

**ESTIMATION OF MORTALITY RATE
DECELERATION PARAMETER**

E.S. Lakshminarayanan¹, U. Kumaran^{2 §}

^{1,2}School of Mathematics
Madurai Kamaraj University
Madurai, INDIA

Abstract: In this paper, in the absence of mortality data an estimation of mortality rate deceleration parameter is provided.

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Key Words: logistic frailty model, mortality rate deceleration parameter, gamma-Gompertz model, heterogeneity, mortality plateaus, age at which onset of mortality deceleration

1. Introduction

In humans and many other organisms the mortality rate $\mu(t)$ increases approximately exponentially with age [14]. Gompertz [10] was the first to recognize this dependency of mortality rate on chronological age and expressed it mathematically by the equation

$$\mu(t) = ae^{bt}, \tag{1}$$

where $\mu(t)$ is the mortality rate at time t , the positive parameter a is the age-independent mortality rate or the initial mortality rate (IMR) and the positive parameter b is the age dependent mortality rate. But Benjamin Gompertz' exponential formula is only a starting point for descriptions of the age trajectory of adult mortality rate in numerous species including our own. Discoveries emerging from the measurement of hazard rates at advanced ages in large populations of model organisms as well as humans have now directed attention onto systematic deviations from Gompertz' formula [18].

The explanations offer to account for deviations at late ages from Gompertz models may be loosely sorted into three categories

1. There is some more general law of mortality to which the Gompertz curve is only an approximation at less advanced ages;
2. There is an essential change in the aging process in the very old;
3. The Gompertz hazard function remains in force at extreme ages, but it is masked by progressive age specific selection operating on heterogenous frailty in the initial population [16].

A closer look at the data reveals, however, that the rate of mortality increase with age slows down at older ages [12, 13]. Similar and more mortality decelerations have been observed for some animal species as well [1, 2, 3]. And the logistic frailty model has been shown to provide a better description of mortality rates in such cases.

In this model, $\mu(t)$ is the baseline, standard, underlying hazard for individuals of frailty one and $S(t)$ is the corresponding survival function. Vaupel et.al(1979a) show that

$$\bar{\mu}(t) = \bar{Z}(t)\mu(t), \quad (2)$$

where $\bar{Z}(t)$ is the average frailty of those alive at age t . For this model $\bar{Z}(t)$ is often to be gamma distributed with mean 1 and variance σ^2 , because this gamma distribution leads to convenient mathematical relationships [19]. In particular, for gamma-distributed frailty

$$\bar{Z}(t) = 1 + \sigma^2 \int_0^t \mu(x) dx. \quad (3)$$

In the presence of mortality data by age, the Logistic Frailty model parameters a , b and σ^2 have been estimated by using the Maximum Likelihood Estimation [4, 5, 6, 7, 11, 8, 17, 9].

In this work, we provide an estimation of σ^2 in the absence of age - specific mortality data.

2. Logistic Frailty Model

If the baseline hazard function is $\mu(t) = ae^{bt}$, then from (2) and (3) we have that the population trajectory of mortality follows the logistic pattern

$$\bar{\mu}(t) = \frac{ae^{bt}}{(1 + \frac{a\sigma^2}{b}(e^{bt} - 1))} \quad (4)$$

leveling off at a value $\frac{b}{\sigma^2}$. Here a is the IMR, b is the age dependent parameter and σ^2 is the mortality deceleration rate parameter. When $\sigma^2 = 0$, the equation (4) reduces to equation (1). This model has the property that $a > 0, b > 0$ and $0 < \sigma^2 < \frac{b}{a}$ [15].

3. Estimation of σ^2

First, observe that

$$\bar{\mu}(t) = \frac{a}{e^{-bt} + \frac{a\sigma^2}{b}(1 - e^{-bt})} = \frac{a}{\frac{a\sigma^2}{b} + e^{-bt}(1 - \frac{a\sigma^2}{b})}. \quad (5)$$

Equation (5) reveals that $\frac{a\sigma^2}{b}$ is numerically comparable to e^{-bt} , i.e, Gompertz mortality function ae^{bt} is comparable to $\frac{b}{\sigma^2}$.

Consider two cases: either $\frac{a\sigma^2}{b} \leq e^{-bt}$ or $\frac{a\sigma^2}{b} \geq e^{-bt}$.

Let

$$\frac{a\sigma^2}{b} \leq e^{-bt}. \quad (6)$$

When σ^2 is very small (the numerical value of σ^2 lies in the neighbourhood of 0), the Logistic frailty mortality function approaches the Gompertz mortality function. When $\sigma^2 > 0$ the Logistic Frailty mortality function coincides with Gompertz mortality function upto some point t (see the Figures 1 and 2).

Consider

$$\frac{ae^{bt}}{1 + \frac{a\sigma^2}{b}(e^{bt} - 1)} = ae^{bt}, \quad t \leq t. \quad (7)$$

Figures 1 and 2 reveal that there exist non-trivial a, b, σ^2 and t which satisfy (7).

Equation (7) can be written in the form

$$\frac{a}{(1 - \frac{a\sigma^2}{b})e^{-bt} + \frac{a\sigma^2}{b}} = \frac{a}{e^{-bt}}.$$

A little algebra gives

$$e^{-bt} + \frac{\frac{a\sigma^2}{b}}{1 - \frac{a\sigma^2}{b}} = \frac{e^{-bt}}{1 - \frac{a\sigma^2}{b}}. \quad (8)$$

On account of (6), we get

$$\frac{1}{1 - \frac{a\sigma^2}{b}} \leq \frac{1}{1 - e^{-bt}}. \quad (9)$$

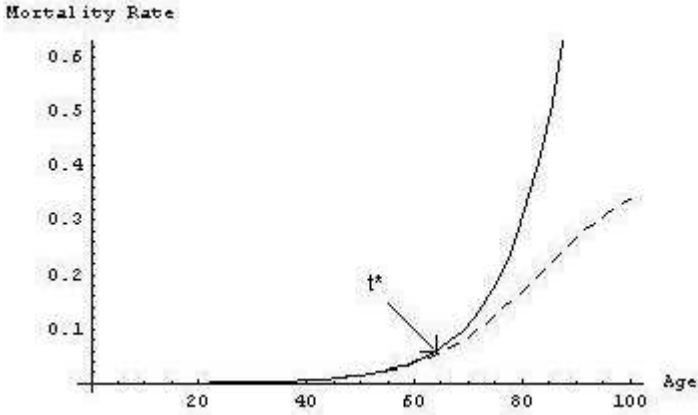


Figure 1

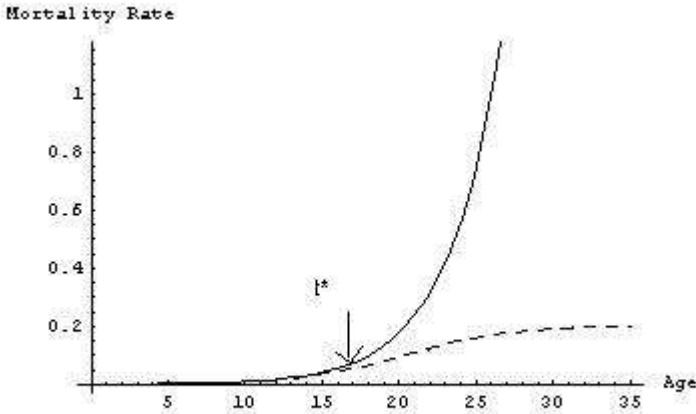


Figure 2: The dotted curve in Figure 1 and 2 represents the logistic frailty model, the other curve represents Gompertz model. With the values $a = 0.0001, b = 0.1, \sigma^2 = 0.25$ [19] we obtain Figure 1 and we obtain Figure 2 by taking $a = 0.00053, b = 0.29, \sigma^2 = 1.42$ [11]. It gives the idea that for any a, b and $\sigma^2 > 0$, the logistic frailty model coincides with Gompertz model and then deviates at t . Here t - the age when observed mortality deviates significantly from the exponential increase.

Using (9), from (8) we get

$$e^{-bt} + \frac{\frac{a\sigma^2}{b}}{1 - \frac{a\sigma^2}{b}} \leq \frac{e^{-bt}}{1 - e^{-bt}}.$$

It follows that

$$\frac{\frac{a\sigma^2}{b}}{1 - \frac{a\sigma^2}{b}} \leq \frac{e^{-2bt}}{1 - e^{-bt}}. \quad (10)$$

In (10) solving for σ^2 , we obtain

$$\sigma^2 \left[\left(e^{-bt} - \frac{1}{2} \right)^2 + \frac{3}{4} \right] \leq \frac{b}{a} e^{-2bt},$$

which gives

$$\sigma^2 \leq \frac{\frac{b}{a} e^{-2bt}}{\left[\left(e^{-bt} - \frac{1}{2} \right)^2 + \frac{3}{4} \right]} \leq \min \left\{ \frac{\frac{b}{a} e^{-2bt}}{\left[\left(e^{-bt} - \frac{1}{2} \right)^2 \right]}, \frac{\frac{b}{a} e^{-2bt}}{\frac{3}{4}} \right\} = \frac{4be^{-2bt}}{3a}.$$

Also observe that

$$\frac{a\sigma^2}{b} \leq \frac{e^{-2bt}}{\left[\left(e^{-bt} - \frac{1}{2} \right)^2 + \frac{3}{4} \right]} = \frac{(s-1)^2}{s^2 - s + 1}$$

has only one real fixed point at $s = 0.43016$, where $s = 1 - e^{-bt}$.

Hence, on account of (6), finally we get

$$\sigma^2 \leq \frac{0.43016b}{a}, \quad \sigma^2 \leq \frac{4be^{-2bt}}{3a}, \quad 0 < t \leq t. \quad (11)$$

In particular, when $\sigma^2 > 1$, $\sigma^2 \leq \frac{4be^{-2bt}}{3a}$ gives $e^{-bt} \geq \sqrt{\frac{3a}{4b}}$, or, equivalently, $t \leq \frac{-1}{b} \ln \left[\sqrt{\frac{3a}{4b}} \right]$.

Thus, we have

Lemma 1. When $\sigma^2 > 1$ it follows from (11) that $\frac{a\sigma^2}{b} \leq 0.43016$ and $\sigma^2 \leq \frac{4be^{-2bt}}{3a}$, where $t \leq \frac{-1}{b} \ln \left[\sqrt{\frac{3a}{4b}} \right]$, which approximates t (see Table 1).

When $\sigma^2 \leq 1$ it follows from (11) that

$$\sigma^2 \leq \frac{4be^{-2bt}}{3a}, \quad \frac{0.56564}{b} < t \leq t. \quad (12)$$

Next consider the case $\frac{a\sigma^2}{b} \geq e^{-bt}$. At this stage we need to investigate the behaviour of $\mu(t)$ when $t > t_m$. In particular, when $t = t_m$ - Maximum Life Span, there are two possible cases: either

$$\mu(t_m) < \frac{b}{\sigma^2} < ae^{bt_m} \quad (13)$$

or

$$\mu(t_m) < ae^{bt_m} < \frac{b}{\sigma^2}. \quad (14)$$

When (13) holds we get

$$\frac{a\sigma^2}{b} \geq e^{-bt_m}.$$

Thus we get the estimation

$$\sigma^2 \geq \frac{b}{a}e^{-bt_m}. \quad (15)$$

Combining (12) and (15) we get

$$\frac{b}{a}e^{-bt_m} \leq \sigma^2 \leq \frac{4be^{-2bt_m}}{3a}. \quad (16)$$

From (16), we obtain

$$t \leq \frac{t_m}{2} + \frac{\ln(\frac{4}{3})}{2b} \quad (17)$$

and hence estimation (12) holds when

$$\frac{0.56564}{b} \leq t \leq \frac{t_m}{2} + \frac{\ln(\frac{4}{3})}{2b}. \quad (18)$$

To compare with experimental data (see Table 1) we show that for a given parameters a , b , and σ^2 , using (12), there exists a $t \leq \frac{\ln(\frac{4b}{3a\sigma^2})}{2b}$ which also satisfies inequality (18), implying that estimation (12) is true.

When (14) holds we get

$$\frac{a\sigma^2}{b} \leq e^{-bt_m},$$

or, equivalently

$$\sigma^2 \leq \frac{b}{a}e^{-bt_m}. \quad (19)$$

To improve the estimation (19), we consider

$$\frac{ae^{bt_m}}{1 + \frac{a\sigma^2}{b}(e^{bt_m} - 1)} < ae^{bt_m}. \quad (20)$$

Species	a	b	σ^2	$\frac{a\sigma^2}{b} < 0.43016$	$t_m(days)$	t	$[\frac{0.56564}{b}, -\frac{\ln[\sqrt{\frac{3a}{4b}}]}{b}]$
Parage Ageria	0.000530	0.290	1.42	0.002595	22.8	10.76	[1.950, 11.3663]
Parage Ageria	0.003800	0.190	1.22	0.024400	22.0	10.53	[2.977, 11.0519]
Parage Ageria	0.001300	0.320	1.87	0.007597	19.0	8.075	[1.768, 9.05256]
Parage Ageria	0.000041	1.700	9.40	0.000227	10.1	2.553	[0.333, 3.21184]
Drosophila Melanogaster	0.000301	0.280	1.78	0.001910	26.3	11.69	[2.020, 12.7199]
Drosophila Melanogaster	0.000200	0.318	2.39	0.001514	26.2	10.66	[1.779, 12.0427]
Drosophila Melanogaster	0.000593	0.309	1.89	0.003628	22.4	9.558	[1.831, 10.5883]
Drosophila Melanogaster	0.000576	0.274	1.90	0.003992	24.8	10.60	[2.064, 11.7746]
Drosophila Melanogaster	0.000448	0.243	1.55	0.002860	27.4	12.64	[2.328, 13.5467]
Drosophila Melanogaster	0.000394	0.285	1.90	0.002624	25.4	10.93	[1.985, 12.0554]

Table 1: (reprinted from [17, 11])

Species	a	b	σ^2	$t_m(days)$	Estimated $\sigma^2 \leq$
Drosophila Melanogaster	0.000034	0.128000	0.860000	63.7	1.09491
Drosophila Melanogaster	0.000163	0.101000	0.520000	60.9	1.31846
Drosophila Melanogaster	0.001195	0.072000	0.450000	53.6	1.27088
Drosophila Melanogaster	0.000251	0.091000	0.430000	60.9	1.42078
Drosophila Melanogaster	0.003407	0.085650	0.319735	34.1	1.35496
Drosophila Melanogaster	0.002218	0.101223	0.336622	34.2	1.43174
Drosophila Melanogaster	0.006267	0.077237	0.137589	28.7	1.34296
Drosophila Melanogaster	0.001548	0.099021	0.132675	37.1	1.62367
Drosophila Melanogaster	0.001929	0.095465	0.111923	35.6	1.65401

Table 2: (reprinted from [17, 9])

A little algebra gives

$$\frac{a}{e^{-bt_m} + \frac{a\sigma^2}{b}(1 - e^{-bt_m})} < \frac{a}{e^{-bt_m}}.$$

It follows that

$$\frac{a\sigma^2}{b}(1 - e^{-bt_m}) > 0,$$

or, equivalently

$$\frac{a\sigma^2}{b}e^{-bt_m} < \frac{a\sigma^2}{b}. \quad (21)$$

On account of (19), we also get

$$\frac{a\sigma^2}{b}e^{-bt_m} < e^{-2bt_m}. \quad (22)$$

In view of (21) and (22), finally we get

$$\frac{a\sigma^2}{b}e^{-bt_m} < \min \left\{ e^{-2bt_m}, \frac{a\sigma^2}{b} \right\}.$$

Notice that in Table 3 only the marked(**) row does not satisfy the equality

$$\min \left\{ e^{-2bt_m}, \frac{a\sigma^2}{b} \right\} = e^{-2bt_m}.$$

4. Conclusion

What is the smallest possible value of $\sigma^2 > 0$? Our findings show that (the pointwise approximation of Logistic frailty model to Gompertz model) when $\frac{a\sigma^2}{b} < e^{-bt_m}$, the degree of smallness of σ^2 cannot be determined in terms of a , b , e^{-bt_m} . Further, it is worth exploring the possibilities of finding an expression for σ^2 by means of other (for instance, areawise) approximations. Also we need to investigate the behaviour of σ^2 with respect to other baseline hazard functions like, Gompertz-Maekham, Weibull, etc.

Species	a	b	σ^2	$t_m(days)$	$\frac{a\sigma^2}{b}e^{-bt_m}$	$\frac{a\sigma^2}{b}$	e^{-2bt_m}
Drosophila Melanogaster	0.000034	0.128000	0.860000	63.7	0.00000007	0.000226	0.00000008
Drosophila Melanogaster	0.000163	0.101000	0.520000	60.9	0.00000179	0.000841	0.00000454
Drosophila Melanogaster	0.001195	0.072000	0.450000	53.6	0.00015742	0.007466	0.00044457
Drosophila Melanogaster	0.000251	0.091000	0.430000	60.9	0.00000465	0.001186	0.00001536
Drosophila Melanogaster	0.003407	0.085650	0.319735	34.1	0.00068550	0.012719	0.00290498
Drosophila Melanogaster	0.002218	0.101223	0.336622	34.2	0.00023141	0.007376	0.00098423
Drosophila Melanogaster	0.006267	0.077237	0.137589	28.7	0.00121651	0.011164	0.01187400
Drosophila Melanogaster	0.001548	0.099021	0.132675	37.1	0.00005265	0.002074	0.00064429
Drosophila Melanogaster	0.001929	0.095465	0.111923	35.6	0.00007558	0.002262	0.00111700

Table 3: (reprinted from [17, 9])

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