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ON SKEW VERSION OF REVERSIBLE RINGS

Hamideh Pourtaherian[1], Isamiddin S. Rakhimov[1,2 §]

[1]Department of Mathematics
Faculty of Science
Universiti Putra Malaysia
Selangor, 43400, MALAYSIA

[2]Institute for Mathematical Research (INSPEM)
Universiti Putra Malaysia
Selangor, 43400, MALAYSIA

Abstract: The paper deals with α-reversible rings and their relationships with well known α-symmetric, α-Armendariz and α-semicommutative rings. We consider a skew version of some classes of rings with respect to a ring endomorphism α.

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1. Introduction

Throughout the paper, R denotes an associative ring with identity and α stands for an endomorphism of R, unless specially noted. Lambek [10] called a ring R symmetric, if $abc = 0$ implies $acb = 0$ for $a, b, c \in R$. According to Cohn [4] a ring R is called reversible provided that $ab = 0$ implies $ba = 0$ for $a, b \in R$. Semicommutative ring is a generalization of a reversible ring. A ring is said to be semicommutative, if it satisfies the following condition: whenever elements a, b in R satisfy $ab = 0$, then $aRb = 0$. Every reduced ring is symmetric [14].

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§Correspondence author

The following implications take place by simple observation [11]:

$$\text{reduced} \Longrightarrow \text{symmetric} \Longrightarrow \text{reversible} \Longrightarrow \text{semicommutative} \Longrightarrow \text{abelian.}$$

In this paper we explore the relationships between several classes of rings, provide examples confirming these relationships and prove some statements about these links.

Let R be a ring with an endomorphism α and an α-derivation δ, that is, δ is an additive map such that $\delta(ab) = \delta(a)b + \alpha(a)\delta(b)$, for all $a, b \in R$. *The Ore extension* $R[x; \alpha, \delta]$ of R is the ring obtained by introducing the new multiplication $xr = \alpha(r)x + \delta(r)$ for any $r \in R$. If $\delta = 0$ then it is denoted by $R[x; \alpha]$ and is called a skew polynomial ring (also Ore extension of endomorphism type). According to Krempa [8], an endomorphism α of a ring R is called rigid, if $a\alpha(a) = 0$ implies $a = 0$ for $a \in R$. In Hong *et al.* [6], a ring R is called α-rigid if there exists a rigid endomorphism α of R. Any rigid endomorphism of a ring is a monomorphism and α-rigid rings are reduced [6]. By Hashemi and Moussavi [5], a ring R is α-compatible if for each $a, b \in R$, $a\alpha(b) = 0$ if and only if $ab = 0$. Ben Yakoub and Louzari [3], called a ring R satisfying the condition (C_α) if whenever $a\alpha(b) = 0$ with $a, b \in R$, then $ab = 0$. Clearly, α-compatible ring satisfies the condition (C_α).

The Armendariz property of rings has been extended to skew polynomial rings in [7]. Following Hong *et al.* [7], a ring R is called α-Armendariz if for $p = \sum_{i=0}^{m} a_i x^i$ and $q = \sum_{j=0}^{n} b_j x^j$ in $R[x; \alpha]$ the condition $pq = 0$ implies $a_i b_j = 0$ for all i and j.

A property $(*)$ of a ring R is said to be *the Hilbert property*, if its polynomial extension possesses the same property $(*)$ [12].

We keep the standard notation Z for the set of all integers numbers.

2. α-Reversibility

In this section we consider a skew version of reversible rings, called α-reversible rings with respect to a ring endomorphism α. Particularly, if α is the identity endomorphism, then it is just a reversible ring. Başer *et al.* [2], called a ring R right(left) α-reversible if whenever $ab = 0$ for $a, b \in R$ then $b\alpha(a) = 0$ $(\alpha(b)a = 0)$. A ring R is called α-reversible if it is both right and left α-reversible.

Observe that every subring S with $\alpha(S) \subseteq S$ of a right α-reversible ring is also right α-reversible.

Clearly, any domain is α-reversible for any endomorphism α. Here is an example showing that there exists an α-reversible ring which is not domain.

Example 1. Let $R = \left\{ \begin{pmatrix} a & 0 \\ b & a \end{pmatrix} | a, b \in Z \right\}$ be a ring with endomorphism α defined by $\alpha\left(\begin{pmatrix} a & 0 \\ b & a \end{pmatrix} \right) = \begin{pmatrix} a & 0 \\ -b & a \end{pmatrix}$. The ring R is α-reversible. Indeed, let $A = \begin{pmatrix} a & 0 \\ b & a \end{pmatrix}$ and $B = \begin{pmatrix} c & 0 \\ d & c \end{pmatrix} \in R$ be such that $AB = 0$. So we have $ac = 0$ and $bc + ad = 0$. It implies that either $a = b = 0$ or $a = c = 0$ or $c = d = 0$. However, in all the cases above $\alpha(B)A = B\alpha(A) = 0$.

The following are examples distinguishing the concept of right and left α-reversibility.

Example 2. Consider a ring $R = \left\{ \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} | a, b, c \in Z \right\}$.

1. Let endomorphism α of R defined by $\alpha\left(\begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \right) = \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix}$.

   Let $A = \begin{pmatrix} a & b \\ 0 & c \end{pmatrix}$, and $B = \begin{pmatrix} a' & b' \\ 0 & c' \end{pmatrix} \in R$, such that $AB = 0$, then we get $aa' = 0$, $cc' = 0$ and $ab' + bc' = 0$.
   $B\alpha(A) = \begin{pmatrix} a'a & 0 \\ 0 & 0 \end{pmatrix} = 0$. Therefore R is right α- reversible but is not left α- reversible.

2. Let endomorphism β of R be defined by $\beta\left(\begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \right) = \begin{pmatrix} 0 & 0 \\ 0 & c \end{pmatrix}$.

   By the same way as above we can see that, R is left β-reversible but is not right β-reversible.

There exist a reversible ring R with endomorphism α such that R is not α-reversible ring.

Example 3. Consider ring $R = Z_2 \oplus Z_2$ with endomorphism $\alpha : R \longrightarrow R$ defined by $\alpha((a, b)) = (b, a)$ with the usual addition and multiplication. The ring R is commutative reduced hence it is reversible. However, R is not α-reversible. Indeed, $(0, 1)(1, 0) = 0$ but $\alpha((1, 0))(0, 1) \neq 0$.

Kwak [9], called an endomorphism α of a ring R, right (respectively, left) symmetric if whenever $abc = 0$ implies $ac\alpha(b) = 0$ (respectively, $\alpha(b)ac = 0$)

for $a, b, c \in R$. A ring R is called right (respectively, left) α-symmetric if there exists a right (respectively, left) symmetric endomorphism α of R. A ring R is α-symmetric if it is both right and left α-symmetric.

Proposition 1. *An α-symmetric ring is α-reversible.*

Proof. Let R be an α-symmetric ring. Suppose that $ab = 0$ for $a, b \in R$. Obviously, $1 \cdot a \cdot b = 0$, since R is right α-symmetric, then $b\alpha(a) = 0$. Hence R is right α-reversible. It is easily can be shown that R is left α-reversible as above. Therefore R is α-reversible. □

Domain $\Longrightarrow$ α-symmetric $\Longrightarrow$ α-reversible.

Theorem 1. *A ring R is α-rigid if and only if R is reduced, α-reversible and α is monomorphism.*

Proof. Let R be an α-rigid ring, then R is reduced and α is monomorphism [6]. It is enough to show that R is α-reversible. If $ab = 0$ for $a, b \in R$, so we have $ba = 0$. Then $a\alpha(ba)\alpha^2(b) = a\alpha(b)\alpha(a\alpha(b)) = 0$. Since R is α-rigid, so $a\alpha(b) = 0$ and so $\alpha(b)a = 0$, hence R is left α-reversible. On the other hand, $b\alpha(ab)\alpha^2(a) = b\alpha(a)\alpha(b\alpha(a)) = 0$, so $b\alpha(a) = 0$, hence R is right α-reversible, therefore R is α-reversible.

Conversely, assume that $a\alpha(a) = 0$ for $a \in R$. It is clear that $\alpha(a)a = 0$ and so $\alpha(a)\alpha(a) = (\alpha(a))^2 = 0$ (by α-reversibility), then $\alpha(a) = 0$ (by reducibility). Since α is a monomorphism we get $a = 0$. Therefore R is α-rigid. □

The following examples show that neither the conditions "α to be a monomorphism" no "R to be reduced" can be dropped.

Example 4. 1. $R = F[x]$ be the polynomial ring over a field F, and α be an endomorphism of R defined by $\alpha(f(x)) = f(0)$ where $f(x) \in R$. R is a commutative domain so it is reduced and α-reversible, but α is not monomorphism, hence R is not α-rigid.

2. Let $R = \left\{ \begin{pmatrix} a & 0 \\ b & a \end{pmatrix} | a, b \in Z \right\}$. and α be an endomorphism of R defined by $\alpha\left(\begin{pmatrix} a & 0 \\ b & a \end{pmatrix}\right) = \begin{pmatrix} a & 0 \\ -b & a \end{pmatrix}$. R is α-reversible but it isn't α-rigid.

Reduced rings are reversible, one may suspect that reduced rings are also α-reversible but the Example 3 above shows that it is not the case.

In the next proposition we give the relationship between α-reversibility and reducibility.

Proposition 2. *Let R be a reduced α-compatible ring then R is α-reversible.*

Proof. Let $a, b \in R$, and $ab = 0$. By hypothesis R is α-compatible so $a\alpha(b) = 0$ and then $\alpha(b)a = 0$ (by reducibility). Hence, R is left α-reversible. The right α-reversibility is obtained similarly. Therefore R is α-reversible. □

Now we consider the relationship between α-reversible and α-Armendariz ring by the following corollaries.

Corollary 1. *Reduced α-Armendariz ring is α-reversible.*

Proof. According to [7] a reduced α-Armendariz ring is α- rigid and α- rigid ring is α-reversible (by Theorem 1). □

Corollary 2. *Let R be a reduced ring with monomorphism α. Then R is α-Armendariz if and only if R is α-reversible.*

Proof. The proof is immediate from Theorem 1. □

The following examples show that neither the conditions "the reducibility" no "α to be monomorphism" can be dropped.

Example 5. Consider a ring $R = \left\{ \begin{pmatrix} a & b \\ 0 & a \end{pmatrix} | a, b \in Z_4 \right\}$ and α be an endomorphism of R defined by $\alpha\left(\begin{pmatrix} a & b \\ 0 & a \end{pmatrix}\right) = \begin{pmatrix} a & -b \\ 0 & a \end{pmatrix}$. It is observed that R isn't reduced because $\begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix}$ is nonzero nilpotent element. The ring R is α-reversible. Indeed, let $A = \begin{pmatrix} a & b \\ 0 & a \end{pmatrix}, B = \begin{pmatrix} a' & b' \\ 0 & a' \end{pmatrix} \in R$ with $AB = 0$. So we have, $aa' = 0$ and $ab' + ba' = 0$. From $aa' = 0$ we obtain $a = 0$ or $a' = 0$ or $a = a' = 2$. Therefore the following cases occur: either $(a = b = 0)$ or $(a = a' = 0)$ or $(a = 0, b = a' = 2)$ or $(a' = b' = 0)$ or $(a' = 0, a = b' = 2)$ or $(a = a' = 2, b = b')$ or $(a = a' = 2, b' = 0, b = 2)$ or $(a = a' = 2, b' = 2, b = 0)$ or $(a = a' = 2, b' = 1, b = 3)$ or $(a = a' = 2, b' = 3, b = 1)$.

These imply that $\alpha(B)A = B\alpha(A) = 0$. Therefore R is α-reversible, but is not α-Armendariz, because for $p = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} + \begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix} x \in R[x; \alpha]$, we have $p^2 = 0$ but $\begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix}\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} \neq 0$.

Example 6. Consider the reduced ring $R = Z_2[x]$ with the endomorphism α defined by $\alpha(f(x)) = f(0)$, where $f(x) \in R$. Then R is a domain, so it is α-reversible. Since α is not monomorphism the ring R is not α-Armendariz.

Başer *et al.* [1] defined the notion of α- semicommutative ring with the endomorphism α as a generalization of α-rigid ring. An endomorphism α of a ring R is called semicommutative if $ab = 0$ implies $aR\alpha(b) = 0$ for $a, b \in R$. A ring R is called α-semicommutative if there exists a semicommutative endomorphism α of R.

Proposition 3. *A reduced α-reversible ring is α-semicommutative.*

Proof. Let R be a reduced α-reversible ring. Let $ab = 0$ for $a, b \in R$ and c be an arbitrary element of R. Then $\alpha(b)a = 0$ (by α-reversibility) and $\alpha(b)ac = 0$. Hence, $ac\alpha(b) = 0$ (by reducibility). Therefore R is α-semicommutative. □

Remind that a ring R satisfies the condition (C_α) if whenever $a\alpha(b) = 0$ with $a, b \in R$, then $ab = 0$.

Proposition 4. *An α-reversible ring that satisfies the condition (C_α) is α-semicommutative.*

Proof. Suppose that R be an α-reversible ring with (C_α) condition and $ab = 0$ for $a, b \in R$. Let c be an arbitrary element of R. Hence $\alpha(b)a = 0$ (by α-reversibility), so $\alpha(b)ac = 0$, then $ac\alpha^2(b) = 0$ (by α-reversibility). Since R satisfies the condition (C_α) we get $ac\alpha(b) = 0$. Therefore R is α-semicommutative. □

Corollary 3. *An α-reversible α-compatible ring is α-semicommutative.*

The proof is obvious.

The next Theorem gives the relationship between reversible and α-reversible rings.

Theorem 2. *Let R be an α-compatible ring. Then one has*

1. *R is symmetric if and only if R is α-symmetric ring.*

2. *R is reversible if and only if R is α-reversible ring.*

3. *R is semicommutative if and only if R is α-semicommutative.*

Proof. 1. Let R be a symmetric ring and $abc = 0$, for $a, b, c \in R$. Then $acb = 0$ (by symmetricity) and $ac\alpha(b) = 0$ (by α-compatibility), hence R is right

α-symmetric. Since R is symmetric so it is reversible then we have $\alpha(b)ac = 0$, hence R is left α-symmetric. Therefore R is α-symmetric ring.

Conversely, let R be an α-symmetric ring and $abc = 0$ for $a, b, c \in R$. So we have $ac\alpha(b) = 0$, and $acb = 0$ (by α-compatibility). Therefore R is symmetric ring.

2. Let R be a reversible ring and $ab = 0$, for $a, b \in R$. Then $ba = 0$ and $b\alpha(a) = 0$(by α-compatibility). Therefore R is right α-reversible.

On the other hand, $ab = 0$ we have $a\alpha(b) = 0$, so $\alpha(b)a = 0$(by reversibility) hence R is left α-reversible. Therefore R is α-reversible.

Conversely, let $ab = 0$ for $a, b \in R$, then $b\alpha(a) = 0$ (by right α-reversibility) and $ba = 0$(by α-compatibility). Therefore R is reversible.

3. Let R be a semicommutative ring and $ab = 0$ for $a, b \in R$, so $aRb = 0$. Since R is α-compatible it implies that $aR\alpha(b) = 0$. Therefore R is α-semicommutative ring. The "only if" part is obvious. □

By the example 3, we see that the condition "α-compatibility" is not superfluous.

In [3] the authors proved that, for a ring R with (C_α) condition, R is reversible (respectively, symmetric) if and only if R is α- reversible (respectively, α-symmetric). Hong *et al.* [7], proved that an α- Armendariz ring satisfies the condition (C_α).

Combining these two results we can state the following

Theorem 3. *An α-Armendariz ring R is reversible (respectively, symmetric) if and only if R is α- reversible (respectively, α-symmetric).*

As an immediate consequence of this theorem we get

Corollary 4. *An α-rigid ring R is reversible (respectively, symmetric) if and only if R is α- reversible (respectively, α-symmetric).*

2.1. Extension of α-Reversible Ring

Let R_i be a ring and α_i be an endomorphism of R_i for each $i \in \Gamma$. Clearly, the product $\prod_{i\in\Gamma} R_i$ of R_i with endomorphism $\bar{\alpha} : \prod_{i\in\Gamma} R_i \longrightarrow \prod_{i\in\Gamma} R_i$ defined by $\bar{\alpha}((a_i)) = (\alpha_i(a_i))$ is $\bar{\alpha}$-reversible if and only if each R_i is α_i-reversible.

2.1.1. Polynomial Ring

For reduced rings α-reversibility is the Hilbert property. The endomorphism α of a ring R can be extended to endomorphism $\bar{\alpha}$ of $R[x]$ by

$$\bar{\alpha}\left(\sum_{i=0}^{n} a_i x^i\right) = \sum_{i=0}^{n} \alpha(a_i) x^i.$$

Now we show that how the notion of α-reversibility can be extended from R to $R[x]$.

Theorem 4. *For a reduced ring R with an endomorphism α, R is α-reversible if and only if $R[x]$ is $\bar{\alpha}$-reversible.*

Proof. Suppose that R is α-reversible. Let $f(x) = \sum_{i=0}^{n} a_i x^i$ and $g(x) = \sum_{j=0}^{m} b_j x^j \in R[x]$ be such that $f(x)g(x) = 0$. We show that $\bar{\alpha}(g(x))f(x) = g(x)\bar{\alpha}(f(x)) = 0$. Indeed, from $f(x)g(x) = 0$ we have the following system of equations:

$$\begin{aligned} (0) &: \quad a_0 b_0 = 0, \\ (1) &: \quad a_0 b_1 + a_1 b_0 = 0, \\ (2) &: \quad a_0 b_2 + a_1 b_1 + a_2 b_0 = 0, \\ &\quad \vdots \end{aligned} \tag{1}$$

We make use the reducibility of R. Then $b_0 a_0 = 0$. If we multiply (1) from the left hand side by b_0, then we get $(b_0 a_1)^2 = 0$. Hence, $b_0 a_1 = 0$, and $a_1 b_0 = 0$. The left α-reversibility of R implies that $\alpha(b_0)a_1 = 0$. From (1) we get $a_0 b_1 = 0$ and then $\alpha(b_1)a_0 = 0$. If we multiply (2) from the left hand side by b_0, then $a_2 b_0 = 0$, so $\alpha(b_0)a_2 = 0$. Now we multiply (2) again from the left hand side by b_1, we obtain $a_1 b_1 = 0$. Hence $\alpha(b_1)a_1 = 0$. The equation (2) becomes $a_0 b_2 = 0$, so $\alpha(b_2)a_0 = 0$. Continuing this way it can be shown eventually that

$$\bar{\alpha}(g(x))f(x) = (\alpha(b_0) + \alpha(b_1)x + ... + \alpha(b_m)x^m)(a_0 + a_1 x + ... + a_n x^n) = 0.$$

Therefore $R[x]$ is left $\bar{\alpha}$-reversible. Similarly we can prove that $R[x]$ is right $\bar{\alpha}$-reversible, so the polynomial ring $R[x]$ is $\bar{\alpha}$-reversible. The converse is clear. □

2.1.2. Skew Polynomial Ring

Now we show that how the notion of α-reversibility can be extended from R to $R[x;\alpha]$. The following Theorem is an generalization of results of Rege and Chhawchharia [13], and Hong *et al.* [7].

Theorem 5. *Let R be an α-Armendariz ring. Then the following holds true.*

1. *R is reversible if and only if $R[x;\alpha]$ is reversible.*
2. *R is symmetric if and only if $R[x;\alpha]$ is symmetric.*

Proof. Let $p = \sum_{i=0}^{m} a_i x^i$, $q = \sum_{j=0}^{n} b_j x^j$ and $h = \sum_{k=0}^{l} c_k x^k$ be elements of $R[x;\alpha]$.

1. Any subring of symmetric (respectively, reversible) ring is symmetric (respectively, reversible), therefore the "only if" part is obvious. Suppose that R is reversible ring and $pq = 0$. Since R is α- Armendariz ring one gets $a_i b_j = 0$ for all i and j. From the α- reversibility of R we obtain $b_j\alpha(a_i) = 0$ for all i and j, then from the reversibility of R we have $\alpha(a_i)b_j = 0$ for all i and j. Hence $b_j\alpha^2(a_i) = 0$ for all i and j. Continuing this process, we can see that $b_j\alpha^j(a_i) = 0$ for all i and j. This implies that $qp = \sum_{t=0}^{m+n} \sum_{i+j=t} b_j\alpha^j(a_i)x^t = 0$. Therefore $R[x;\alpha]$ is reversible.
2. If $hpq = 0$, then $c_k a_i b_j = 0$ for all i, j and k. Due to symmetricity property $a_i c_k b_j = 0$ for all i, j and k. Then $c_k b_j a_i = 0$ for all i, j and k (by the reversibility), and $a_i\alpha(c_k)\alpha(b_j) = 0$ (by the α- reversibility). It implies that $\alpha(c_k)\alpha(b_j)a_i = 0$, hence $a_i\alpha^2(c_k)\alpha^2(b_j) = 0$ for all i, j and k. Continuing this process, we get $a_i\alpha^t(c_k)\alpha^t(b_j) = 0$ for any non-negative integer t, then $\alpha^{t+1}(b_j)a_i\alpha^t(c_k) = 0$ (by the α- reversibility), hence $a_i\alpha^t(c_k)\alpha^{t+1}(b_j) = 0$ (by the reversibility) for all i, j and k. Applying this process, we obtain that $a_i\alpha^t(c_k)\alpha^s(b_j) = 0$ for any $s \geq t$ and for all i, j and k. Then $phq = \sum_{e=0}^{m+n+l} \sum_{i+j+k=e} a_i\alpha^i(c_k)\alpha^{i+k}(b_j)x^e = 0$. Therefore $R[x;\alpha]$ is symmetric.

□

The semicommutativity is not the Hilbert property, in general. There exist a skew polynomial ring $R[x;\alpha]$ over a semicommutative ring R which is not semicommutative.

Example 7. Consider $R = \{(a, b) \in Z \oplus Z; a \equiv b(mod 2)\}$ with usual componentwise addition and multiplication. It is easy to see that R is a commutative reduced ring. Let $\alpha : R \longrightarrow R$ be the endomorphism defined by $\alpha((a, b)) = (a, 0)$. Obviously R is semi-commutative, however $R[x; \alpha]$ is not. Indeed, for $p = (0, b)x \in R[x; \alpha]$ $(b \neq 0)$, $p^2 = 0$ but $p(a, c)p \neq \bar{0}$.

Theorem 6. *Let R be an α-rigid ring. Then the following holds.*

1. *R is semicommutative if and only if $R[x; \alpha]$ is semicommutative.*
2. *R is reversible if and only if $R[x; \alpha]$ is reversible.*
3. *R is symmetric if and only if $R[x; \alpha]$ is symmetric.*

Proof. 1. A subring of semicommutative ring is semicommutative, therefore the "only if" part is obvious.

Let now $p = \sum\limits_{i=0}^{m} a_i x^i$, $q = \sum\limits_{j=0}^{n} b_j x^j$ and $h = \sum\limits_{k=0}^{l} c_k x^k$ be elements of $R[x; \alpha]$ with $pq = 0$. Since R is α-Armendariz ring one gets $a_i b_j = 0$ for all i and j, then $a_i c_k b_j = 0$ for all i, j, k (by the semicommutativity). Moreover, due to [6] the ring R is reduced and therefore it is reversible. So we have $c_k b_j a_i = 0$ for all i, j, k. Hence $a_i \alpha(c_k)\alpha(b_j) = 0$ for all i, j, k (by the α- reversibility). By the similar arguments those in the proof of Theorem 5, we obtain $a_i \alpha^t(c_k)\alpha^s(b_j) = 0$ for any $s \geq t$ and for all i, j, k. Then $phq = \sum\limits_{r=0}^{m+n+l} \sum\limits_{i+j+k=r} a_i \alpha^i(c_k)\alpha^{i+k}(b_j)x^r = 0$. Therefore $R[x; \alpha]$ is semicommutative.

2. It is a immediate consequence of Theorem 5.

3. It follows from the second part of Theorem 5.

□

Corollary 5. *Let R be a reduced α-compatible ring. Then the following statements are equivalent.*

1. *R is semicommutative (resp., symmetric, reversible)*
2. *R is α-semicommutative (resp., α-symmetric, α-reversible).*
3. *$R[x; \alpha]$ is semicommutative (resp., symmetric, reversible).*

Proof. (1)$\Leftrightarrow$ (2) due to Theorem 2.

The statement (1)$\Leftrightarrow$ (3) follows from Theorem 6 and the fact that reduced α-compatible rings are α-rigid. □

Theorem 7. *Let R be an α-Armendariz ring. Then the following statements are equivalent.*

1. *R is reversible (resp., symmetric)*
2. *R is α-reversible (resp., α-symmetric).*
3. *$R[x;\alpha]$ is reversible (resp., symmetric).*

Proof. Part (1)$\Leftrightarrow$ (2) follows from Theorem 3.

The statement (1)$\Leftrightarrow$ (3) is a consequence of Theorem 5. □

2.1.3. Trivial Extension of Ring

For an endomorphism α of a ring R, let $\bar{\alpha}$ stand for the endomorphism of $T(R,R)$ defined by $\bar{\alpha}\left(\begin{pmatrix} a & b \\ 0 & a \end{pmatrix}\right) = \begin{pmatrix} \alpha(a) & \alpha(b) \\ 0 & \alpha(a) \end{pmatrix}$. For an α-reversible ring R, the trivial extension $T(R,R)$ need not to be an $\bar{\alpha}$-reversible ring by the next example.

Example 8. Consider a ring $R = \left\{\begin{pmatrix} a & b \\ 0 & a \end{pmatrix} | a,b \in Z\right\}$ with an endomorphism α defined by $\alpha\left(\begin{pmatrix} a & b \\ 0 & a \end{pmatrix}\right) = \begin{pmatrix} a & b \\ 0 & -a \end{pmatrix}$.

Clearly, R is α-reversible. Indeed, let $A = \begin{pmatrix} a & b \\ 0 & a \end{pmatrix}, B = \begin{pmatrix} c & d \\ 0 & c \end{pmatrix} \in R$ such that $AB = 0$, so $ac = 0$ and $ad + bc = 0$.

$\alpha(B)A = \begin{pmatrix} ca & cb+da \\ 0 & -ca \end{pmatrix} = 0$ and $B\alpha(A) = \begin{pmatrix} ca & 2bc \\ 0 & -ca \end{pmatrix} = 0$. Therefore R is α-reversible.

The extension $T(R,R)$ is not $\bar{\alpha}$-reversible. Consider the following two elements of $T(R,R)$.

$$A = \begin{pmatrix} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} -1 & 1 \\ 0 & -1 \end{pmatrix} \\ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \end{pmatrix},$$

$$B = \begin{pmatrix} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} -1 & 1 \\ 0 & 1 \end{pmatrix} \\ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \end{pmatrix}.$$

It is clear that $AB = 0$, however $\bar{\alpha}(B)A \neq 0$. Therefore $T(R,R)$ is not $\bar{\alpha}$-reversible.

Proposition 5. *Let R be a reduced ring. If R is an α-reversible ring, then $T(R,R)$ is an $\bar{\alpha}$-reversible ring.*

Proof. Let $A = \begin{pmatrix} a & b \\ 0 & a \end{pmatrix}, B = \begin{pmatrix} c & d \\ 0 & c \end{pmatrix} \in T(R,R)$ such that $AB = 0$. So we have $ac = 0$ and $ad + bc = 0$. By the α-reversibility $\alpha(c)a = 0$. The $cad + cbc = 0$ follows $cb = 0$ and $ad = 0$ (by the reducibility). Therefore $\alpha(c)b = 0$ and so $\alpha(d)a = 0$.These all imply that

$$\bar{\alpha}(B)A = \begin{pmatrix} \alpha(c)a & \alpha(c)b + \alpha(d)a \\ 0 & \alpha(c)a \end{pmatrix} = 0.$$

Therefore$T(R,R)$ is a left $\bar{\alpha}$-reversible ring. Similarly we can prove that $T(R,R)$ is a right $\bar{\alpha}$-reversible, hence $T(R,R)$ is an $\bar{\alpha}$-reversible □

Corollary 6. *If R is an α-rigid ring, then $T(R,R)$ is an $\bar{\alpha}$-reversible ring.*

Proof. The result follows from Theorem 1 and proposition 5. □

2.1.4. Classical Quotient Ring

Let R be a ring with automorphism α. Suppose that there exists the classical left quotient $Q(R)$ of R. Then $\bar{\alpha}$ defined by $\bar{\alpha}(b^{-1}a) = (\alpha(b))^{-1}\alpha(a)$ is an automorphism of $Q(R)$.

Proposition 6. *Suppose that there exists the classical left quotient $Q(R)$ of a ring R with automorphism α. Then R is α-reversible if and only if $Q(R)$ is $\bar{\alpha}$-reversible ring.*

Proof. Suppose that R is α-reversible. Let $b^{-1}a, c^{-1}d \in Q(R)$ such that $b^{-1}ac^{-1}d = 0$. Thus $\alpha(c^{-1}d)b^{-1}a = 0$, then

$$(\alpha(c))^{-1}\alpha(d)b^{-1}a = 0.$$

So $\bar{\alpha}(c^{-1}d)b^{-1}a = 0$, therefore $Q(R)$ is right $\bar{\alpha}$-reversible. Similarly one can show that $Q(R)$ is left $\bar{\alpha}$-reversible, therefore $Q(R)$ is $\bar{\alpha}$-reversible.

The converse is clear if one makes use the fact that $R \subset Q(R)$. □

Let R be a ring with automorphism α and S a multiplicatively closed subset of R consisting of central regular elements. Let $\bar{\alpha}$ be the automorphism of $S^{-1}R$ defined by $\bar{\alpha}(b^{-1}a) = (\alpha(b))^{-1}\alpha(a)$.

The proof of the following proposition is similar that of Proposition 6.

Proposition 7. *A ring R with automorphism α is α-reversible if and only if $S^{-1}R$ is $\bar{\alpha}$-reversible*

Recall that, the ring of Laurent polynomials over a ring R, denoted by $R[x, x^{-1}]$, consists of all formal sums $\sum_{i=m}^{n} a_i x^i$ with usual addition and multiplication, where $a_i \in R$ and m, n are integers (possibly negative). For an automorphism α of R, let $\bar{\alpha}$ be the automorphism of $R[x, x^{-1}]$ defined by $\bar{\alpha}(f(x)) = \sum_{i=m}^{n} \alpha(a_i)x^i$, where $f(x) \in R[x, x^{-1}]$.

Corollary 7. *For a ring R with automorphism α, the polynomial ring $R[x]$ is $\bar{\alpha}$-reversible if and only if $R[x, x^{-1}]$ is $\bar{\alpha}$-reversible.*

Proof. Suppose that $R[x]$ is $\bar{\alpha}$-reversible. Let $S = \{1, x, x^2, ...\}$. Clearly S is a multiplicatively closed subset of $R[x]$. It is clear that $R[x, x^{-1}] = S^{-1}R[x]$. Since $R[x]$ is $\bar{\alpha}$-reversible so is $S^{-1}R[x]$ by Proposition 7. Therefore $R[x, x^{-1}]$ is $\bar{\alpha}$-reversible. The converse is obvious. □

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