1)-(H_5)$ are satisfied, $p_{\tau_k} \in (0,1),\ k \in N$ and*

$$(ii) \quad \liminf_{t\to\infty} \int\limits_{\sigma(t)}^{t} p(s)ds > \frac{1}{e} \limsup_{t\to\infty} \prod_{\sigma(t)<\tau_k<t} (1 - p_{\tau_k}).$$

Then, all solutions of equation (E_1) are oscillatory.

Proof. Proceeding as in the beginning of the proof of the Theorem 1, we may conclude first, that $z(t)$ satisfies (9) under the conditions in (8). Further, we shall prove that the function $w(t) = \dfrac{z(\sigma(t))}{z(t)}$ is bounded for $t \geq T_0$. Denote

$L = \limsup_{t\to\infty} \prod_{\sigma(t)<\tau_k<t} (1-p_{\tau_k})$. Then, by (ii), for every $t \geq T_0$ there exists a point $\tilde{t} \in (\sigma(t), t)$, such that

$$\int_{\sigma(t)}^{\tilde{t}} p(s)ds \geq \frac{L}{2e}, \qquad \int_{\tilde{t}}^{t} p(s)ds > \frac{L}{2e}. \tag{10}$$

If we integrate (9) from $\tilde{t}$ to t, we have

$$z(t) - z(\tilde{t}) - \sum_{\tilde{t}<\tau_k<t} \Delta z(\tau_k) + \int_{\tilde{t}}^{t} p(s)z(\sigma(s))ds = 0.$$

Taking into account the down-jumps of $z(t)$ and its decreasing positive nature, we get

$$z(\tilde{t}) \geq +z(\sigma(t)) \int_{\tilde{t}}^{t} p(s)ds > z(\sigma(t))\frac{L}{2e}. \tag{11}$$

If we integrate (9) from $\sigma(t)$ to $\tilde{t}$, we have

$$z(\sigma(t)) - z(\tilde{t}) + \sum_{\sigma(t)<\tau_k<\tilde{t}} \Delta z(\tau_k) = \int_{\sigma(t)}^{\tilde{t}} p(s)z(\sigma(s))ds.$$

Taking into account the down-jumps of $z(t)$ and its decreasing positive nature, we get

$$z(\sigma(t)) \geq \inf_{s\in[\sigma(t),\tilde{t}]} z(\sigma(s)) \int_{\sigma(t)}^{\tilde{t}} p(s)ds \geq z(\sigma(\tilde{t})) \int_{\sigma(t)}^{\tilde{t}} p(s)ds > z(\sigma(\tilde{t}))\frac{L}{2e}. \tag{12}$$

From (11) and (12), it follows $z(\tilde{t}) \geq z(\sigma(\tilde{t}))\left(\frac{L}{2e}\right)^2$, which proves that $w(t)$ is bounded from above for every $t \geq T_0$. Moreover, in view of (8), we have $w(t) = \dfrac{z(\sigma(t))}{z(t)} > 1$, i.e. $w(t)$ is bounded from bellow for every $t \geq T_0$, as well. Hence, $1 < w(t) < +\infty,\ t \geq T_0$.

Now, if we divide (9) by $z(t) > 0$ and integrate from $\sigma(t)$ to t, we obtain

$$\ln\frac{z(\sigma(t))}{z(t)} + \sum_{\sigma(t)<\tau_k<t} \ln\frac{z(\tau_k+0)}{z(\tau_k)} = \int_{\sigma(t)}^{t} p(s)w(s)ds \tag{13}$$

But,

$$z(\tau_k + 0) - z(\tau_k) \leq -p_{\tau_k} z(\sigma(\tau_k)) \leq -p_{\tau_k} z(\tau_k),$$

whence

$$\frac{z(\tau_k + 0)}{z(\tau_k)} \leq 1 - p_{\tau_k}.$$

From this and (13), using the notation $w_l = \liminf\limits_{s\to\infty} w(s)$, it follows

$$\ln\left(w(t) \prod_{\sigma(t)<\tau_k<t} (1 - p_{\tau_k}) \right) \geq w_l \int\limits_{\sigma(t)}^{t} p(s)ds,$$

which implies

$$\liminf_{t\to\infty} \int\limits_{\sigma(t)}^{t} p(s)ds \leq \frac{1}{e} \limsup_{t\to\infty} \prod_{\sigma(t)<\tau_k<t} (1 - p_{\tau_k}).$$

The last inequality contradicts the condition (ii) of the theorem. Therefore, it is impossible for (9) to have an eventually positive solution. This contradicts (8) and completes the proof.

Theorem 3. *Let the hypotheses $(H_1) - (H_5)$ are satisfied. Suppose also that:*

(iii) $\liminf\limits_{t\to+\infty} \left(\int\limits_{h(\sigma(t))}^{t} c(\sigma(r))p(r)dr + \sum\limits_{h(\sigma(t))<\tau_k<t} c_{\sigma(\tau_k)} p_{\tau_k} \right) > 1.$

Then, the equation (E_1) is oscillatory.

Proof. Assume, for the sake of contradiction, that equation (E_1) has a non-oscillatory solution. Since the negative of a solution of (E_1) is again a solution of (E_1), it suffices to prove the theorem considering an eventually positive solution of (E_1). So, let us suppose that $y(t)$ is a solution of (E_1) and there exists a number $T_0 > 0$, such that $y(t) > 0$ for $t \geq \rho(T_0)$. From Lemma 1, it follows that $z(t)$, which is defined by $(*)$, satisfies the conditions in (8).

From the fact that eventually $y(t) - c(t)y(h(t)) = z(t) > 0$, it follows $y(t) > z(t)$, as well as $y(t) > c(t)y(h(t))$. Obviously, $y(\sigma(t)) > c(\sigma(t))y(h(\sigma(t))) > c(\sigma(t))z(h(\sigma(t)))$, i.e.

$$y(\sigma(t)) > c(\sigma(t))z(h(\sigma(t))). \tag{14}$$

Multiplying both sides of (14) by $-p(t) < 0$ we obtain

$$z'(t) = -p(t)y(\sigma(t)) < -c(\sigma(t))p(t)z(h(\sigma(t))), \qquad \text{i.e.}$$

$$z'(t) + c(\sigma(t))p(t)z(h(\sigma(t))) < 0. \tag{15}$$

Observe that from (14) we have also $c_{\sigma(\tau_k)}z(h(\sigma(\tau_k)) < y(\sigma(\tau_k)),\ k \in N.$

Multiplying by $-p_{\tau_k} < 0,\ k \in N$ both sides of the last inequality, we obtain

$$-c_{\sigma(\tau_k)}p_{\tau_k}z(h(\sigma(\tau_k))) > -p_{\tau_k}y(\sigma(\tau_k)) = \Delta z(\tau_k), \quad k \in N, \qquad \text{i.e.}$$

$$\Delta z(\tau_k) + c_{\sigma(\tau_k)}p_{\tau_k}z(h(\sigma(\tau_k))) < 0,\ k \in N. \tag{16}$$

Denote $s(t) = h(\sigma(t)),\ q(t) = c(\sigma(t))p(t),\ q_{\tau_k} = c_{\sigma(\tau_k)}p_{\tau_k}$. From (15) and (16), it follows that the positive function $z(t)$ satisfies the delay impulsive differential inequality

$$z(t)' + q(t)z(s(t)) < 0,\ t \neq \tau_k,\ k \in N$$

$$\Delta z(\tau_k) + q_{\tau_k}z(s(\tau_k)) < 0,\ k \in N,$$

which has the form of $(N_{2,\leq})$. But, the conclusion obtained under the condition (iii) of the theorem and in view of Lemma 3(b), contradicts (8). The proof is complete.

Theorem 4. *Let the hypotheses* $(H_1) - (H_5)$ *are satisfied. Suppose also that:*

$(i\ddot{v})$ $\sigma(t) < h(t) < t$, *for every* $t \in R^+$;

$(\ddot{v})$ $\exists\ r_0 < 1 :\ r(t) = c(\sigma(t))\dfrac{p(t)}{p(h(t))} \leq r_0$ *and* $r'(t) \geq 0,\ t \in R^+$,

$r_{\tau_k} = c_{\sigma(\tau_k)}\dfrac{p_{\tau_k}}{p_{h(\tau_k)}},\ k \in N$;

$$(\ddot{v}i)\ \liminf_{t\to\infty}\left(\int\limits_{h^{-1}(\sigma(t))}^{t}\frac{p(s)}{1 - r(h^{-1}(\sigma(s)))}ds + \sum_{h^{-1}(\sigma(t))<\tau_k<t} p_{\tau_k}\right) > 1,\ k \in N.$$

Then, the equation (E_1) *is oscillatory.*

Proof. Assume for the sake of the contradiction, that $y(t)$ is an eventually positive solution of the equation (E_1), i.e. $y(t)$ is a solution of (E_1) and there exists a number $T_0 > 0$, such that $y(t) > 0$ for $t \geq \rho(T_0)$. Then, in view of Lemma 1, the function $z(t)$, defined by $(*)$, is a decreasing to zero eventually positive function with down-jumps, i.e.

$$z(t) > 0, \qquad \Delta z(\tau_k) \leq 0, \qquad z'(t) < 0, \qquad \lim_{t\to+\infty} z(t) = 0 \tag{17}$$

Let set a second auxiliary function

$$w(t) = z(t) - r(t)z(h(t)), \qquad \Delta w(\tau_k) = \Delta z(\tau_k) - r_{\tau_k}\Delta z(h(\tau_k)),\ k \in N. \quad (**)$$

Obviously, we have

$$w'(t) = z'(t) - r(t)z'(h(t)) - r'(t)z(h(t)). \tag{18}$$

But, $z(t)$ satisfies (E_1*), by Lemma 4. Therefore, (18) yields

$$w'(t) = -p(t)z(\sigma(t)) - r'(t)z(h(t)) < 0.$$

Moreover, $\Delta w(\tau_k) = \Delta z(\tau_k) - r_{\tau_k}\Delta z(h(\tau_k)) = -p_{\tau_k}z(h(\tau_k)) \leq 0$. Hence, $w(t)$ is a decreasing function with not strict down-jumps and obviously $L = \lim\limits_{t\to+\infty} w(t)$ does exist.

Assume $L < 0$, i.e. $w(t)$ is an eventually negative function. Then, by condition $(\ddot{v})$, we have $0 > w(t) = z(t) - r(t)z(h(t)) \geq z(t) - r_0 z(h(t))$, which in view of (17) and the monotonicity of $w(t)$, leads us to a contradiction. Therefore, $L \geq 0$ and we conclude

$$w(t) \geq 0, \qquad w(t) \leq z(t). \tag{19}$$

Further, using the fact, that $1 - r(t) > 0$, utilizing the Lemma 1 and counting on the decreasing positive nature of $w(t)$, we find $w(t) = z(t) - r(t)z(h(t)) \leq (1 - r(t))\, z(h(t))$, which can be rewritten as

$$w(h^{-1}(\sigma(t))) \leq \left(1 - r(h^{-1}(\sigma(t)))\right) z(\sigma(t)).$$

If we multiply the last inequality by $\dfrac{-p(t)}{1 - r(h^{-1}(\sigma(t)))} < 0$, we obtain

$$\frac{-p(t)}{1 - r(h^{-1}(\sigma(t)))} w(h^{-1}(\sigma(t))) \geq -p(t)z(\sigma(t)) = w'(t) + r'(t)z(h(t)),$$

or more precisely

$$w'(t) + \frac{p(t)}{1 - r(h^{-1}(\sigma(t)))} w(h^{-1}(\sigma(t))) \leq -r'(t)z(h(t)) < 0. \tag{20}$$

Moreover, from (E_1*), (20), the condition $(i\ddot{v})$ and the monotonicity of $w(t)$, it follows

$$\Delta w(\tau_k) = -p_{\tau_k}z(\sigma(\tau_k)) \leq -p_{\tau_k}w(\sigma(\tau_k)) < -p_{\tau_k}w(h^{-1}\sigma((\tau_k))), \ k \in N. \tag{21}$$

Finally, from (20) and (21) we can conclude, that the eventually positive function $w(t)$ satisfies the delay impulsive differential inequality

$$w'(t) + \frac{p(t)}{1 - r(h^{-1}(\sigma(t)))} w(h^{-1}(\sigma(t))) < 0$$

$$\Delta w(\tau_k) + p_{\tau_k} w(h^{-1}\sigma((\tau_k))) < 0, \;\; k \in N,$$

which has the form of $(N_{2,\leq})$. But, the conclusion obtained under the condition $(\ddot{v}i)$ of the theorem and in view of Lemma 3(b), contradicts (19). The proof is complete.

Corollary 1. *Let the conditions of Theorem 1, or Theorem 2, or Theorem 3, or Theorem 4 hold. Then:*

(i) the inequality $(N_{1,\leq})$ *has no eventually positive solutions;*

(ii) the inequality $(N_{1,\geq})$ *has no eventually negative solutions.*

The proof of the corollary is carried out analogously to the proofs of the respective theorems.

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