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A METHOD FOR FRACTIONAL PROGRAMMING

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1. Introduction

In this paper we consider the fractional programming problem:

$$\max_{x\in D} \frac{f(x)}{g(x)}\ , \tag{1.1}$$

where $D = \{x \in R^n \mid Ax \leq b\}$ and $g(x) > 0$ on D. $g, f : D \longrightarrow R$ are scalar functions.

Problem (1.1) has many applications in economics and engineering. For instance, problems such as minimization of average cost function [3] and maximization of consumption per a capital [3] belong to class of fractional programming.

Depending on type of functions f and g, problem (1.1) can be split into:

- (1.1) is called a linear fractional program if all functions $f(x)$ and $g(x)$ are affine;
- (1.1) is said to be a quadratic fractional program if $f(x)$ and $g(x)$ are quadratic functions;
- (1.1) is called a concave-convex fractional program if $f(x)$ is concave and $g(x)$ is convex;

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• (1.1) is called a convex-concave fractional program if $f(x)$ is convex and $g(x)$ is concave.

The well known existing methods for solving problem (1.1) are variable transformation [5], direct nonlinear programming approach [1], and parametric approach [2]. So far less attention paid to convex-concave fractional programming. We will reduce it to quasiconvex maximization problem and then apply global optimality conditions [4].

2. Global Optimality Conditions

Consider convex-concave fractional programming problem

$$\max_{x \in D} \left\{ \varphi(x) = \frac{f(x)}{g(x)} \right\} , \tag{2.1}$$

$f(x)$ and $g(x)$ are differentiable functions, D is a convex subset in R^n, and $f(x)$ is convex on D and $g(x)$ is concave on D, $f(x) > 0$ and $g(x) > 0$ for all $x \in D$. Introduce the level set of the function $\varphi(x)$ for a given $C > 0$.

$$L(\varphi, C) = \{x \in D |\ \varphi(x) \leq C\}$$

Lemma 2.1. *The set $L(\varphi, C)$ is convex.*

Proof. Since $g(x) > 0$ on D, then $\varphi(x) \leq C, \forall x \in D$ can be written as follows:

$$f(x) - Cg(x) \leq 0, \forall x \in D.$$

Clearly, a set defined by

$$M = \{x \in D |\ f(x) - Cg(x) \leq 0\}$$

is convex which implies convexity of $L(\varphi, C)$.

Definition 2.1. A function $f: \ R^n \to D$ is said to be quasiconvex if

$$f(\alpha x + (1 - \alpha)y) \leq \max\{f(x), f(y)\}$$

holds for all $x, y \in R^n$ and $\alpha \in [0, 1]$.

Lemma 2.2. (see [4]) *The function $f(x)$ is quasiconvex if and only if the set $L(f, C)$ is convex for all $C \in R$.*

Then it is clear that the function $\varphi(x)$ is quasiconvex on D.

The optimality condition for problem (1.1) will be formulated as follows (see [4]):

Theorem 2.1. *Let z be a solution to problem (1.1), and let $E_C(\varphi) = \{y \in R^n |\ \varphi(y) = C\}$.*
Then

$$\langle \varphi'(y), x - y \rangle \leq 0 \tag{2.2}$$

for all $y \in E_{\varphi(z)}(\varphi)$ and $x \in D$.

If in addition $\varphi'(y) \neq 0$ holds for all $y \in E_{\varphi(z)}(\varphi)$, then condition (2.2) is sufficient for $z \in D$ to be a global solution to problem (1.1).

Condition (2.2) can be simplified as:

$$\sum_{i=1}^{n} \left\{ \frac{\partial f(y)}{\partial x_i} g(y) - \frac{\partial g(y)}{\partial x_i} f(y) \right\} \left(\frac{x_i - y_i}{g^2(y)} \right) \leq 0,$$

for all $y \in E_{\varphi(z)}(\varphi)$ and $x \in D$.

Lemma 2.3. *If for feasible points $x, y \in D$, the inequality*

$$\langle \varphi'(y), x - y \rangle > 0$$

holds then $\varphi(x) \geq \varphi(y)$, where φ' denotes the gradient and $\langle , \rangle$ denotes the scalar product of two vectors.

Proof. On the contrary, assume that $\varphi(x) < \varphi(y)$. Since φ is quasiconvex, we have

$$\varphi(\alpha x + (1 - \alpha)y) \leq \max\{\varphi(x), \varphi(y)\} = \varphi(y).$$

By Taylor's formula, there is a neighborhood of the point y on which

$$\varphi(y + \alpha(x - y)) - \varphi(y) = \alpha \left(\langle \varphi'(y), x - y \rangle + \frac{o(\alpha||x - y||)}{\alpha} \right) \leq 0,$$

$\alpha > 0$. Taking into account that

$$\frac{o(\alpha||x - y||)}{\alpha} \xrightarrow{\alpha \to 0} 0$$

We obtain $\langle \varphi'(y), x - y \rangle \leq 0$ which contradicts $\langle \varphi'(y), x - y \rangle > 0$.

This completes the proof.

3. Algorithm and Approximation Set

Definition 3.1. The set $A(z)$ defined for a given m by

$$A_z^m = \{y^1,\ y^2,\ \ldots,\ y^m|\ y^i \in E_{\varphi(z)}(\varphi) \cap D,\ i = 1, 2, \ldots, m\}$$

is called an approximation set.

Lemma 3.1. *If there are a point $y^i \in A_z^m$ and a feasible point $z \in D$ such that*

$$\langle \varphi'(y^j),\ u^j - y^j \rangle > 0$$

then $\varphi(u^j) > \varphi(z)$, where $\langle \varphi'(y^j), u^j \rangle = \max_{x \in D} \langle \varphi'(y^j), x \rangle$.

The proof follows immediate from Lemma 2.3.

Now we can construct an algorithm for solving problem (1.1) approximately.

Algorithm

Step 1. Choose $x^k \in D$, $k := 0$. $z^k = argloc \max_{x \in D} \varphi(x)$, and m is given.

Step 2. Construct an approximation set $A_{z^k}^m$ at z^k.

Step 3. Solve Linear programming problems:

$$\max_{x \in D} \langle \varphi'(y^i), x \rangle\ ,\ i = 1, 2, \ldots, m$$

Let u^i be solutions to above problems:

$$\langle \varphi'(u^i), x \rangle = \max_{x \in D} \langle \varphi'(y^i), x \rangle, \quad i = 1, 2, \ldots, m.$$

Step 4. Compute η_k :

$$\eta_k = \max_{1 \le i \le m} \langle \varphi'(y^i), u^i - y^i \rangle = \langle \varphi'(y^j), u^j - y^j \rangle.$$

Step 5. If $\eta_k > 0$ then $x^{k+1} := u^j$, $k := k + 1$ and go to Step1.

Step 6. Terminate, z^k is an approximate global solution.

Theorem 3.1. *If $\eta_k > 0$ for all $k = 0, 1, \ldots,$ then the sequence $\{z^k\}$ constructed by the Algorithm is a relaxation sequence, i.e,*

$$f(z^{k+1}) \ge f(z^k),\ k = 0, 1, \ldots .$$

The proof follows from Lemmas 2.3 and Lemma 3.1.

4. Numerical Experiments

In order to implement the proposed algorithm, we consider the problem of the following type:

$$\max_{x\in D}\left\{\varphi(x)=\frac{\langle Ax,x\rangle+\langle b,x\rangle+k}{\langle Cx,x\rangle+\langle d,x\rangle+e}\right\},$$

where $D=\{x\in R^n|\ \alpha_i\le x_i\le\beta_i,\ i=1,2,\dots,n\}$, $k=2000, e=3000$.

Elements of the approximation set are defined as:

$$y^i=z^k+\alpha h^i,\ i=1,2,\dots,m.$$

Where z^k is a local solution found by the conditional gradient method starting from an arbitrary feasible point $x^k\in D$. Vectors h^i are generated randomly. Parameter α can be found from the equation $\varphi(y^i)=\varphi(z^k)$ in the following:

$$\alpha=\frac{\langle(\varphi(z^k)C-A)h^i,z^k\rangle+\langle(\varphi(z^k)C-A)z^k+\varphi(z^k)d-b-\varphi(z^k)e,h^i\rangle}{\langle(\varphi(z^k)C-A)h^i,h^i\rangle}.$$

The following problems have been solved numerically by the proposed algorithm and in all cases the global solutions are found.

Problem 1. Let

$$A=\begin{pmatrix}1&2\\-1&3\end{pmatrix},\quad C=\begin{pmatrix}-2&1\\-1&-4\end{pmatrix},\quad b=\begin{pmatrix}2\\1\end{pmatrix}\quad d=\begin{pmatrix}1\\5\end{pmatrix},$$

$$D=\{-1\le x_1\le 3,\ \ -2\le x_2\le 4\}.$$

Solution. $x^*=(-1,4),\ f(x^*)=0,6970.$

Problem 2. Let

$$A=\begin{pmatrix}1&-2&-1\\-1&3&0\\4&1&2\end{pmatrix},\quad C=\begin{pmatrix}2&1&3\\0&2&1\\-1&1&-1\end{pmatrix},\quad b=\begin{pmatrix}3\\2\\4\end{pmatrix},\quad d=\begin{pmatrix}2\\1\\1\end{pmatrix}.$$

$$D_1=\{1\le x_1\le 3,\ 2\le x_2\le 5,\ 1\le x_3\le 4\}.$$

Solution. $x^*=(1,5,4),\ f(x^*)=0,7190.$

$$D_2=\{-1\le x_1\le 3,-2\le x_2\le 4,-1\le x_3\le 5\}.$$

Solution. $x^*=(3,-2,5),\ f(x^*)=0,7255.$

Problem 3. Let

$$A = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 1 & 1 \\ 1 & 1 & 3 & 1 \\ 1 & 1 & 1 & 4 \end{pmatrix}, \ C = \begin{pmatrix} -2 & 1 & 1 & 2 \\ 1 & -1 & -1 & -2 \\ 1 & -1 & -3 & 1 \\ 2 & -2 & 1 & -9 \end{pmatrix},$$

$$b = \begin{pmatrix} 2 \\ -2 \\ 3 \\ -4 \end{pmatrix}, \ d = \begin{pmatrix} 1 \\ -2 \\ 3 \\ -1 \end{pmatrix}.$$

$$D = \{-2 \leq x_1 \leq 2, -1 \leq x_2 \leq 4, -1 \leq x_3 \leq 5, -3 \leq x_4 \leq 1\}.$$

Solution. $x^* = (2, 4, 5, 1), \ \ f(x^*) = 0,7688.$

Problem 4. Let

$$A = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 2 & 1 & 1 & 1 \\ 1 & 1 & 3 & 1 & 1 \\ 1 & 1 & 1 & 4 & 1 \\ 1 & 1 & 1 & 1 & 5 \end{pmatrix}, \quad C = \begin{pmatrix} -2 & 1 & 1 & 2 & 1 \\ 1 & -1 & -1 & -2 & -1 \\ 1 & -1 & -3 & 1 & 1 \\ 2 & -2 & 1 & -9 & -2 \\ 1 & -1 & 1 & -6 & -4 \end{pmatrix},$$

$$b = (1, -8, -3, \ 2, \ 5)', \quad d = (-2, \ 2, \ 5, -6, \ 4)',$$

$$D = \{-3 \leq x_1 \leq 2, -2 \leq x_2 \leq 3, -1 \leq x_3 \leq 5, -1 \leq x_4 \leq 4, -4 \leq x_5 \leq 1\},$$

Solution. $x^* = (2, 3, 4, 1, 6), \quad f(x^*) = 0,8666.$

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