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THE UNIT GROUPS OF *FG* OF GROUPS WITH ORDER 12

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Abstract: Note that there are five mutually non-isomorphic groups of order 12: two decomposable Abelian groups $G_1 \cong C_3 \times C_4$ and $G_2 \cong C_3 \times K_4$; three indecomposable non-Abelian groups A_4, Q_{12} and D_{12}. For a finite field F, the structure of $\mathscr{U}(FA_4)$ was determined by R.K. Sharma, J.B. Srivastava and M. Khan in 2007. In this paper, we determine the structure of the unit group $\mathscr{U}(FG)$ of the group ring FG, where F is a finite field, and G= G_1, G_2, Q_{12} or D_{12}.

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1. Introduction

Let FG be the group ring of a group G over a field F. For a normal subgroup H of G, the natural homomorphism $G \to G/H : g \mapsto gH$ can be extended to an F-algebra homomorphism from FG to $F(G/H)$ defined by $\sum_{g\in G} \mathrm{a}_g \mathrm{g} \mapsto \sum_{g\in G} \mathrm{a}_g\, gH$, for $\mathrm{a}_g \in F$. The kernel of this homomorphism, denoted by $\omega(H)$, is the ideal of FG generated by $\{h-1 \mid h \in H\}$ in FG. Of course, it is clear that $FG/\omega(H) \cong F(G/H)$. The ideal $\omega(G)$ is known as the augmentation ideal of

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the group ring FG, which is also denoted by $\omega(FG)$. It can be seen that $\omega(H) = \omega(FH)FG = FG\ \omega(FH)$. For convenience, we use $\omega^n(H)$ to denote $(\omega(H))^n$. Also $FG/\omega(\mathrm{G})\cong F$ implies that the Jacobson radical, $J(FG)$, is contained in $\omega(FG)$. It is known that for an ideal $I \subseteq J(FG)$, the natural homomorphism from FG to FG/I induces an epimorphism from the unit group of FG, $\mathscr{U}(FG)$, to $\mathscr{U}(FG/I)$ with kernel $1 + I$, so that $\mathscr{U}(FG)/(1 + I)\cong\mathscr{U}(FG/I)$.

The lower central chain of G is given by

$G = \gamma_1(\mathrm{G}) \supseteq \gamma_2(G) \supseteq \cdots \supseteq \gamma_{n+1}(\mathrm{G}) \supseteq \cdots$

where $\gamma_{m+1}(\mathrm{G}) = (\gamma_m(\mathrm{G}),\mathrm{G})$, for $m \geq 1$. For $g_1, g_2 \in G$, the commutator $(g_1, g_2) = g_1^{-1}g_2^{-1}g_1g_2$. The group G is said to be nilpotent of class n if $\gamma_{n+1}(G) = (1)$ and $\gamma_n(G) \neq (1)$.

Note that there are two Abelian groups of order 12: $G_1 \cong C_3 \times C_4$ and $G_2 \cong C_3 \times K_4$, both of them are decomposable. And there are three non-Abelian groups of order 12, all of them are indecomposable:

$$A_4 = \langle a, b \mid a^3 = 1, b^2 = 1, (ab)^3 = 1\rangle$$
$$= \{1,\ a,\ a^2,\ b,\ ba,\ ba^2,\ ab,\ aba,\ aba^2,\ a^2b,\ a^2ba,\ a^2ba^2\},$$
$$Q_{12} = \langle a, b \mid a^6 = 1, b^2 = a^3, ab = ba^{-1}\rangle$$
$$= \{1,\ a,\ a^2,\ a^3,\ a^4,\ a^5,\ b,\ ab,\ a^2b,\ a^3b,\ a^4b,\ a^5b\},\ \text{and}$$
$$D_{12} = \langle a, b \mid a^6 = 1, b^2 = 1, ab = ba^{-1}\rangle$$
$$= \{1,\ a,\ a^2,\ a^3,\ a^4,\ a^5,\ b,\ ab,\ a^2b,\ a^3b,\ a^4b,\ a^5b\}.$$

As we know, A_4, Q_{12} and D_{12} are the alternative group of degree 4, the quaternion group of order 12 and the dihedral group of order 12, respectively.

Given a group ring RG and a finite subset X of the group G, we shall denote $\widehat{X}$ the following element of RG: $\widehat{X} = \sum\limits_{x\in X} x$. In addition, the all distinct conjugate classes of Q_{12} and D_{12} both are $\mathscr{C}_1 = \{1\}$, $\mathscr{C}_2 = \{a, a^5\}$, $\mathscr{C}_3 = \{a^2, a^4\}$, $\mathscr{C}_4 = \{a^3\}$, $\mathscr{C}_5 = \{b, a^2b, a^4b\}$, $\mathscr{C}_6 = \{ab, a^3b, a^5b\}$. By [6, Theorem 3.6.2], $\{\widehat{\mathscr{C}_1}, \widehat{\mathscr{C}_2}, \widehat{\mathscr{C}_3}, \widehat{\mathscr{C}_4}, \widehat{\mathscr{C}_5}, \widehat{\mathscr{C}_6}\}$ forms a basis of the center $\mathscr{Z}(FQ_{12})$ and it also forms a basis of $\mathscr{Z}(FD_{12})$, where $\widehat{\mathscr{C}_i}$ denotes the class sum, i =1, 2, ... , 6.

On the study of unit groups of integral group rings ZG there are many literatures, such as [6], [7], [11], [12], et al). Recently, the structure of the unit group $\mathscr{U}(FG)$ over a finite field F were studied for some little groups. In 2007 and 2008, Khan, Sharma and Srivastava determined the structure of the unit group $\mathscr{U}(FG)$ for groups $G = A_4$ and S_4 in [16] and [9], respectively. From 2008, Gildea and his cooperators have made many investigations on the unit group of $\mathscr{U}(FG)$ mainly for the special case that the order of G is divided by the characteristic of F(for details see [1-5]). In this paper we completely determine the structure of the unit group $\mathscr{U}(FG)$ over any finite field F and all groups of order 12.

In what follows F is a finite field, $M(n,\ F)$ is the algebra of all $n\times n$ matrices over F, $GL(n,\ F)$ is the general linear group of degree n over F, char(F) is the characteristic of F, F^* is the multiplicative group of all nonzero elements of F, F_n is the extension field of F with degree n, C_n is the cyclic group of order n and C_n^k is the direct product of k copies of C_n.

2. Lemmas

Lemma 2.1. (see [8, Lemma 1.17]) *Let G be a locally finite p-group, and let F be a field of characteristic p. Then $J(FG) = \omega(FG)$.*

Lemma 2.2. (see [8, Proposition 8.1.20]) *Let R be a commutative noetherian ring and let G be an arbitrary group. Then there exists finitely many indecomposable rings $R_1, R_2, \cdots, R_n$ such that $RG \cong R_1G\times R_2G\times\cdots\times R_nG$. In particular, $\mathscr{U}(RG) \cong \mathscr{U}(R_1G) \times \mathscr{U}(R_2G) \times \cdots \times \mathscr{U}(R_nG)$.*

Lemma 2.3. (see [10, Lemma 4.11]) *If a left (resp., right) ideal $I \subseteq R$ is nil, then $I \subseteq J(R)$.*

Lemma 2.4. (see [13, Exercise 4, Page 134]) *Let R be a commutative ring and let $G,\ H$ be groups. Then $R(G \times H) \cong (RG)H$ (the group ring of H over the ring RG).*

Lemma 2.5. (see [13, Theorem 2.6.8]) (*Wedderburn-Artin*) *A ring R is semisimple if and only if it is a direct sum of finite number of matrix algebras over division rings:*

$$R \cong M(n_1, D_1) \oplus M(n_2, D_2) \oplus \cdots \oplus M(n_r, D_r).$$

Lemma 2.6. (see [13, Corollary 3.4.8]) *Let G be a finite group and let F be a field. Then, FG is semisimple if and only if char$(F) \nmid |G|$.*

Lemma 2.7. (see [14, Exercise 3.6, Page 154]) *Let p be an odd prime, and let:*

(i) If $p \equiv 3 (\text{mod } 4)$, then for any integer a we have $a^2 \not\equiv -1 \pmod p$;

(ii) If $p \equiv 1 (\text{mod } 4)$, then there exists an integer a such that $a^2 \equiv -1 \pmod p$.

Lemma 2.8. (see [15, Theorem 7.2.7]) *Let H be a normal subgroup of G with $[G : H] = n < \infty$. Then $(J(KG))^n \subseteq J(KH)KG \subseteq J(KG)$. If in addition $n \neq 0$ in K, then $J(KG) = J(KH)KG$.*

Lemma 2.9. (see [15, Exercise 9, Page 29]) *Let F be a field of characteristic $\neq 2$ and let K_4 be the Klein's four-group. Then $FK_4 \cong F \oplus F \oplus F \oplus F$.*

3. The unit groups of FG_1 and FG_2

In this section, we give the structures of $\mathscr{U}(FG_1)$ and $\mathscr{U}(FG_2)$, where $G_1 \cong C_3 \times C_4$, $G_2 \cong C_3 \times K_4$.

Lemma 3.1. *Let p be a prime and F an extension field of Z_p with order n. If $p \equiv 3 \pmod 4$ and n is odd, then $x^2 = -1$ has no solution in F.*

Proof. (i) If $n = 1$, by Lemma 2.7, we know that $x^2 = -1$ has no solution in Z_p.

(ii) Assume that $n \geq 3$. If $x^2 = -1$ has a solution, saying β, then β is not in Z_p and $\beta^2 + 1 = 0$. Thus

$$Z_p \subset Z_p[\beta] \subset F \quad \text{and} \quad [Z_p[\beta] : Z_p] = 2.$$

which contradicts to $[F : Z_p] = n$ and n is odd.

This completes the proof of Lemma 3.1. □

Lemma 3.2. *Let p be a prime and F a finite field of characteristic p with $|F| = p^n$. If $p \equiv 3 \pmod 4$ and n is odd, then $x^2 = 1$ has exactly eight roots in FC_4. Therefore $\mathscr{U}(FC_4)$ has exactly seven elements of order two.*

Proof. Let $C_4 = \langle \tau \mid \tau^4 = 1 \rangle = \{1, \tau, \tau^2, \tau^3\}$. $\forall\, x = \beta_0 + \beta_1\tau + \beta_2\tau^2 + \beta_3\tau^3 \in FC_4$, then

$$x^2 = 1 \Longleftrightarrow \begin{cases} \beta_0^2 + 2\beta_1\beta_3 + \beta_2^2 = 1 \\ \beta_1^2 + 2\beta_0\beta_2 + \beta_3^2 = 0 \\ \beta_0\beta_1 + \beta_2\beta_3 = 0 \\ \beta_0\beta_3 + \beta_1\beta_2 = 0 \end{cases}$$

$$\Longleftrightarrow \begin{cases} (\beta_0 + \beta_2)^2 + (\beta_1 + \beta_3)^2 = 1 \\ (\beta_0 - \beta_2)^2 - (\beta_1 - \beta_3)^2 = 1 \\ (\beta_0 + \beta_2)(\beta_1 + \beta_3) = 0 \\ (\beta_0 - \beta_2)(\beta_1 - \beta_3) = 0 \end{cases} \quad \text{in} \quad F$$

Then β_i $(i = 0, 1, 2, 3)$ need to satisfy the following systems of congruence equations simultaneously:

$$(1) \begin{cases} (\beta_0 + \beta_2)^2 + (\beta_1 + \beta_3)^2 = 1 \\ (\beta_0 + \beta_2)(\beta_1 + \beta_3) = 0 \end{cases}$$

$$\Longleftrightarrow \begin{cases} \beta_0 + \beta_2 = 0 \\ \beta_1 + \beta_3 = \pm 1 \end{cases} \text{or} \begin{cases} \beta_0 + \beta_2 = \pm 1 \\ \beta_1 + \beta_3 = 0 \end{cases} \quad \text{and}$$

$$(2) \begin{cases} (\beta_0 - \beta_2)^2 - (\beta_1 - \beta_3)^2 = 1 \\ (\beta_0 - \beta_2)(\beta_1 - \beta_3) = 0 \end{cases}$$

$$\Longleftrightarrow \begin{cases} \beta_0 - \beta_2 = 0 \\ (\beta_1 - \beta_3)^2 = -1 \end{cases} \text{or} \begin{cases} \beta_0 - \beta_2 = \pm 1 \\ \beta_1 - \beta_3 = 0 \end{cases}$$

By Lemma 3.1, we know that $(\beta_1 - \beta_3)^2 = -1$ has no solution in F.

Then the above systems of congruence equations are equivalent to the followings:

$$(i)\begin{cases}\beta_1+\beta_3=0\\ \beta_0+\beta_2=\pm1\\ \beta_1-\beta_3=0\\ \beta_0-\beta_2=\pm1\end{cases}\qquad (ii)\begin{cases}\beta_0+\beta_2=0\\ \beta_1+\beta_3=\pm1\\ \beta_1-\beta_3=0\\ \beta_0-\beta_2=\pm1\end{cases}$$

By (i), we have the following four solutions:

$$\begin{cases}\beta_0=1\\ \beta_1=0\\ \beta_2=0\\ \beta_3=0\end{cases}\quad \begin{cases}\beta_0=-1\\ \beta_1=0\\ \beta_2=0\\ \beta_3=0\end{cases}\quad \begin{cases}\beta_0=0\\ \beta_1=0\\ \beta_2=1\\ \beta_3=0\end{cases}\quad \begin{cases}\beta_0=0\\ \beta_1=0\\ \beta_2=-1\\ \beta_3=0\end{cases}$$

By (ii), we have the following four solutions:

$$\begin{cases}\beta_0=-\beta_2=\pm2^{-1}\\ \beta_1=\beta_3=\pm2^{-1}\end{cases}$$

By verification, $x=1,\ -1,\ \tau^2,\ -\tau^2,\ 2^{-1}(1+\tau-\tau^2+\tau^3),\ 2^{-1}(1+\tau-\tau^2+\tau^3)$, $2^{-1}(1+\tau-\tau^2+\tau^3)$ and $2^{-1}(1+\tau-\tau^2+\tau^3)$ are solutions of $x^2=1$ in FC_4. Hence, $x^2=1$ has exactly 8 solutions in FC_4. □

Lemma 3.3. *Let F be a finite field of characteristic $p(\neq 2)$ with $|F|=p^n$ and $C_4=\langle\tau\rangle$ a cyclic group of order 4. Then*

$$FC_4\cong\begin{cases}F\oplus F\oplus F\oplus F, & \text{if } p\equiv 1 \ (mod\ 4) \text{ or } n \text{ is even;}\\ F_2\oplus F\oplus F, & \text{if } p\equiv 3\ (mod\ 4) \text{ and } n \text{ is odd.}\end{cases}$$

Proof. Suppose $x=\beta_0+\beta_1\tau+\beta_2\,\tau^2+\beta_3\tau^3\in FC_4$, for $\beta_i\in F$, $i=0,\ 1,\ 2,\ 3$.

If $p\equiv 1 \pmod 4$, then $p^n\equiv 1 \pmod 4$, for all n. Thus

$x^{p^n}=(\beta_0+\beta_1\tau+\beta_2\tau^2+\beta_3\tau^3)^{p^n}=\beta_0^{p^n}+\beta_1^{p^n}(\tau)^{p^n}+\beta_2^{p^n}(\tau^2)^{p^n}+\beta_3^{p^n}(\tau^3)^{p^n}=x.$

Also, when $p\equiv 3 \pmod 4$ and n is even, then $p^n\equiv 1 \pmod 4$, so that $x^{p^n}=x$. Hence, $FC_4\cong F\oplus F\oplus F\oplus F$, if $p\equiv 1 \pmod 4$ or n is even.

If $p\equiv 3 \pmod 4$ and n is odd, we get $p^n\equiv 3 \pmod 4$. Then $p^{2n}\equiv 1 \pmod 4$ which implies $x^{p^{2n}}=x$. Hence, $FC_4\cong F_2\oplus F\oplus F$ or $FC_4\cong F_2\oplus F_2$. Clearly, $x^2=1$ has 8 roots in $F_2\oplus F\oplus F$ and it has 4 roots in $F_2\oplus F_2$. So, by Lemma 3.2, we must have $FC_4\cong F_2\oplus F\oplus F$.

This completes the proof of Lemma 3.3. □

Theorem 3.4. *Let $\mathscr{U}(FG_1)$ be the unit group of FG_1 over a finite field F of characteristic p and $|F|=p^n$, where $G_1\cong C_3\times C_4=C_{12}=\langle a\rangle$.*

(1) *If $p=2$, then*

$$\mathscr{U}(FG_1)\cong\begin{cases}C_2^{3n}\times C_4^{3n}\times C_{2^n-1}^3 & \text{if } n \text{ is even;}\\ C_2^{3n}\times C_4^{3n}\times C_{2^{2n}-1}\times C_{2^n-1} & \text{if } n \text{ is odd.}\end{cases}$$

(2) *If $p=3$, then*

$$\mathscr{U}(FG_1) \cong \begin{cases} C_3^{8n} \times C_{3^n-1}^4 & \textit{if } n \textit{ is even;} \\ C_3^{8n} \times C_{3^{2n}-1} \times C_{3^n-1}^2 & \textit{if } n \textit{ is odd.} \end{cases}$$

(3) *If* $p > 3$, *then*

$$\mathscr{U}(FG_1) \cong \begin{cases} C_{p^n-1}^{12} & \textit{if } p \equiv 1 (\textit{mod } 12) \textit{ or } n \textit{ is even;} \\ C_{p^{2n}-1}^4 \times C_{p^n-1}^4 & \textit{if } p \equiv 5 (\textit{mod } 12) \textit{ and } n \textit{ is odd.} \\ C_{p^{2n}-1}^3 \times C_{p^n-1}^6 & \textit{if } p \equiv 7 (\textit{mod } 12) \textit{ and } n \textit{ is odd.} \\ C_{p^{2n}-1}^5 \times C_{p^n-1} & \textit{if } p \equiv 11 (\textit{mod } 12) \textit{ and } n \textit{ is odd.} \end{cases}$$

Proof. Let $H = 1 + J(FG_1)$, where $J(FG_1)$ is the Jacobson radical of the group ring FG_1. F^* is the multiplicative group of F, which is a cyclic group of order $p^n - 1$, we also denote it by C_{p^n-1}.
(1) Let $\mathrm{char}(F) = 2$ with $|F| = 2^n$. Set $C_4 = \{1, a, a^2, a^3\}$, then we have $G_1/C_4 \cong C_3$. Thus, by Lemma 2.1 and Lemma 2.8, $J(FG_1) = J(FC_4)FG_1 = \omega(FC_4)FG_1 = \omega(C_4)$. So, $FG_1/J(FG_1) \cong F(G_1/C_4) \cong FC_3$, where $C_3 = \langle\bar{\sigma}\rangle$ is a cyclic group of order 3. Hence, from the ring epimorphism $FG_1 \to FC_3$, we get a group epimorphism $\phi : \mathscr{U}(FG_1)) \to \mathscr{U}(FC_3)$ and $Ker\phi = H = 1 + J(FG_1) = 1 + \omega(C_4)$.

Since C_3 is a subgroup of C_{12}, the ring monomorphism $FC_3 \to FG_1$, $\alpha_0+\alpha_1\bar{\sigma}+\alpha_2\bar{\sigma}^2 \to \alpha_0+\alpha_1 a^4+\alpha_2 a^8$, reduces a group homomorphism

$$\theta : \mathscr{U}(FC_3) \to \mathscr{U}(FG_1).$$

And we can verify that $\phi\theta = 1_{\mathscr{U}(FC_3)}$. Thus $\mathscr{U}(FG_1)$ is an extension of $\mathscr{U}(FC_3)$ by H, so

$$\mathscr{U}(FG_1) \cong H \times \mathscr{U}(FC_3).$$

Suppose $x = \alpha_0+\alpha_1\bar{\sigma}+\alpha_2\bar{\sigma}^2 \in FC_3$, for $\alpha_i \in F$. If n is even, then $3 \mid (2^n-1)$, and consequently, $x^{2^n} = (\alpha_0+\alpha_1\bar{\sigma}+\alpha_2\bar{\sigma}^2)^{2^n} = \alpha_0^{2^n}+\alpha_1^{2^n}(\bar{\sigma})^{2^n}+\alpha_2^{2^n}(\bar{\sigma}^2)^{2^n} = x$ so that $o(x) \mid (2^n - 1)$, where $x \in \mathscr{U}(FC_3)$. Thus $FC_3 \cong F \oplus F \oplus F$, if n is even. When n is odd, $3 \nmid (2^n - 1)$, but $3 \mid (2^{2n} - 1)$ and as above, $x^{2^{2n}} = x$, for any $x \in FC_3$. Hence, if n is odd, $FC_3 \cong F_2 \oplus F$.

On the other hand,

$$\omega(C_4) = \{\textstyle\sum_{i=0}^{11} \alpha_i a^i | \sum_{i=0}^{3} \alpha_{3i+j} = 0, j = 0, 1, 2,\ \alpha_i \in F\}.$$

By calculation, we have that

$$\{\alpha^2 \mid \alpha \in \omega(C_4)\} = \{\textstyle\sum_{i=0}^{5}(\alpha_i^2 + \alpha_{6+i}^2)a^{2i} | \sum_{i=0}^{3} \alpha_{3i+j} = 0, j = 0, 1, 2,\ \alpha_i \in F\},$$

and $\alpha^4 = 0$, for any $\alpha \in \omega(C_4)$.

For every $x \in H = 1 + J(FG_1) = 1 + \omega(C_4)$, set $x = 1 + \alpha, \alpha \in \omega(C_4)$, we get $x^2 = (1 + \alpha)^2 = 1 + \alpha^2$ and $x^4 = (1 + \alpha)^4 = 1 + \alpha^4 = 1$. Hence, H is an Abelian 2-group and has exponent 4. So $H \cong C_2^k \times C_4^l$ for some non-negative integers k, l. In addition, $FG_1/J(FG_1) \cong FC_3$, we know that $\dim_F J(FG_1) =$

9. So $|H| = |\omega(C_4)| = |J(FG_1)| = 2^{9n}$ and so $2^k \times 4^l = |H| = 2^{9n}$. Next, we need to determine k and l. Set

$$S = \{\alpha \in \omega(C_4) \mid \alpha^2 = 0, \text{ and } \exists \beta \in \omega(C_4) \text{ such that } \alpha = \beta^2\}.$$

By direct computation, we have that

$S = \{\sum_{i=0}^{2} \alpha_{2i}(a^{2i} + a^{6+2i})|, \alpha_{2i} \in F\}$, and therefore $|S| = 2^{3n}$, which implies that $l = 3n$ and then $k = 3n$. Thus $H \cong C_2^{3n} \times C_4^{3n}$. So, the proof of statement (1) is finished.

(2) Let char(F)= 3 and $|F| = 3^n$. Obviously we have $G_1/C_3 \cong C_4$. Then $F(G_1/C_3) \cong FC_4$. Since $J(FG_1) = J(FC_3)FG_1 = \omega(FC_3)FG_1 = \omega(C_3)$, then $FG_1/J(FG_1) \cong FC_4$, where $C_4 = \langle \bar{\tau} \rangle$ is a cyclic group of order 4. By Lemma 3.3, we know that

$$\mathscr{U}(FC_4) \cong \begin{cases} F^* \times F^* \times F^* \times F^* & \text{if n is even;} \\ F_2^* \times F^* \times F^* & \text{if n is odd.} \end{cases}$$

Let $C_3 = \{1, \sigma, \sigma^2\}$. We know that $\omega(C_3) = \langle g - 1 : g \in C_3 \rangle$, $\omega^2(C_3) = \langle (g_1 - 1)(g_2 - 1) : g_1, g_2 \in C_3 \rangle = (1 + \sigma + \sigma^2)FG_1$ and $\omega^3(C_3) = 0$. For any $x \in H$, set $x = 1 + \alpha, \alpha \in J(FG_1) = \omega(C_3)$, we have $x^3 = (1+\alpha)^3 = 1 + 3\alpha + 3\alpha^2 + \alpha^3$ $= 1$. Hence, H is an elementary Abelian 3-group.

In addition, $FG_1/J(FG_1) \cong FC_4$, we know that $\dim_F J(FG_1) = 8$. So, $|H|$ $= |J(FG_1)| = 3^{8n}$ and $H \cong C_3^{8n}$. Since $(3, 3^k - 1) = 1$, it is easy to see that $\mathscr{U}(FG_1) \cong H \times \mathscr{U}(FC_4)$. Hence the statement (2) is hold.

(3) Let char(F) $= p > 3$ and $|F| = p^n$.

First, by Lemma 2.4, we have $FG_1 \cong F(C_3 \times C_4) \cong (FC_3)C_4$. We have known that

$$FC_3 \cong \begin{cases} F \oplus F \oplus F, & \text{if } p \equiv 1 \pmod 3 \text{ or } n \text{ is even;} \\ F_2 \oplus F, & \text{if } p \equiv 2 \pmod 3 \text{ and } n \text{ is odd.} \end{cases}$$

Then, we have

$$FG_1 \cong \begin{cases} (F \oplus F \oplus F)C_4, & \text{if } p \equiv 1 \pmod 3 \text{ or } n \text{ is even;} \\ (F_2 \oplus F)C_4, & \text{if } p \equiv 2 \pmod 3 \text{ and } n \text{ is odd.} \end{cases}$$

By Lemma 2.2, we have

$$FG_1 \cong \begin{cases} FC_4 \oplus FC_4 \oplus FC_4, & \text{if } p \equiv 1 \pmod 3 \text{ or } n \text{ is even;} \\ F_2C_4 \oplus FC_4, & \text{if } p \equiv 2 \pmod 3 \text{ and } n \text{ is odd.} \end{cases}$$

By Lemma 3.3, we get

$$FC_4 \cong \begin{cases} F \oplus F \oplus F \oplus F, & \text{if } p \equiv 1 \pmod 4 \text{ or } n \text{ is even;} \\ F_2 \oplus F \oplus F, & \text{if } p \equiv 3 \pmod 4 \text{ and } n \text{ is odd.} \end{cases}$$

And then, we have

$$F_2C_4 \cong F_2 \oplus F_2 \oplus F_2 \oplus F_2.$$

Hence, if $p \equiv 1 \pmod{12}$ or n is even, then

$$FG_1 \cong F \oplus F \oplus F \oplus F \oplus F \oplus F \oplus F \oplus F \oplus F \oplus F \oplus F \oplus F;$$

if $p \equiv 5 \pmod{12}$ and n is odd, then

$$FG_1 \cong F_2 \oplus F_2 \oplus F_2 \oplus F_2 \oplus F \oplus F \oplus F \oplus F;$$

if $p \equiv 7 \pmod{12}$ and n is odd, then

$$FG_1 \cong F_2 \oplus F_2 \oplus F_2 \oplus F \oplus F \oplus F \oplus F \oplus F \oplus F.$$

if $p \equiv 11 \pmod{12}$ and n is odd, then

$$FG_1 \cong F_2 \oplus F_2 \oplus F_2 \oplus F_2 \oplus F_2 \oplus F \oplus F.$$

This completes the proof of the Theorem 3.4. □

Theorem 3.5. *Let $\mathscr{U}(FG_2)$ be the unit group of FG_2 over a finite field F of positive characteristic p and $|F| = p^n$, where $G_2 \cong C_3 \times K_4$.*

(1) If $p = 2$, then

$$\mathscr{U}(FG_1) \cong \begin{cases} C_2^{9n} \times C_{2^n-1}^3 & \text{if } n \text{ is even;} \\ C_2^{9n} \times C_{2^{2n}-1} \times C_{2^n-1} & \text{if } n \text{ is odd.} \end{cases}$$

(2) If $p = 3$, then

$$\mathscr{U}(FG_2) \cong C_3^{8n} \times C_{3^n-1}^4.$$

(3) If $p > 3$, then

$$\mathscr{U}(FG_1) \cong \begin{cases} C_{p^n-1}^{12} & \text{if } p \equiv 1 (\text{mod } 3) \text{ or } n \text{ is even;} \\ C_{p^{2n}-1}^4 \times C_{p^n-1}^4 & \text{if } p \equiv 2 (\text{mod } 3) \text{ and } n \text{ is odd.} \end{cases}$$

Proof. Let $H = 1 + J(FG_2)$, where $J(FG_2)$ is the Jacobson radical of the group ring FG_2. F^* is the multiplicative group of F, which is a cyclic group of order $p^n - 1$, we also denote it by C_{p^n-1}.

(1) Let $\text{char}(F) = 2$ with $|F| = 2^n$. Since $G_2 \cong C_3 \times K_4$, we have $G/K_4 \cong C_3$. Thus, by Lemma 2.1 and Lemma 2.8, $J(FG_2) = J(FK_4)FG_2 = \omega(FK_4)FG_2 = \omega(K_4)$. Hence, $FG_2/J(FG_2) \cong F(G_2/K_4) \cong FC_3$. From the ring epimorphism $FG_2 \to FC_3$, we get a group epimorphism $\phi : \mathscr{U}(FG_2)) \to \mathscr{U}(FC_3)$ and $Ker\phi = H = 1 + J(FG_2) = 1 + \omega(K_4)$.

Since C_3 is a subgroup of G_2, the ring monomorphism $FC_3 \to FG_2$, $\alpha_0+\alpha_1\sigma+\alpha_2\sigma^2 \to \alpha_0+\alpha_1\sigma+\alpha_2\sigma^2$, reduces a group homomorphism

$$\theta : \mathscr{U}(FC_3) \to \mathscr{U}(FG_2).$$

And we can verify that $\phi\theta = 1_{\mathscr{U}(FC_3)}$. Thus $\mathscr{U}(FG_2)$ is an extension of $\mathscr{U}(FC_3)$ by H, therefore

$$\mathscr{U}(FG_2) \cong H \times \mathscr{U}(FC_3).$$

Set $K_4 = \{a_0, a_1, a_2, a_3\}$ and $C_3 = \langle \sigma \mid \sigma^3 = 1 \rangle$, then

$$\omega(K_4) = \{\textstyle\sum_{i=0}^{3} \sum_{j=0}^{2} \alpha_{ij} a_i \sigma^j | \sum_{i=0}^{3} \alpha_{ij} = 0, j = 0, 1, 2,\ \alpha_{ij} \in F\}.$$

By calculation, we have that $\alpha^2 = 0$ for any $\alpha \in \omega(K_4)$.

For every $x \in H = 1 + J(FG_1) = 1 + \omega(K_4)$, set $x = 1 + \alpha, \alpha \in \omega(K_4)$, we get $x^2 = (1+\alpha)^2 = 1$. Hence, H is an elementary Abelian 2-group. In addition, $FG_2/J(FG_2) \cong FC_3$, we know that $\dim_F J(FG_2) = 9$. So $|H| = |\omega(K_4)| = |J(FG_2)| = 2^{9n}$ and so $H \cong C_2^{9n}$.

We have known that

$$FC_3 \cong \begin{cases} F \oplus F \oplus F & \text{if } n \text{ is even;} \\ F_2 \oplus F & \text{if } n \text{ is odd.} \end{cases}$$

Thus, we have finished the proof of (1).

(2) Let $\mathrm{char}(F) = 3$ and $|F| = 3^n$. Obviously we know $G_2/C_3 \cong K_4$. Also, by Lemma 2.1 and Lemma 2.8, $J(FG_2) = J(FC_3)FG_2 = \omega(FC_3)FG_2 = \omega(C_3)$, then $FG_2/J(FG_2) \cong F(G_2/C_3) \cong FK_4$. By Lemma 2.9, we have $FK_4 \cong F \oplus F \oplus F \oplus F$.

The rest part of the proof of (2) is similar to the proof of (1), we omit it.

(3) Let $\mathrm{char}(F) = p > 3$ and $|F| = p^n$, we can conclude that $FG_2 \cong F(C_3 \times K_4) \cong (FC_3)K_4$, by Lemma 2.4. We know that

$$FC_3 \cong \begin{cases} F \oplus F \oplus F, & \text{if } p \equiv 1 (\text{mod } 3) \text{ or } n \text{ is even;} \\ F_2 \oplus F, & \text{if } p \equiv 2 (\text{mod } 3) \text{ and } n \text{ is odd.} \end{cases}$$

Then, by Lemma 2.2, Lemma 2.4 and Lemma 2.9, we have

$$FG_1 \cong \begin{cases} (F \oplus F \oplus F)K_4, & \text{if } p \equiv 1 \ (\text{mod } 3) \text{ or } n \text{ is even;} \\ (F_2 \oplus F)K_4, & \text{if } p \equiv 2 \ (\text{mod } 3) \text{ and } n \text{ is odd.} \end{cases}$$

$$\cong \begin{cases} FK_4 \oplus FK_4 \oplus FK_4, & \text{if } p \equiv 1 \ (\text{mod } 3) \text{ or } n \text{ is even;} \\ F_2K_4 \oplus FK_4, & \text{if } p \equiv 2 \ (\text{mod } 3) \text{ and } n \text{ is odd.} \end{cases}$$

Hence, if $p \equiv 1 \ (\text{mod } 3)$ or n is even,

$$FG_2 \cong F \oplus F \oplus F \oplus F \oplus F \oplus F \oplus F \oplus F \oplus F \oplus F \oplus F \oplus F;$$

if $p \equiv 2 \ (\text{mod } 3)$ and n is odd,

$$FG_2 \cong F_2 \oplus F_2 \oplus F_2 \oplus F_2 \oplus F \oplus F \oplus F \oplus F.$$

This completes the proof of the Theorem 3.5. □

4. The unit groups of FQ_{12} and FD_{12}

In [16], the structure of $\mathscr{U}(FA_4)$ was studied by R.K. Sharma, J.B. Srivastava and M. Khan in 2007. In this section, we give the structures of $\mathscr{U}(FQ_{12})$ and $\mathscr{U}(FD_{12})$.

Lemma 4.1. *Let $p \geq 5$ be a prime and F a finite field of characteristic p with $|F| = p^n$. Then $FS_3 \cong F \oplus F \oplus M(2, F)$.*

Proof. Assume $p \geq 5$. Since $p \nmid |S_3|$, by Lemma 2.5 and Lemma 2.6, we have

$$FS_3 \cong M(n_1, D_1) \oplus M(n_2, D_2) \oplus \cdots \oplus M(n_r, D_r),$$

where D_i's are finite dimentional division algebras over F. Thus D_i's are finite fields, as F is finite. Since FS_3 is non-commutative, there exists a k such that $n_k > 1$. So

$$FS_3 \cong F \oplus F \oplus M(2,F) \text{ or } F_2 \oplus M(2,F).$$

We know that $\mathscr{C}_1 = \{1\}$, $\mathscr{C}_2 = \{a, a^2\}$, and $\mathscr{C}_3 = \{b, ab, a^2b\}$ are all conjugacy classes of S_3, and then $\widehat{\mathscr{C}_1} = 1$, $\widehat{\mathscr{C}_2} = a + a^2$, $\widehat{\mathscr{C}_3} = b + ab + a^2b$, and by [8], $\mathscr{Z}(FS_3) = F\widehat{\mathscr{C}_1} + F\widehat{\mathscr{C}_2} + F\widehat{\mathscr{C}_3}$. It is easy to verify that

$$\widehat{\mathscr{C}_1}^{p^n} = \widehat{\mathscr{C}_1},\ \widehat{\mathscr{C}_2}^{p^n} = \widehat{\mathscr{C}_2}.$$

Since $1 + a + a^2 \in \mathscr{Z}(FS_3)$ and p is odd, we can conclude that

$$\widehat{\mathscr{C}_3}^{p^n} = (b + ab + a^2b)^{p^n} = (1 + a + a^2)^{p^n} b^{p^n} = b + ab + a^2b = \widehat{\mathscr{C}_3}.$$

Thus, we have $x^{p^n} = x$, $\forall\ x \in \mathscr{Z}(FS_3)$ and therefore $FS_3 \cong F \oplus F \oplus M(2,F)$. □

Theorem 4.2. *Let $\mathscr{U}(FQ_{12})$ be the unit group of FQ_{12} of the generalized quaternion group of order 12, Q_{12}, over a finite field F of positive characteristic p and $|F| = p^n$. Let $V_1 = 1 + J(FQ_{12})$, where $J(FQ_{12})$ is the Jacobson radical of the group ring FQ_{12}.*

(1) *If $p = 2$ and $V_2 = 1 + \omega(H)$, where $H = \{1, a^3\}$, then*

(*a*) *$\mathscr{U}(FQ_{12})/V_1 \cong GL(2,F) \times F^*$*

(*b*) *The structure of V_1 is determined as following:*

(*i*) *V_1/V_2 is an elementary Abelian 2−group of order 2^n;*

(*ii*) *V_1 is a nilpotent group of class 2.*

(2) *If $p = 3$, then*

$$(a)\ \mathscr{U}(FQ_{12})/V_1 \cong \begin{cases} F^* \times F^* \times F^* \times F^* & \text{if } n \text{ is even;} \\ F_2^* \times F^* \times F^* & \text{if } n \text{ is odd.} \end{cases}$$

(*b*) *V_1 is an elementary 3-group of order 3^{8n} and a nilpotent group of class 2.*

(*c*) *$\mathscr{U}(FQ_{12})$ is a nilpotent group of class 3.*

(3) *If $p > 3$, then*

(*a*) *$\mathscr{U}(FQ_{12}) \cong GL(2,F) \times GL(2,F) \times F^* \times F^* \times F^* \times F^*$, if $p \equiv 1, 5 (mod\ 12)$ or n is even;*

(*b*) *$\mathscr{U}(FQ_{12}) \cong GL(2,F) \times GL(2,F) \times F_2^* \times F^* \times F^*$, if $p \equiv 7, 11 (mod\ 12)$ and n is odd.*

Proof. (1) Let $p = 2$ and $|F| = 2^n$.
(a) Since $a^3 \in \mathscr{Z}(FQ_{12})$, we know that $H = \{1, a^3\}$ is a normal subgroup of Q_{12}. As $Q_{12}/H \cong S_3$, we have $FS_3 \cong F(Q_{12}/H) \cong FQ_{12}/\omega(H)$ so that $\dim_F(\omega(H)) = 6$. Further, we get $\omega^2(H) = 0$. By Lemma 2.3, we obtain $\omega(H) \subseteq J(FQ_{12})$. Hence, $J(FS_3) \cong J(FQ_{12}/\omega(H)) \cong J(FQ_{12})/\omega(H)$. By calculation, we derive that $J(FS_3) = F\alpha$, where $\alpha = 1 + a + a^2 + b + ab + a^2b$. So $\dim_F J(FS_3) = 1$ and $J(FS_3)^2 = 0$. Thus we have $\dim_F J(FQ_{12}) = 7$ and $(J(FQ_{12})/\omega(H))^2 = 0$. This implies $(J(FQ_{12}))^2 \subseteq \omega(H)$. Hence, $\dim_F(FQ_{12}/J(FQ_{12})) = 5$.

Since $FQ_{12}/J(FQ_{12}) \cong (FQ_{12}/\omega(H))/(J(FQ_{12})/\ \omega(H)) \cong FS_3/J(FS_3)$, and $FS_3/J(FS_3)$ is semisimple, $\dim_F(FS_3/J(FS_3)) = 5$, then $FS_3/J(FS_3) \cong F \oplus F \oplus F \oplus F \oplus F$ or $M(2,F) \oplus F$. Since $\bar{\gamma} = (1-a)ab(1+a) \in FS_3/J(FS_3)$, then $\bar{\gamma} \neq 0$ but $\bar{\gamma}^2 = 0$. Thus, we have $FS_3/J(FS_3) \cong M(2,F) \oplus F$. Hence, $FQ_{12}/J(FQ_{12})$ is non-commutative, $FQ_{12}/J(FQ_{12}) \cong M(2,F) \oplus F$. Thus

$$\mathscr{U}(FQ_{12})/V_1 \cong \mathscr{U}(FQ_{12}/J(FQ_{12})) \cong GL(2,F) \times F^*.$$

(b) Since $(J(FQ_{12}))^2 \subseteq \omega(H)$, for any $y \in V_1/V_2$, let $y = v_1V_2$, where $v_1 \in V_1$, then $y^2 = v_1^2V_2$. Let $v_1 = 1 + x, x \in J(FQ_{12})$, we get $v_1^2 = (1+x)^2 = 1 + x^2 \in V_2$. So $o(y) = 2$. Hence V_1/V_2 is an elementary Abelian 2-group. Further, $|V_1| = |J(FQ_{12})| = 2^{7n}$ and $|V_2| = |\omega(H)| = 2^{6n}$. This proves (i) as $V_1/V_2 = 2^n$.

Now, observe that $(J(FQ_{12}))^2 \subseteq \omega(H) \subseteq \mathscr{Z}(FQ_{12})$ and $\omega^2(H) = 0$, so that $(J(FQ_{12})^4 \subseteq \omega^2(H) = 0$. For $\zeta, \eta \in J(FQ_{12})$, since $(1+\zeta)(1-\zeta+\zeta^2-\zeta^3) = 1-\zeta^4 = 1$, then $(1+\zeta)^{-1} = 1 - \zeta + \zeta^2 - \zeta^3$. Hence $(1+\zeta)^{-1} \equiv 1-\zeta \pmod{\mathscr{Z}(FQ_{12})}$. Similarly $(1+\eta)^{-1} \equiv 1-\eta \pmod{\mathscr{Z}(FQ_{12})}$. Then we have

$$\begin{aligned}(1+\zeta, 1+\eta) &\equiv (1+\zeta)^{-1}\ (1+\zeta)^{-1}\ (1+\zeta)\ (1+\eta) && \bmod \mathscr{Z}(FQ_{12})\\ &\equiv (1-\zeta)(1-\eta)\ (1+\zeta)\ (1+\eta) && \bmod \mathscr{Z}(FQ_{12})\\ &\equiv (1-\zeta-\eta)\ (1+\zeta+\eta) && \bmod \mathscr{Z}(FQ_{12})\\ &\equiv 1 && \bmod \mathscr{Z}(FQ_{12})\end{aligned}$$

Thus $\gamma_2(V_1) \subseteq \mathscr{Z}(FQ_{12})$ and hence $\gamma_3(V_1) = (1)$. Consequently, V_1 is a nilpotent group of class 2.

(2) Let $p = 3$ and $|F| = 3^n$.

(a) It is easy to see that the commutator subgroup of Q_{12} is $Q_{12}' = \{1,\ a^2,\ a^4\}$. Since $Q_{12}/Q_{12}' \cong C_4$, we have $FC_4 \cong F(Q_{12}/Q_{12}') \cong FQ_{12}/\omega(Q_{12}')$ so that $\dim_F\omega(Q_{12}') = 8$. Further, $\omega(Q_{12}')^2 = (1 + a_2 + a_4)\ FQ_{12}$ and $\omega^3(Q_{12}') = 0$, by Lemma 2.3, we have $\omega(Q_{12}') \subseteq J(FQ_{12})$. Hence, $J(FC_4) \cong J(FQ_{12}/\omega(Q_{12}')) \cong J(FQ_{12})/\omega(Q_{12}')$.

Since $\mathrm{char}(F) = 3$, by Lemma 3.3, we have

$$FC_4 \cong \begin{cases} F \oplus F \oplus F \oplus F & \text{if } n \text{ is even;}\\ F_2 \oplus F \oplus F & \text{if } n \text{ is odd.}\end{cases}$$

Thus we have $\dim_F J(FC_4) = 0$ and $J(FC_4) = 0$. Hence, $J(FQ_{12}) \subseteq \omega(Q_{12}')$. So $J(FQ_{12}) = \omega(Q_{12}')$. Hence

$$FQ_{12}/J(FQ_{12}) \cong \begin{cases} F \oplus F \oplus F \oplus F & \text{if } n \text{ is even;}\\ F_2 \oplus F \oplus F & \text{if } n \text{ is odd.}\end{cases}$$

Thus,

$$\mathscr{U}(FQ_{12})/V_1 \cong \mathscr{U}(FQ_{12}/J(FQ_{12})) \cong \begin{cases} F^* \times F^* \times F^* \times F^* & \text{if } n \text{ is even;}\\ F_2^* \times F* \times F* & \text{if } n \text{ is odd.}\end{cases}$$

(b) Observe that $\omega^2(Q'_{12}) = (1 + \mathrm{a}^2 + \mathrm{a}^4)\ FQ_{12} \subseteq \mathscr{Z}(FQ_{12})$ and $\omega^3(Q'_{12}) = 0$. $\forall\ \zeta, \eta \in \omega(Q'_{12})$, we have

$$\begin{aligned} (1+\zeta,\, 1+\eta) &\equiv (1+\zeta)^{-1}\,(1+\eta)^{-1}\,(1+\zeta)\,(1+\eta) && \mod \mathscr{Z}(FQ_{12}) \\ &\equiv (1-\zeta)\,(1-\eta)\,(1+\zeta)\,(1+\eta) && \mod \mathscr{Z}(FQ_{12}) \\ &\equiv (1-\zeta-\eta)\,(1+\zeta+\eta) && \mod \mathscr{Z}(FQ_{12}) \\ &\equiv (1+2\zeta+2\eta)\,(1+\zeta+\eta) && \mod \mathscr{Z}(FQ_{12}) \\ &\equiv 1 && \mod \mathscr{Z}(FQ_{12}) \end{aligned}$$

Thus $\gamma_2(V_1) \subseteq \mathscr{Z}(FQ_{12})$ and $\gamma_3(V_1) = (1)$. Hence, V_1 is a nilpotent group of class 2.

Also, $\dim_F J(FQ_{12}) = 8$, then $|V_1| = 3^{8n}$. Since $J(FQ_{12})^3 = \omega^3(Q'_{12}) = 0$, then the order of any nontrivial element of V_1 is 3. So V_1 is an elementary 3−group of order 3^{8n}.

(c) Since $\mathscr{U}(FQ_{12})/V_1 \cong \mathscr{U}(FC_4)$ is an Abelian group, we have $\mathscr{U}(FQ_{12})' \subseteq V_1$, therefore $\mathscr{U}(FQ_{12})'' \subseteq V_1' \subseteq \mathscr{Z}(FQ_{12})$. Hence $\mathscr{U}(FQ_{12})$ is a nilpotent group of class 3.

(3) Assume $p > 3$. Since $p \nmid |Q_{12}|$, by Lemma 2.5 and Lemma 2.6, we have

$$FQ_{12} \cong M(n_1, D_1) \oplus M(n_2, D_2) \oplus \cdots \oplus M(n_r, D_r)$$

where D_i's are finite dimentional division algebras over F. Thus D_i's are finite fields, as F is finite. Since FQ_{12} is non-commutative, there exists a k such that $\mathrm{n}_k > 1$, so that n_k will be either 2 or 3. Further, $\dim_F \mathscr{Z}(FQ_{12}) = 6$, all of the possible structures of the group ring FQ_{12} are given by

$$\begin{aligned} FQ_{12} \cong\ & M(2,F) \oplus M(2,F) \oplus F \oplus F \oplus F \oplus F \ \textit{or} \\ & M(2,F_2) \oplus F \oplus F \oplus F \oplus F \ \text{or} \\ & M(2,F_2) \oplus F_2 \oplus F \oplus F \quad \text{or} \\ & M(2,F_2) \oplus F_2 \oplus F_2 \quad \text{or} \\ & M(2,F_2) \oplus F_3 \oplus F \quad \text{or} \\ & M(2,F_2) \oplus F_4 \quad \text{or} \\ & M(2,F) \oplus M(2,F) \oplus F_2 \oplus F \oplus F \quad \text{or} \\ & M(2,F) \oplus M(2,F) \oplus F_2 \oplus F_2 \quad \text{or} \\ & M(2,F) \oplus M(2,F) \oplus F_3 \oplus F \quad \text{or} \\ & M(2,F) \oplus M(2,F) \oplus F_4 \end{aligned}$$

Obviously, every element of $\mathscr{Z}(FQ_{12})$ is an F-linear combination of conjugacy class sums of Q_{12}. Observe the following regarding these conjugacy class sums equations:

$$\widehat{\mathscr{C}_1}^{p^n} = \widehat{\mathscr{C}_1}, \widehat{\mathscr{C}_3}^{p^n} = \widehat{\mathscr{C}_3} \text{ and } \widehat{\mathscr{C}_4}^{p^n} = \widehat{\mathscr{C}_4}$$

Next, we need to show other equations.

(i) If $p \equiv 1 \pmod{12}$, then $p^n \equiv 1 \pmod{12}$, for all n. Then we compute that

$$\widehat{\mathscr{C}_2}^{p^n} = (a+a^5)^{p^n} = a \cdot a^{p^n-1} + a^5 \cdot (a^5)^{p^n-1} = a + a^5 = \widehat{\mathscr{C}_2}$$

$$\widehat{\mathscr{C}_5}^{p^n} = [b(1+a^2+a^4)]^{p^n} = b^{p^n-1} \cdot [b(1+a^2+a^4)] = b(1+a^2+a^4) = \widehat{\mathscr{C}_5}$$

$$\widehat{\mathscr{C}_6}^{p^n} = [ab(1+a^2+a^4)]^{p^n} = (ab)^{p^n-1} \cdot [ab(1+a^2+a^4)] = ab(1+a^2+a^4) = \widehat{\mathscr{C}_6}$$

Thus $x^{p^n} = x$, for all $x \in \mathscr{Z}(FQ_{12})$. In particular, if $x \in \mathscr{U}(\mathscr{Z}(FQ_{12}))$, then $o(x) \mid (p^n - 1)$. Hence

$$FQ_{12} \cong M(2,F) \oplus M(2,F) \oplus F \oplus F \oplus F \oplus F.$$

(ii) If $p \equiv 5,\ 7,\ 11 \pmod{12}$ and n is even, then clearly $p^n \equiv 1 \pmod{12}$. Hence, in this case , we also have

$$FQ_{12} \cong M(2,F) \oplus M(2,F) \oplus F \oplus F \oplus F \oplus F.$$

(iii) If $p \equiv 5 \pmod{12}$ and n is odd, note that $o(a) = 6$, $o(b) = 4$, and $n = 2k+1$, where $k = (n-1)/2$, then we can verify that:

$$\widehat{\mathscr{C}_i}^{p^n} = \widehat{\mathscr{C}_i}, \text{ for } 1 \le i \le 6.$$

Hence, in this case, we also have

$$FQ_{12} \cong M(2,F) \oplus M(2,F) \oplus F \oplus F \oplus F \oplus F.$$

(iv) If $p \equiv 7,\ 11 \pmod{12}$ and n is odd, then we can verify that:

$$\widehat{\mathscr{C}_i}^{p^{2n}} = \widehat{\mathscr{C}_i}, \text{ for } 1 \le i \le 6, \text{ but } \widehat{\mathscr{C}_5}^{p^n} \ne \widehat{\mathscr{C}_5}.$$

So, in this case, $x^{p^{2n}} = x$, for any $x \in \mathscr{Z}(FQ_{12})$, i.e., $o(x) \mid (p^{2n} - 1)$, for any $x \in \mathscr{U}(\mathscr{Z}(FQ_{12}))$. Thus the occurrences of F_3 and F_4 in the presentation of FQ_{12} are impossible. Hence the possible presentations of the group ring FQ_{12} are the followings:

$$\begin{aligned} FQ_{12} \cong\ & M(2,F_2) \oplus F \oplus F \oplus F \oplus F \quad \text{or} \\ & M(2,F_2) \oplus F_2 \oplus F \oplus F \quad \text{or} \\ & M(2,F_2) \oplus F_2 \oplus F_2 \quad \text{or} \\ & M(2,F) \oplus M(2,F) \oplus F_2 \oplus F \oplus F \quad \text{or} \\ & M(2,F) \oplus M(2,F) \oplus F_2 \oplus F_2 \end{aligned}$$

Since $FQ_{12}/\omega(H) \cong FS_3$ and FQ_{12} is semisimple, by Lemma 4.1, we have that $FQ_{12} \cong M(2,F) \oplus M(2,F) \oplus F_2 \oplus F \oplus F$.

Hence, we have

(*a*) $\mathscr{U}(FQ_{12}) \cong GL(2,F) \times GL(2,F) \times F^* \times F^* \times F^* \times F^*$, if $p \equiv 1, 5 \pmod{12}$ or n is even;

(*b*) $\mathscr{U}(FQ_{12}) \cong GL(2,F) \times GL(2,F) \times F_2^* \times F^* \times F^*$, if $p \equiv 7, 11 \pmod{12}$ and n is odd.

This completes the proof of the Theorem 4.2. □

Theorem 4.3. *Let $\mathscr{U}(FD_{12})$ be the unit group of FD_{12} of the dihedral group of order 12, D_{12}, over a finite field F of positive characteristic p and*

$|F| = p^n$. *Let* $V_1 = 1 + J(FD_{12})$, *where* $J(FD_6)$ *is the Jacobson radical of the group ring* FD_{12}.

(1) *If* $p = 2$, *then*

(a) $\mathscr{U}(FD_{12})/V_1 \cong GL(2, F) \times F^*$;

(b) *The structure of* V_1 *is determined as:*

(i) V_1/V_2 *is an elementary Abelian 2−group of order* 2^n;

(ii) V_1 *is a nilpotent group of class 2.*

(2) *If* $p = 3$, *then*

(a) $\mathscr{U}(FD_{12})/V_1 \cong F^* \times F^* \times F^* \times F^*$;

(b) V_1 *is an elementary 3-group of order* 3^{8n} *and a nilpotent group of class 2;*

(c) $\mathscr{U}(FD_{12})$ *is a nilpotent group of class 3.*

(3) *If* $p > 3$, *then*

$\mathscr{U}(FD_{12}) \cong GL(2, F) \times GL(2, F) \times F^* \times F^* \times F^* \times F^*$.

Proof. (1)(a) Let char(F) = 2 with $|F| = 2^n$. We know that $a^3 \in \mathcal{Z}(FD_{12})$ and $H = \{1, a^3\}$ is a normal subgroup of D_{12}. It is easy to verify that $D_{12}/H \cong S_3$. Then we get that $\mathscr{U}(FD_{12})/\mathrm{V}_1 \cong GL(2, F) \times F^*$, which is similar to (1) (a) of Theorem 4.2.

The proof of (b) is Similar to the proof of (1) (b) of Theorem 4.2.

(2)(a) Let char(F) = 3 and $|F| = 3^n$. It is easy to see that the commutator subgroup of D_{12} is $D'_{12} = \{1,\ a^2,\ a^4\}$ and $D_{12}/D'_{12} \cong K_4$. By Lemma 2.8 and Lemma 2.1, we have that $J(FD_{12}) = J(FD'_{12})FD_{12} = \omega(FD'_{12})FD_{12} = \omega(D'_{12})$, so $FD_{12}/J(FD_{12}) \cong FD_{12}/\omega(D'_{12}) \cong FK_4 \cong F \oplus F \oplus F \oplus F$ by Lemma 2.9.

Thus

$$\mathscr{U}(FD_{12})/V_1 \cong \mathscr{U}(FD_{12}/J(FD_{12})) \cong F^* \times F^* \times F^* \times F^*.$$

In addition, a similar argument of (2)(b) and (c) of Theorem 4.2 can be applied to here.

(3) Assume $p > 3$ and $|F| = p^n$. Since p $\nmid |D_{12}|$, by Lemma 2.5 and Lemma 2.6, we have

$$FD_{12} \cong M(n_1, C_1) \oplus M(n_2, H) \oplus \cdots \oplus M(n_r, C_r)$$

where C_i's are finite dimentional division algebras over F. Thus C_i's are finite fields, as F is finite. Since FD_{12} is non-commutative, there exists a k such that $n_k > 1$, so that n_k will be either 2 or 3. Further, since $\dim_F \mathcal{Z}(FD_{12})$ = 6, we will get the following possibilities:

$$\begin{aligned} FD_{12} \cong\ & M(2, F) \oplus M(2, F) \oplus F \oplus F \oplus F \oplus F \quad \text{or} \\ & M(2, F_2) \oplus F \oplus F \oplus F \oplus F \quad \text{or} \\ & M(2, F_2) \oplus F_2 \oplus F \oplus F \quad \text{or} \end{aligned}$$

$$M(2,F_2)\oplus F_2\oplus F_2 \quad \text{or}$$
$$M(2,F_2)\oplus F_3\oplus F \quad \text{or}$$
$$M(2,F_2)\oplus F_4 \quad \text{or}$$
$$M(2,F)\oplus M(2,F)\oplus F\oplus F\oplus F\oplus F \quad \text{or}$$
$$M(2,F)\oplus M(2,F)\oplus F_2\oplus F\oplus F \quad \text{or}$$
$$M(2,F)\oplus M(2,F)\oplus F_2\oplus F_2 \quad \text{or}$$
$$M(2,F)\oplus M(2,F)\oplus F_3\oplus F \quad \text{or}$$
$$M(2,F)\oplus M(2,F)\oplus F_4$$

We know that every element of $\mathcal{Z}(FD_{12})$ is an F-linear combination of conjugacy class sums of D_{12}. Since $p>3$, we have $p\equiv 1$ or $5 \pmod 6$ and then $p^n\equiv 1$ or $5 \pmod 6$ for any positive integer n. By straightforward computing, we get that:

$$\widehat{\mathscr{C}_i}^{P^n}=\widehat{\mathscr{C}_i}, \text{ for any } 1\le i\le 6.$$

Thus $x^{p^n}=x$, for any $x\in\mathcal{Z}(FD_{12})$. In particular, if $x\in\mathscr{U}(\mathcal{Z}(FD_{12}))$, then $o(x)\mid(p^n-1)$. Hence

$$FD_{12}\cong M(2,F)\oplus M(2,F)\oplus F\oplus F\oplus F\oplus F.$$

Hence, in this case, we have

$\mathscr{U}(FD_{12})\cong GL(2,F)\times GL(2,F)\times F^*\times F^*\times F^*\times F^*$.

This completes the proof of the Theorem 4.3. □

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References

[1] L. Creedon, J. Gildea, The structure of the unit group of the group algebra $F_{3^k}D_6$, *Int. J. Pure Appl. Math.* **45**, No. 2 (2008), 315-320.

[2] J. Gildea, On the Order of $\mathscr{U}(F_{p^k}D_{2p^m})$, *Int. J. Pure Appl. Math.* **46**, No. 2 (2008), 267-272.

[3] J. Gildea, The structure of $\mathscr{U}(F_{5^k}D_{20})$, *International Electronic Journal of Alebra*, **8** (2010), 153-160.

[4] J. Gildea, The structure of the unit group of the group algebra $F_{3^k}(C_3 \times D_6)$, *Comm. Algebra*, **38** (2010), 3311-3317.

[5] J. Gildea, The structure of the unit group of the group algebra of Paulis‘s group over any field of characteristic 2, *International Jounal of Algebra and Computation*, **20**, No. 5 (2010), 721-729.

[6] I. Hughes, K.R. Pearson, The group of units of the integral group ring ZS_3, *Canda. Math. Bull.*, **15** (1972), 85-92.

[7] E. Jespers, M. M. Parmenter, Units of group rings of groups of order 16, *Glasgow Math. J.*, **35** (1993), 367-379.

[8] G. Karpilovsky, *Unit Groups of Classical Rings*, Oxford, Clarendon Press (1988).

[9] M. Khan, R.K. Sharma, J.B. Srivastava, The unit group of FS_4, *Acta Math. Hungar.*, **118**, No-s: 1-2 (2008), 105-113.

[10] T.Y. Lam, *A First Course in Noncommutative Rings*, New York, Springer Verlag (1991).

[11] Y.L. Li, M.M. Parmenter, Central units of the integral group ring ZA_5, *Proc. Amer. Math. Soc.*, **125**, No. 1 (1997), 61-65.

[12] C.P. Milies The group of units of the integral group ring ZD_4, *Bol. Soc. Brasileira de Mat.*, **4** (1972), 85-92.

[13] C.P. Milies, S.K. Sehgal, *An Introduction to Group Rings*, Netherlands, Kluqver Academic Publishers (2002).

[14] C.D. Pan, C.B. Pan, *Elementary Number Theorey*, Second Edition, Beijing, Beijing University Publishing Company (2005).

[15] D.S. Passman, *The Algrbraic Structure of Group Rings*, New York, Wiley-Interscience, John Wiley Sons (1977).

[16] R.K. Sharma, J.B. Srivastava, M. Khan, The unit group of FA_4, *Publ. Math. Debrecen.*, **71** (2007), 21-26.