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OPTTIMAL ALLOCATION OF REDUNDANCY IN A MIXED SYSTEM

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Abstract: This paper investigates *optimal allocation of reduncancy in a mixed system with n stages in series where components are in parallel at each stage.* The method described in this paper gives stonger convergence criteria and finds optimal solutions which may be nonintegral and can then be used to generate optimal integral solutions in polynomial time.

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1. Introduction

The reliability of a system has been defined variously in the literature. According to the National Aeronautics and Space Administration (NASA) (see [4]) the reliability of a system is "the probability of a device performing adequately for the period of time intended under the operating conditions encountered." Many industrial system designers are routinely faced with the problem of ensuring that a given complex system adequately performs its intended function within a specified interval of time. Usually the failure characteristic of some systems are used to determine the economic feasibility of the system. "To increase

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the overall reliability of a complex system, reduncancies are often designed into the system", see [4].

2. Statement of Problem

The problem of optimal allocation of reduncancy in a mixed system with n stages in series where components are in parallel at each stage can be stated as follows:

$$\begin{cases} \min \quad C^T x = \sum_{i=1}^{n} c_i x_i, & (a) \\ \text{s.t.} \quad R_s(x) = \prod_{i=1}^{n} (1 - \rho_i^{x_i}) \geq R_r, & (b) \\ \quad x > 0, & \end{cases} \tag{2.1}$$

where $i = 1, 2, 3, \cdots, n$, n is the number of stages of a system, each x_i is a redundant unit corresponding to the i-th stage (subsystem), ρ_i is component unreliability for the i-th stage, $R_s(x)$ is system reliability as a function of the n-vector, x, of decision variables in n-dimensional Euclidean space, R^n, c_i is component cost($c_i > 0$), $C_s(x)$ is system cost as a function of x, and R_r is minimum required system reliability ($0 < R_r < 1$).

3. Computational Procedure

Definition 1. Let R_+^n be the set of all ordered n-tuples of positive real numbers, $R_s(x)$ and R_r be as defined in (2.1). Define the set K as

$$K := \{ x \in R_+^n \mid R_s(x) \geq R_r \}.$$

To describe the solution method, we make the following observations:

Observation 3.1. 1. In practice the data may be readily available and the task facing the design engineer could be to determine the number of redundant components that would minimize the the overall system cost; hence, it is justified to assume that the system cost vector, c, the vector, r, of component reliabilities for the system, and R_r, the pre-determined minimum system reliability have been provided.

2. C being the cost vector, its components are rational; thus it can be rescaled to an integral vector without altering the integral and nonintegral optimal feasible vectors.

To find find the nonintegral optimal solution, x^*,we relax the integrality constraint so that the variables are treated as though they were continuous variables.

Lemma 3.2. *The set* $B := \{x \in K \mid C^T x \leq C^T x^{(SI)}\}$ *is compact.*

Proof. Note that $\mathbf{B}$ is nonvoid and closed.
Since $c_i > 0\,(1 \leq i \leq n)$, $\mathbf{B}$ is bounded. □

Theorem 2. *Let* $\mathbf{K}$ *be as in Definition 3.1 and* $\mathbf{B}$ *be as in Lemma 3.2. Then the nonintegral solution,* x^*, *to* **(2.1)** *lies in* $\partial\mathbf{K}$, *the boundary of* $\mathbf{K}$. *Furthermore, there exists an optimal integral vector,* $\widetilde{x}$, *in* $\mathbf{B}$.

Proof. Let $\neg\mathbf{B}$ denote the completement of the set $\mathbf{B}$ and note that $C^T x \geq C^T x^{(SI)}$ for any $x \in \mathbf{K} \cap \neg\mathbf{B}$. Therefore,

$$\min_{x \in \mathbf{B}} C^T x = \min_{x \in \mathbf{K}} C^T x.$$

Now, observe that $C^T x$ is a continuous function on the compact set, $\mathbf{B}$. Hence, x^* exists. Since $\nabla_{\mathbf{x}}(C^T x) = (c_i) \in \mathbf{R}^{\mathbf{n}}_{+}$, it follows that $\nabla_{\mathbf{x}}(C^T x)$ $\neq \mathbf{0}$ for all elements x in $\mathbf{K}$. Thus, $C^T x$ has no stationary point in the interior of $\mathbf{K}$. Hence, x^* must be on the boundary of $\mathbf{K}$. Next, note that the integral feasible vector $x^{(SI)}$ is in $\mathbf{B}$; consequently, $\widetilde{x}$ is also in $\mathbf{B}$. □

In the literature, many presentations of nonlinear optimization use a standard notation in which an inequality constraint is written as $g(x) \leq 0$, for an appropriate function g. To conform to this practice, we re-write the constraint on the system reliability as

$$R_r - \prod_{i=1}^{n} (1 - \rho_i^{x_i}) \leq 0$$

and introduce the function g definded as

$$g(x) = R_r - \prod_{i=1}^{n} (1 - \rho_i^{x_i}).$$

We will use Lagrange multipliers to locate possible local minimum points. But before we do, we need to check regularity, a technical condition on the constraints. For our problem, this condition applies only to points on the boundary of the feasible set. For such points we need only check that $\nabla\ \mathbf{g} \neq \mathbf{0}$.

Lemma 3.3. *Every feasible point of* (2.1) *is regular.*

Proof. Notice that

$$\nabla_{\mathbf{x_i}} \mathbf{g} = (\ln \rho_i)\, \rho_i^{x_i} \prod_{i \neq j} \left(1 - \rho_j^{x_j}\right).$$

Since $x > 0$ for all elements of x of $\mathbf{K}$, and $0 < \rho_i < 1$, we see that $\nabla g \neq \mathbf{0}$. □

Proposition 3.4. (Necessary Optimality Conditions) *Suppose that* x^* *is a nonintegral optimal solution of (2.1). Then there exists a* $\boldsymbol{\lambda}^* \geq \mathbf{0}$ *such that*

$$\nabla \mathcal{L}(x^*, \boldsymbol{\lambda}^*) = \begin{bmatrix} \nabla_x \mathcal{L}(x^*, \boldsymbol{\lambda}^*) \\ \nabla_{\boldsymbol{\lambda}} \mathcal{L}(x^*, \boldsymbol{\lambda}^*) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix},$$

where

$$\mathcal{L}(x, \boldsymbol{\lambda}) = \sum_{i=1}^{n} c_i x_i + \boldsymbol{\lambda}\, g(x)$$

is the Lagrangian function.

A proof of this Proposition 3.4 can be found in any standard text on nonlinear programming such as [3], and is therefore omitted here.

We now develop some lemmas and propositions in preparation for the main result,Theorem (3), of this paper.

Lemma 3.5. *If* $\nabla \mathcal{L}(x^*, \boldsymbol{\lambda}^*) = 0$, *then*

$$\prod_{i=1}^{n} (\lambda a_i R_r) = R_r \prod_{i=1}^{n} (c_i + \lambda a_i R_r),$$

where $a_i = -\ln(\rho_i)$.

Proof. By Lemma 3.3 every feasible point x^* of (2.1) is regular, and so by Proposition 3.4 there is a corresponding nonnegative Lagrange multiplier $\boldsymbol{\lambda}^*$ such that x^* and $\boldsymbol{\lambda}^*$ solve

$$\nabla \mathcal{L}(x, \boldsymbol{\lambda}) = \begin{bmatrix} \nabla_x \mathcal{L}(x, \boldsymbol{\lambda}) \\ \nabla_{\boldsymbol{\lambda}} \mathcal{L}(x, \boldsymbol{\lambda}) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}. \tag{3.1}$$

Now computing $\nabla \mathcal{L}(x, \boldsymbol{\lambda})$ explicitly, we see that equation (3.1) is equivalent to

$$\begin{bmatrix} c_i - \lambda a_i \rho_i^{x_i} \prod_{j \neq i} \left(1 - \rho_j^{x_j}\right) \\ \prod_{i=1}^{n} (1 - \rho_i^{x_i}) \end{bmatrix} = \begin{bmatrix} 0 \\ R_r \end{bmatrix}. \tag{3.2}$$

Let $z_i = \rho_i^{x_i}$. Then using the second equation in equation (3.2), we get

$$c_i - z_i \left(c_i + \lambda a_i R_r \right) = 0. \tag{3.3}$$

Now from equation (3.3) we get:

$$z_i = \frac{c_i}{c_i + \lambda a_i R_r} \quad \text{which implies that}$$

$$-z_i = \frac{\lambda a_i R_r}{c_i + \lambda a_i R_r}, \quad i = 1, 2, 3, \ldots, n. \tag{3.4}$$

Thus, the second equation in equation (3.2) becomes

$$\prod_{i=1}^{n} \left(\frac{\lambda a_i R_r}{c_i + \lambda a_i R_r} \right) = R_r, \tag{3.5}$$

which is equivalent to

$$\prod_{i=1}^{n} \left(\lambda a_i R_r \right) = R_r \prod_{i=1}^{n} \left(c_i + \lambda a_i R_r \right). \tag{3.6}$$

□

The following proposition is a generalization of the symmetric functions of n positive numbers.

Proposition 3.6. *Let c_i, a_i, and α be positive numbers $(i = 1, 2, 3, \cdots, n)$. Then*

$$\prod_{i=1}^{n} \left(c_i + \alpha a_i \right) = \sum_{i=0}^{n} A_{n-i} \alpha^{n-i}, \tag{3.7}$$

where

$$A_n = \prod_{i=1}^{n} a_i, \quad A_{n-1} = \sum_{1=1}^{n} \left(c_i \prod_{\substack{j=1 \\ j \neq i}}^{n} a_j \right), \quad A_{n-2} = \sum_{1 < i_1 < i_2 \leq n}^{n} \left(c_{i_1} c_{i_2} \prod_{\substack{j=1 \\ j \notin \{i_1, i_2\}}}^{n} a_j \right)$$

$$\vdots$$

$$A_{n - \lfloor \frac{n}{2} \rfloor} = \sum_{1 < i_1 < i_2 < \cdots < i_{\lfloor \frac{n}{2} \rfloor} \leq n}^{n} \left(\prod_{s}^{\lfloor \frac{n}{2} \rfloor} c_{i_s} \prod_{\substack{j=1 \\ j \notin \{i_1, i_2, \cdots, \lfloor \frac{n}{2} \rfloor\}}}^{n} a_j \right)$$

$$A_{n-\left(\lfloor \frac{n}{2}\rfloor+1\right)} = \sum_{1<i_1<i_2<\cdots<i_{\left(\lfloor \frac{n}{2}\rfloor+1\right)}\leq n} \left(\prod_{s=1}^{\lfloor \frac{n}{2}\rfloor+1} a_{i_s} \prod_{\substack{j=1 \\ j\notin\{i_1,i_2,\cdots,\lfloor \frac{n}{2}\rfloor+1\}}}^{n} c_j \right)$$

$$\vdots$$

$$A_2 = \sum_{1\leq i_1<i_2\leq n} a_{i_1}a_{i_2} \prod_{\substack{j=1 \\ j\notin\{i_1,i_2\}}}^{n} c_j, \quad A_1 = \sum_{1=1}^{n} a_i \prod_{\substack{j=1 \\ j\neq i}}^{n} c_j, \quad A_0 = \prod_{j=1}^{n} c_j.$$

Proof. The proof is by induction on n. Suppose $n = 2$. Then, we have

$$\begin{aligned}\prod_{i=1}^{2}(c_i+\alpha a_i) &= (c_1+\alpha a_1)(c_2+\alpha a_2) \\ &= c_1c_2+\alpha(c_1a_2+a_1c_2)+\alpha^2 a_1a_2 \\ &= \left(\prod_{i=1}^{2} a_i\right)\alpha^2 + \left(\sum_{i=1}^{2} c_i \prod_{\substack{j=1 \\ j\neq i}}^{2} a_j\right)\alpha + \prod_{i=1}^{2} c_i.\end{aligned}$$

Thus, the proposition holds for $n = 2$. Next, suppose that the proposition holds for $n-1$. We must show that it also holds for n. That is, we assume that the following also holds true for the proposition:

$$\prod_{i=1}^{n-1}(c_i+\alpha a_i) = \sum_{i=0}^{n-1} A'_{n-1-i}\alpha^{n-1-i}, \tag{3.8}$$

where

$$A'_{n-1} = \prod_{i=1}^{n-1} a_i, \quad A'_{n-2} = \sum_{1=1}^{n-1}\left(c_i \prod_{\substack{j=1 \\ j\neq i}}^{n-1} a_j \right),$$

$$A'_{n-3} = \sum_{1<i_1<i_2\leq n-1} \left(c_{i_1}c_{i_2} \prod_{\substack{j=1 \\ j\notin\{i_1,i_2\}}}^{n-1} a_j \right)$$

$\vdots$

$$A'_{n-1-\lfloor\frac{n-1}{2}\rfloor} = \sum_{1<i_1<i_2<\cdots<i_{\lfloor\frac{n-1}{2}\rfloor}\leq n-1}^{n-1} \left(\prod_{s}^{\lfloor\frac{n-1}{2}\rfloor} c_{i_s} \prod_{\substack{j=1\\ j\notin\{i_1,i_2,\cdots,\lfloor\frac{n-1}{2}\rfloor\}}}^{n-1} a_j \right)$$

$$A'_{n-1-\left(\lfloor\frac{n-1}{2}\rfloor+1\right)} = \sum_{1<i_1<i_2<\cdots<i_{\left(\lfloor\frac{n-1}{2}\rfloor+1\right)}\leq n-1} \left(\prod_{s=1}^{\lfloor\frac{n-1}{2}\rfloor+1} a_{i_s} \prod_{\substack{j=1\\ j\notin\{i_1,i_2,\cdots,\lfloor\frac{n-1}{2}\rfloor+1\}}}^{n-1} c_j \right)$$

$$\vdots$$

$$A'_2 = \sum_{1\leq i_1<i_2\leq n-1} a_{i_1}a_{i_2} \prod_{\substack{j=1\\ j\notin\{i_1,i_2\}}}^{n-1} c_j, \quad A'_1 = \sum_{1=1}^{n-1} a_i \prod_{\substack{j=1\\ j\neq i}}^{n-1} c_j, \quad A'_0 = \prod_{j=1}^{n-1} c_j.$$

Now, to show that the proposition also holds true for n, we observe that

$$\prod_{i=1}^{n} (c_i + \alpha a_i) = \left(\prod_{i=1}^{n-1} (c_i + \alpha a_i) \right) (c_n + \alpha a_n). \tag{3.9}$$

Using the induction hypothesis, the r.h.s. of equation (3.9) becomes

$$\left(\sum_{i=0}^{n-1} A'_{n-1-i} \alpha^{n-1-i} \right) (c_n + \alpha_n) \tag{3.10}$$

$$= \left(A'_{n-1}\alpha^{n-1} + \cdots + A'_{n-1-\lfloor\frac{n-1}{2}\rfloor} \alpha^{n-1-\lfloor\frac{n-1}{2}\rfloor} \right) (c_n + \alpha a_n)$$
$$+ \left(A'_{n-1-\left(\lfloor\frac{n-1}{2}\rfloor+1\right)} \alpha^{n-1-\left(\lfloor\frac{n-1}{2}\rfloor+1\right)} + \cdots + A'_2\alpha^2 + A'_1\alpha + A'_0 \right) (c_n + \alpha a_n).$$

But also, equation (3.10) simplifies to

$$\left[\left(\prod_{i=1}^{n-1} a_i \right) \alpha^{n-1} \right] \alpha a_n + \left[\left(\sum_{1=1}^{n-1} c_i \prod_{\substack{j=1\\ j\neq i}}^{n-1} a_j \right) \alpha^{n-2} \right] (c_n + \alpha a_n)$$

$$+ \left[\left(\sum_{1\leq i_1<i_2\leq n-1} c_{i_1}c_{i_2} \prod_{\substack{j=1\\ j\notin\{i_1,i_2\}}}^{n-1} a_j \right) \alpha^{n-3} \right] (c_n + \alpha a_n)$$

$$\vdots$$

$$+\left[\left(\sum_{1<i_1<i_2<\cdots<i_{\lfloor\frac{n-1}{2}\rfloor}\le n-1}\left(\prod_{s=1}^{\lfloor\frac{n-1}{2}\rfloor} c_{i_s} \prod_{\substack{j=1\\ j\notin\{i_1,i_2,i_3,\cdots,\lfloor\frac{n-1}{2}\rfloor\}}}^{n-1} a_j\right)\right)\alpha^{n-1-\lfloor\frac{n-1}{2}\rfloor}\right](c_n+\alpha a_n)$$

$$+\left[\left(\sum_{1\le i_1<\cdots<i_{(\lfloor\frac{n-1}{2}\rfloor+1)}\le n-1}\left(\prod_{s=1}^{\lfloor\frac{n-1}{2}\rfloor+1} a_{i_s} \prod_{\substack{j=1\\ j\notin\{i_1,i_2,\cdots,(\lfloor\frac{n-1}{2}\rfloor+1)}}^{n-1} c_j\right)\right)\alpha^{n-1-\left(\lfloor\frac{n-1}{2}\rfloor+1\right)}\right](c_n+\alpha a_n)$$

$$\vdots$$

$$+\left[\left(\sum_{1\le i_1<i_2\le n-1} a_{i_1}a_{i_2}\prod_{\substack{j=1\\ j\notin\{i_1,i_2\}}}^{n-1} c_j\right)\alpha\right](c_n+\alpha a_n)+\left[\left(\sum_{1=1}^{n-1} a_i\prod_{\substack{j=1\\ j\ne i}}^{n-1} c_j\right)\right](c_n+\alpha a_n)+\left(\prod_{j=1}^{n-1} c_j\right)c_n,$$

which is equivalent to

$$\left(\prod_{i=1}^{n} a_i\right)\alpha^n+\left(c_1\prod_{j=2}^{n} a_j+c_2\,a_1\prod_{j=3}^{n} a_j+\cdots+c_n\prod_{j=1}^{n-1} a_j\right)\alpha^{n-1} \tag{3.11}$$

$$+\left(c_1\,c_2\prod_{j=3}^{n} a_j+c_1\,c_3\,a_2\prod_{j=4}^{n} a_j+\cdots+c_{n-1}\,c_n\prod_{j=1}^{n-2} a_j\right)\alpha^{n-2}$$

$$\vdots$$

$$+\left[c_{i_1}\,c_{i_2}\,c_{i_3}\cdots c_{i_{\lfloor\frac{n}{2}\rfloor}}\left(\prod_{j=\lfloor\frac{n}{2}\rfloor+1}^{n} a_j\right)+\cdots+c_{i_n}\left(\prod_{j=1}^{\lfloor\frac{n}{2}\rfloor-1} a_j\right)\right]\alpha^{n-\lfloor\frac{n}{2}\rfloor}$$

$$+\left[a_{i_1}\,a_{i_2}\,a_{i_3}\cdots a_{i_{(\lfloor\frac{n}{2}\rfloor+1)}}\left(\prod_{j=\lfloor\frac{n}{2}\rfloor+2}^{n} c_j\right)+\cdots+a_{i_n}\prod_{j=1}^{\lfloor\frac{n}{2}\rfloor} c_j\right]\alpha^{n-1-\lfloor\frac{n}{2}\rfloor}$$

$$\vdots$$

$$+\left(a_{i_1}\,a_{i_2}\prod_{j=3}^{n} c_j+a_{i_1}\,a_{i_3}\,c_{i_2}\prod_{j=4}^{n} c_j+\cdots+a_{n-1}\,a_n\prod_{j=1}^{n-2} c_j\right)\alpha^2$$

$$+\left(a_{i_1}\prod_{j=2}^{n}c_j+\cdots+a_{i_n}\prod_{j=1}^{n-1}c_j\right)\alpha+\prod_{j=1}^{n}c_j.$$

Now comparing coefficients of like terms, we see that the coefficients in equation (3.11) are identically the same as the $A_{n-i}(i=1,2,3,\cdots,n)$, in equation (3.7). □

Proposition 3.7. *Let* λ, R_r, a_i *and* c_i $(i=1,2,3,\cdots,n)$ *be positive numbers. Then*

$$\prod_{i=1}^{n}(\lambda a_i R_r)-R_r\prod_{i=1}^{n}(c_i+\lambda R_r a_i)=\sum_{i=0}^{n}b_{n-1}\lambda^{n-i},$$

where

$$b_n=R_r^{n-1}(1-R_r)\prod_{i=1}^{n}a_i,\quad b_{n-1}=\frac{b_n}{1-R_r}\sum_{1=1}^{n}\left(\frac{c_i}{a_i}\right),$$

$$b_{n-2}=\frac{b_n}{R_r(1-R_r)}\sum_{1<i_1<i_2\leq n}\left(\frac{c_{i_1}c_{i_2}}{a_{i_1}a_{i_2}}\right),$$

$$\vdots$$

$$b_{n-\lfloor\frac{n}{2}\rfloor}=\left(\frac{b_n}{R_r^{\lfloor\frac{n}{2}\rfloor-1}(1-R_r)}\right)\sum_{1<i_1<i_2<\cdots<i_{\lfloor\frac{n}{2}\rfloor}\leq n}\left(\prod_{s=1}^{\lfloor\frac{n}{2}\rfloor}\left(\frac{c_{i_s}}{a_{i_s}}\right)\right),$$

$$b_{n-(\lfloor\frac{n}{2}\rfloor+1)}=b_0R_r^{\lfloor\frac{n}{2}\rfloor}\sum_{1<i_1<i_2<\cdots<i_{(\lfloor\frac{n}{2}\rfloor+1)}\leq n}\left(\prod_{s=1}^{\lfloor\frac{n}{2}\rfloor+1}\left(\frac{a_{i_s}}{c_{i_s}}\right)\right),$$

$$\vdots$$

$$b_2=b_0R_r^2\sum_{1\leq i_1<i_2\leq n}\left(\frac{a_{i_1}a_{i_2}}{c_{i_1}c_{i_2}}\right),\quad b_1=b_0R_r\sum_{1=1}^{n}\left(\frac{a_i}{c_i}\right),$$

and $b_0=\prod_{1=1}^{n}c_i.$

Proof. Since

$$\prod_{i=1}^{n}(\lambda a_i R_r) \;=\; R_r\prod_{i=1}^{n}(c_i+\lambda a_i R_r)\,,and\left(R_r^n\prod_{i=1}^{n}a_i\right)\lambda^n \;=\; R_r\prod_{i=1}^{n}(c_i+\lambda a_i R_r)\,,$$

it follows that

$$\left(R_r^{n-1}\prod_{i=1}^{n}a_i\right)\lambda^n \;=\; \prod_{i=1}^{n}(c_i+\lambda a_i R_r)\,.$$

Now set P_n equal to

$$\left(R_r^{n-1}\prod_{i=1}^{n}a_i\right)\lambda^n \;-\; \prod_{i=1}^{n}(c_i+\lambda a_i R_r)\,.$$

Then Proposition 3.7 implies that

$$P_n(\lambda) \;=\; \left(R_r^{n-1}\prod_{i=1}^{n}a_i\right)\lambda^n - \left[\left(\prod_{i=1}^{n}a_i\right)(\lambda R_r)^n \;+\; \sum_{i=1}^{n}A_{n-1}(\lambda R_r)^{n-1}\right], \quad (3.12)$$

where, except that $\lambda\ R_r$ is in the place of α, the $A_{n-i}(i=1,2,3,\cdots,n)$ are as before. Now comparing coefficients and doing little bit of algebra, this ends the proof. □

Observation 3.8. 1. All intermediate coefficients of P_n depend on multiples of b_n and b_0, a coding advantage.

2. Except for b_n, all the numerical coefficients of P_n are negative.

Theorem 3. (The Main Theorem) *The vector x^* and the positive multiplier λ^* that satisfy Proposition 3.4 (the necessary optimality conditions) are unique. Furthermore, λ^* is the only positive zero of $P_n(\lambda)$.*

Proof. Since it follows from Lemma 3.5 and Proposition 3.7 that the x^* and λ^* that satisfy the necessary minimality conditions are precisely the same x^* and λ^* that solve $P_n(\lambda)$, all that we need to show is that $P_n(\lambda)$ has a unique positive zero. Since except for b_n all numerical coefficients of P_n are negative and $P_n(0)=-b_0<0,$, it follows from the theory of equations that

$P_n(\lambda) = 0$ has exactly one positive root. Let λ^* denote this unique positive zero of $P_n(\lambda)$. Now observe from construction that

$$z_i^* = \rho_i^{x_i^*} \text{ and } z_i^* = \frac{c_i}{c_i + \lambda^* a_i R_r}.$$

And so we get $x^* = \frac{\ln(z_i^*)}{\ln(\rho_i)}$, which is equivalent to

$$x_i^* = \frac{\ln(c_i) - \ln(c_i + \lambda^* a_i R_r)}{\ln(\rho_i)}.$$

Finally, noting the uniqueness of λ^* and the fact that ρ_i, c_i, R_r and a_i are fixed, it follows that x^* is unique. □

Remark 3.1. 1. The unique positive root, λ^*, of $P_n(\lambda) = 0$ can easily be obtained using Newton's method with initial value $\lambda_0 = \prod_{i=1}^{n} c_i$.

2. Sufficient optimality conditions are not required for problems of the type (2.1) since there is only one acceptable solution of the Lagrange multiplier equation and all feasible points of (2.1) are regular.

3. Other nonlinear optimization techniques are also applicable but the method described in this paper gives much stronger convergence criteria.

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